Modeling of Coupled Circuits

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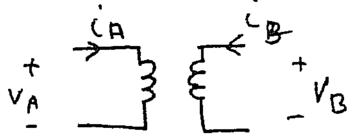


Fig 1: Transformer Windings

$$\begin{bmatrix} V_A \\ V_B \end{bmatrix} = \begin{bmatrix} L_{AA} & L_{AB} \\ L_{BA} & L_{BB} \end{bmatrix} \frac{d}{dt} \begin{bmatrix} i_A \\ i_B \end{bmatrix} \quad \text{— differential Equation for Coupled Circuit.} \qquad (1)$$

Let us evaluate L_{AA}, L_{BB} --- etc for a typical Transformer.

Ex: Consider a 100 MVA transformer with a primary/secondary rated voltage of 100kV/50kV. Let the leakage inductance be 10% (distributed equally on both sides) and the magnetizing current is 1%.

Thus:

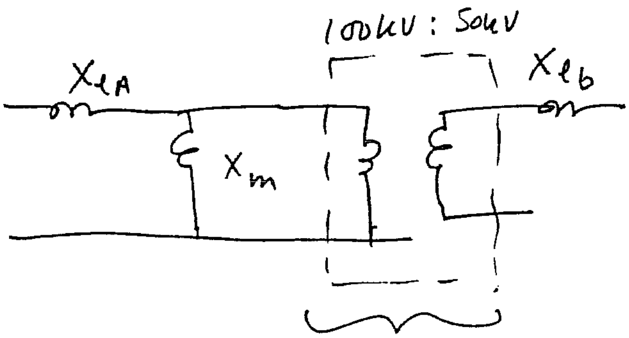


Fig 2: Transformer Equivalent Circuit

$I_{base} = \dfrac{MVA\ base}{kV_{base}} = 1\,kA$

$\therefore I_m = 1\% \times I_{base}$
$= 0.01\,kA$

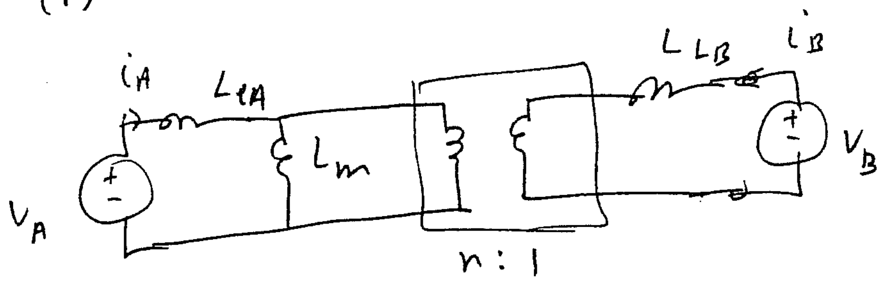
$\therefore X_m = \dfrac{kV_{base}}{I_m} = 10\,k\Omega$

$Z_{base} = \dfrac{kV_{base}^2}{MVA_{base}} = 100\,\Omega.$ $\quad \therefore X_{lA} = \dfrac{10\%}{2} \times 100\,\Omega$
$= 5\,\Omega.$

$X_{lB} = \dfrac{10\%}{2} \times 25\,\Omega$
$= 1.25\,\Omega$

$\therefore L_{lA} = 13.26\,mH$, $L_{lB} = 3.32\,mH$, $L_m = 26.526\,mH$
assuming $\omega_0 = 377$ rad/s

Let us now try to obtain the circuit equations in the form (1).



We see that

$$V_A = L_{\ell A}\frac{di_A}{dt} + L_m \frac{d}{dt}\left(i_A + \frac{i_B}{n}\right)$$

$$V_B = \frac{1}{n}\left[L_m \frac{d}{dt}\left(i_A + \frac{i_B}{n}\right)\right] + L_{LB}\cdot\frac{d}{dt} i_B$$

Thus:

$$\begin{bmatrix} V_A \\ V_B \end{bmatrix} = \begin{bmatrix} L_{\ell A} + L_m & \frac{L_m}{n} \\ \frac{L_m}{n} & \frac{L_m}{n^2} + L_{\ell B} \end{bmatrix} \frac{d}{dt} \begin{bmatrix} i_A \\ i_B \end{bmatrix}$$

Thus:

$$L_{\ell A} + L_m = L_{AA}, \quad \frac{L_m}{n} = L_{AB} = L_{BA}$$

$$\frac{L_m}{n^2} + L_{\ell B} = L_{BB}.$$

Obtaining an admittance matrix formulation

$$\begin{bmatrix} V_A \\ V_B \end{bmatrix} = \mathcal{L} \frac{d}{dt} \begin{bmatrix} i_A \\ i_B \end{bmatrix} \qquad - (1).$$

Thus:

$$\underbrace{\begin{bmatrix} i_A \\ i_B \end{bmatrix}}_{\Downarrow\ \underline{i}(t)} = \mathcal{L}^{-1} \int_{t-\Delta t}^{t} \begin{bmatrix} V_A \\ V_B \end{bmatrix} dt + \begin{bmatrix} i_A(t-\Delta t) \\ i_B(t-\Delta t) \end{bmatrix}$$

<u>By Trapezoidal Rule</u>

$$\underline{i}(t) = \mathcal{L}^{-1} \frac{\underline{v}(t) + \underline{v}(t-\Delta t)}{2} \cdot \Delta t + \underline{i}(t-\Delta t)$$

$$= \frac{\mathcal{L}^{-1}\Delta t}{2} v(t) + \left[i(t-\Delta t) + \frac{\mathcal{L}^{-1} \underline{v}(t-\Delta t)\Delta t}{2} \right]$$

$$= \underbrace{S}_{\text{Admittance matrix.}} \underline{v}(t) + \underbrace{\underline{I}_h(t-\Delta t)}_{\text{History Term}} . \qquad -(2)$$

Now each element of $\underline{i}(t)$ is the current in a branch connected (say) between nodes k and m. Similarly each element of $\underline{v}$ is $v_k - v_m$ where v_k and v_m are node voltages. We now write the KCL equations at node k or m by including the term $+i_{km}$ at node k or $-i_{km}$ at node m.

Note that $i_{km} = s_1' v_1 + s_2' v_2 + \cdots + s' v_n + i_{hkm}(t-\Delta t)$ from eqn. 2. Thus at node k,

$$0 = -g_{1k} v_1 - g_{2k} v_2 \cdots + (g_{k1} + g_{k2} + \cdots + g_{kn}) v_k - \cdots - g_{nk} v_n$$
$$- J_k + I_{hk}(t-\Delta t)$$
$$+ (+s_1' v_1 + s_2' v_2 \cdots + s_n' v_n) + i_{hkm}(t-\Delta t) \qquad - 3$$

where as before, g_{1k}, .. etc. are the admittances to other branches, J_k is the current injection from external sources and $I_{hk}(t-\Delta t)$ is the history term injection.

The coefficients s_1' etc. are from the appropriate row of (2) corresponding to transformer branch currents ... into transformer branches connected at node k.

Thus:

$$(-g_{1k}-s_1')\,v_1 + (-g_{2k}-s_2')\,v_2 \cdots + (g_{k1}+\cdots g_{kn}-s_k')\,v_k + (-s_n'-g_{nk})\,v_n = J_k - I_{hk}(t-\Delta t) + i_{hkm}(t-\Delta t).$$

Similarly all other KCL equations are developed, and we obtain for the network, the familiar form:

$$[Y]\,\underline{v}(t) = \underline{J}(t) - \underline{I}_h(t-\Delta t) \qquad -(5)$$

which is then solved in the usual manner.

Example:

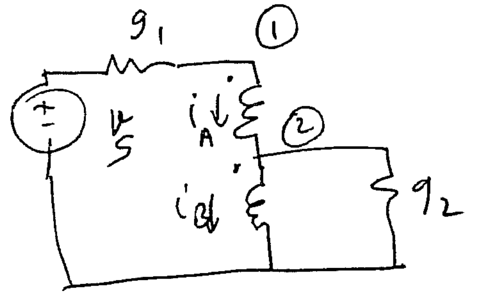


Fig 4: Coupled Circuit

$$\begin{pmatrix} i_A \\ i_B \end{pmatrix} = \begin{bmatrix} L_{AA} & L_{AB} \\ L_{BA} & L_{BB} \end{bmatrix}^{-1} \frac{\Delta t}{2} \cdot \begin{pmatrix} v_A \\ v_B \end{pmatrix} + \underbrace{\begin{bmatrix} I_{LA}(t-\Delta t) \\ I_{LB}(t-\Delta t) \end{bmatrix}}_{\text{In the form of Eqn. 2.}} \qquad (6)$$

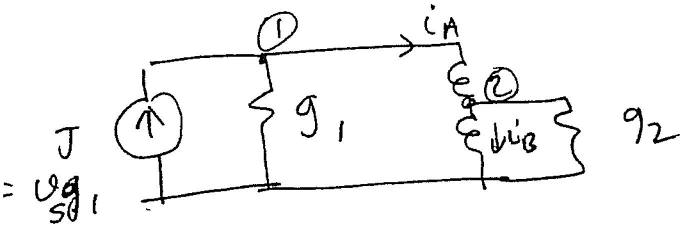
We assign node numbers as shown (1 & 2) in Fig 4.
The history terms in the above equation have the form:

(6)

where $I_{LA}(t-\Delta t) = i_A(t-\Delta t) + S_{AA}\left[V_1(t-\Delta t) - V_2(t-\Delta t)\right] + S_{AB}\left(V_2(t-\Delta t)\right)$

$$I_{LB}(t-\Delta t) = i_B(t-\Delta t) + S_{BA}\left[V_1(t-\Delta t) - V_2(t-\Delta t)\right] + S_{BB}\, V_2(t-\Delta t). \quad (7)$$

Let us write KCL at node ① and ②



$$\left.\begin{aligned} -J + g_1 V_1 + i_A &= 0 \\ -i_A + i_B + g_2 V_2 &= 0 \end{aligned}\right] \quad (8)$$

expanding for $i_A = S_{AA} V_A + S_{AB} V_B + I_{LA}(t-\Delta t)$

and for $i_B = S_{BA} V_A + S_{BB} V_B + I_{LB}(t-\Delta t)$

and realizing that $V_A = V_1 - V_2$, $V_B = V_2$, we get:

$$-J + g_1 V_1 + S_{AA}(V_1 - V_2) + S_{AB} \cdot V_2 + I_{LA}(t-\Delta t) = 0$$

$$-S_{AA}(V_1 - V_2) - S_{AB} V_2 - I_{LA}(t-\Delta t) + S_{BA}(V_1 - V_2) + S_{BB} V_2 + I_{LB}(t-\Delta t) = 0$$

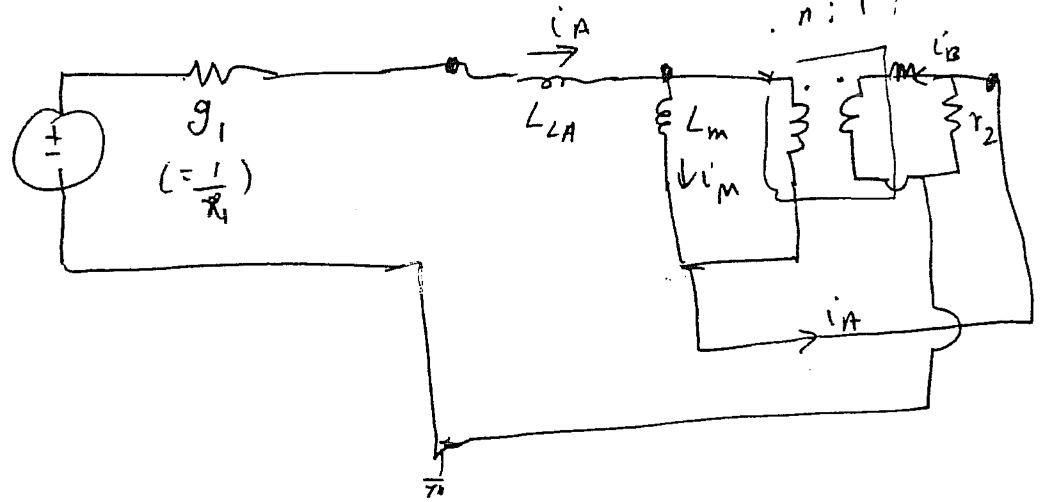
or $$\begin{bmatrix} g_1 + S_{AA} & -S_{AA} - S_{AB} \\ -S_{AA} + S_{BA} & S_{AA} - S_{AB} - S_{BA} + S_{BB} \end{bmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{bmatrix} J - I_{LA}(t-\Delta t) \\ + I_{LA}(t-\Delta t) - I_{LB}(t-\Delta t) \end{bmatrix}$$

— (9)

or $[Y] V(t) = \underline{I}(t-\Delta t)$ (6)

and $V(t)$ can be evaluated in the standard manner. The history terms for the next time-step are evaluated from (7) and the currents in the windings from (6).

State Variable Formulation



$R_2 = \frac{1}{g_2}$

Choose i_A, i_B as state variables. Only 2 can be chosen because the three are not linearly independent. Then $i_m = i_A + \frac{i_B}{n}$

VL equations give:

$$V_S - R_1 i_A - L_m \frac{di_m}{dt} + R_2 (i_A - i_B) = 0$$
$$- L_{LA} \frac{di_A}{dt}$$

ho $(i_A - i_B) R_2 - L_B \frac{di_B}{dt} = L_m \frac{di_m}{dt} \cdot \frac{1}{n}$

Thus:

$$v_S - R_1 i_A - L_{LA}\frac{di_A}{dt} - L_m \frac{d}{dt}\left(i_A + \frac{i_B}{n}\right) + R_2 (i_A - i_B) = 0$$

$$(i_A - i_B) R_2 - L_{LB}\frac{di_B}{dt} = \frac{1}{n} \cdot L_m \left[i_A + \frac{i_B}{n} \right]$$

Thus:

$$\begin{bmatrix} -L_m - L_{LA} & -\frac{L_m}{n} \\ -\frac{L_m}{n} & -L_{LB} - \frac{L_m}{n^2} \end{bmatrix} \frac{d}{dt}\begin{bmatrix} i_A \\ i_B \end{bmatrix} = \begin{bmatrix} -V_S \\ 0 \end{bmatrix} + \begin{bmatrix} R_1 + R_2 & -R_2 \\ R_2 & -R_2 \end{bmatrix}\begin{bmatrix} i_A \\ i_B \end{bmatrix}$$

or

$$\begin{pmatrix} \dot{i}_A \\ \dot{i}_B \end{pmatrix} = \left[\mathcal{L}\right]^{-1}\left[R\right]\begin{pmatrix} i_A \\ i_B \end{pmatrix} + \begin{bmatrix} -V_S \\ 0 \end{bmatrix}$$

$$\dot{\underline{X}} = A\,\underline{X} + B\underline{u}$$