Extremal problems for affine cubes of integers

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Abstract

A collection H of integers is called an *affine d-cube* if there exist $d+1$ positive integers $x_0, x_1, \ldots, x_d$ so that

$$H = \left\{ x_0 + \sum_{i \in I} x_i : I \subseteq \{1, 2, \ldots, d\} \right\}.$$

We address both density and Ramsey-type questions for affine d-cubes. Regarding density results, upper bounds are found for the size of the largest subset of $\{1, 2, \ldots, n\}$ not containing an affine d-cube. In 1892 Hilbert published the first Ramsey-type result for affine d-cubes by showing that for any positive integers r and d, there exists a least number $n = h(d, r)$ so that for any r-coloring of $\{1, 2, \ldots, n\}$, there is a monochromatic affine d-cube. Improvements for upper and lower bounds on $h(d, r)$ are given for $d > 2$.

1 Introduction

In this section, we give a brief survey of results related to partition and density problems for affine cubes of integers. For a more detailed survey of these extremal problems, see [16], for example. In Section 2, we state two results (Theorems 2.3 and 2.5) and contrast them to known bounds. The proofs of these follow in the remaining sections.

For a set X we use the standard notations $\mathcal{P}(X) = \{Y : Y \subseteq X\}$ and $[X]^s = \{S \subset X : |S| = s\}$. It will often be convenient to use $X = [n] = [1, n] = \{1, 2, \ldots, n\}$. An arithmetic progression of length k will be denoted by AP_k.

1.1 Arithmetic progressions

In 1927, van der Waerden published his well known partition theorem for integers (see also [36] for history of its proof).

Theorem 1.1 (van der Waerden [35]) *For every pair of positive integers k and r, there exists a least $n = W(k,r)$ so that for any partition of $[1,n]$ into r classes, one class contains an AP_k.*

For more proofs of van der Waerden's theorem we refer the reader to any of, for example, [3], [13], [15], [19], [20], or [30]. The function $\mathrm{W}(k,r)$ is primitive recursive (see [30]), and aside from a few small values, not much more is known about upper bounds for $\mathrm{W}(k,r)$.

In 1946, Behrend proved a (lower bound) density result for subsets of $[1,n]$ not containing any AP_3. (See [9] and [10] for related results.)

Theorem 1.2 (Behrend [1]) *There exists a constant c so that for m sufficiently large, there exists an AP_3-free set $B \subset [1,m]$ with*

$$|B| \geq me^{-c\sqrt{\ln m}} = m^{1-o(1)}.$$

1.2 Affine cubes

In 1892, Hilbert [22] proved the first non-trivial partition Ramsey-type theorem. This theorem, which preceded the celebrated Schur and van der Waerden theorems, states that in any finite coloring of the set of positive integers, there exists a monochromatic "affine d-cube" (defined below).

Definition 1.3 *A collection H of integers is called a d-*dimensional affine cube*, or simply, an* affine d-cube *if and only if there exist $d+1$ positive integers $x_0, x_1, \ldots, x_d$ so that*

$$H = \left\{ x_0 + \sum_{i \in I} x_i : I \subseteq [1,d] \right\}.$$

If all sums are distinct, that is, if $|H| = 2^d$, then we say that the affine d-cube is replete.

If an affine d-cube H is generated by $x_0, x_1, \ldots, x_d$, then we write $H = H(x_0, x_1, \ldots, x_d)$. For example, $H(1,1,1) = \{1,2,3\}$, while a replete affine 2-cube is $H(1,3,9) = \{1,4,10,13\}$. We note that $H(x_0, x_1, \ldots, x_d)$ may differ from, say, $H(x_1, x_0, \ldots, x_d)$.

Hilbert originally proved that if the positive integers are colored with finitely many colors, then one color class contains a (monochromatic) d-cube. By compactness, a finite version also holds.

Theorem 1.4 (Hilbert [22]) *For every r, d, there exists a least number $h(d,r)$ so that for every coloring*

$$\chi : [h(d,r)] \to [1,r],$$

there exists an affine d-cube monochromatic under χ.

Theorem 1.4 now follows from van der Waerden's Theorem since an AP_{d+1} given by $\{a, a+k, a+2k, \ldots, a+dk\}$ is also an affine d-cube $H(a,k,k,\ldots,k)$.

There have been many, in some sense, strengthened versions of Hilbert's result by eliminating the x_0 in Definition 1.3. For a set $\{x_1, \ldots, x_d\}$, the set of all finite non-empty sums $\{\sum_{i \in I} x_i : I \subseteq [1,d], I \neq \emptyset\}$ is sometimes called a *projective d-cube* (see, for example, [26]) or *finite sum* set. For an infinite set $\{x_1, x_2, \ldots\}$, the set of all finite non-empty sums is said to be an infinite-dimensional projective cube or a *Hindman set*.

Rado [27], and later independently, Sanders [28] and Folkman (see [15] or [17]), showed that for any r and d, there exists a least number $n = \mathrm{FRS}(r,d)$ so that for any partition of $[1,n]$ into r classes, one class contains a projective d-cube. The case $d = 2$ is the celebrated Schur's Theorem [29]. Hindman [23] settled a conjecture of Graham and Rothschild [18] by showing that upon any finite coloring of the positive integers, there exists a monochromatic infinite-dimensional projective cube. Any known upper bounds on $\mathrm{FRS}(r,d)$ are tower functions (see [34] or [16] for discussion) and for large d, there is a huge gap between upper and lower bounds. In Theorem 2.5, we give relatively tight bounds for $h(d,r)$.

2 Results

In 1969, Szemerédi proved that if a set A of positive integers has positive upper density, (i.e., $\overline{\lim\limits_{n\to\infty}} |A \cap [1,n]|/n > 0$) then A contains an AP_4. In the same paper (Lemma $p(\delta, l)$, p. 93), Szemerédi gave a density version of Theorem 1.4; this density version has since become known as "Szemerédi's cube lemma" (cf. [20]).

Theorem 2.1 (Szemerédi's cube lemma [32]) *For any $0 < \epsilon < 1$ and positive integer d, there exists $n_0 = n_0(\epsilon, d)$ so that for all $n \geq n_0$, if $A \subset [1, n]$ and $|A| > \epsilon n$ then A contains an affine d-cube.*

Szemerédi's proof was by induction on d, and although no explicit bounds on n_0 are mentioned, the argument shows that $n_0(\epsilon, d)$ is much larger than $(2/\epsilon)^{2^d}$.

Another proof of (a strengthened version of) Szemerédi's cube lemma was found by Graham et al [15] and [20]. Lovász [25] (Problem 14.12) gives two proofs, one elegant proof using Ramsey's theorem, and another, based on the same idea as in [15] and [20] which actually proves the following:

Theorem 2.2 *Let $d \geq 2$ be given. Then there exists $n_0 = n_0(d)$ so that for every $n \geq n_0$, if $A \subseteq [1, n]$ satisfies*

$$|A| \geq (4n)^{1-\frac{1}{2^{d-1}}},$$

then A contains an affine d-cube.

We point out that in the proof of Theorem 2.2 no attempt was made to find the best possible constant. In Section 3, we modify this proof further and obtain the following improvement in the constant and are explicit about bounds on $n_0(d)$.

Theorem 2.3 *Let $d \geq 3$ be given. Then there exists $n_0 = n_0(d) \leq 2^{\frac{d2^{d-1}}{2^{d-1}-1}-1}$ so that for every $n \geq n_0$, if $A \subseteq [1, n]$ satisfies*

$$|A| \geq 2^{1-\frac{1}{2^{d-1}}}(\sqrt{n}+1)^{2-\frac{1}{2^{d-2}}},$$

then A contains an affine d-cube.

For reference, we note a modest improvement over Theorem 2.2.

Corollary 2.4 *For $d \geq 3$ there exists $n_0 \leq (\frac{2^d-2}{\ln 2})^2$ so that for every $n \geq n_0$, if $A \subseteq [1,n]$ satisfies*

$$|A| \geq 2n^{1-\frac{1}{2^{d-1}}}$$

then A contains an affine d-cube.

We leave it to the reader to check that the hypotheses of Corollary 2.4 indeed satisfy the hypotheses in Theorem 2.3. In Section 5 we give yet another proof of Szemerédi's cube lemma based on an extremal result for hypergraphs.

In [2] it was shown that $h(2,r) = (1+o(1))r^2$; the lower bound uses Singer sets (cyclic difference sets arising from a finite projective plane, see also [31]) and the upper bound follows from well known bounds for B_2-sets, or Sidon sets (see also [6]). Also in [2], it was noted that there exist constants c_1 and c_2 so that

$$r^{c_1 d} \leq h(d,r) \leq r^{(c_2)^d}, \tag{1}$$

where $c_2 \sim 2.6$ follows from Hilbert's original proof (using Fibonacci numbers). Section 6 is devoted to the proof of the following improvement of both bounds in (1) for $d > 2$.

Theorem 2.5 *For any integers $d \geq 3$ and $r \geq 2$,*

$$r^{(1-o(1))(2^d-1)/d} \leq h(d,r) \leq (2r)^{2^{d-1}}$$

where $o(1) \to 0$ as $r \to \infty$.

3 Proof of Theorem 2.3

In the proof of Theorem 2.3, we will use the following lemma without proof.

Lemma 3.1 *If $n \geq 2$ and $b \geq (\sqrt{n}+1)/2$, then $\frac{\binom{b}{2}}{n-1} \geq \frac{b^2}{2(\sqrt{n}+1)^2}$.*

Proof of Theorem 2.3: Fix $d \geq 3$ and $n \geq 2^{\frac{d2^{d-1}}{2^{d-1}-1}-1}$. Let $A \subseteq [1,n]$ satisfy

$$|A| \geq 2^{1-\frac{1}{2^{d-1}}}(\sqrt{n}+1)^{2-\frac{1}{2^{d-2}}}. \tag{2}$$

For any real x, we use the standard notation $A - x = \{a - x : a \in A\}$. For each $i \in [1, n-1]$, define

$$A_i = A \cap (A - i). \tag{3}$$

Since

$$\sum_{i=1}^{n-1} |A_i| = \binom{|A|}{2}, \tag{4}$$

we can find $i_1 \in [1, n-1]$ so that $|A_{i_1}| \geq \frac{\binom{|A|}{2}}{n-1}$. Similarly, there exists i_2 so that for $A_{i_1,i_2} = A_{i_1} \cap (A_{i_1} - i_2)$, we have $|A_{i_1,i_2}| \geq \frac{\binom{|A_{i_1}|}{2}}{n-1}$. Continuing this process, there exist $i_1, i_2, \ldots, i_{d-1}$ so that for each $k = 1, \ldots, d-2$, we recursively define $A_{i_1,i_2,\ldots,i_{k+1}} = A_{i_1,i_2,\ldots,i_k} \cap (A_{i_1,i_2,\ldots,i_k} - i_k)$ satisfying

$$|A_{i_1,i_2,\ldots,i_k}| \geq \frac{\binom{|A_{i_1,i_2,\ldots,i_{k-1}}|}{2}}{n-1}. \tag{5}$$

Observe that $A_{i_1} = \{x : x \in A, x + i_1 \in A\}$ and

$$A_{i_1,i_2} = \{x : x \in A, x + i_1 \in A, x + i_2 \in A, x + i_1 + i_2 \in A\}$$

determines an affine 2-cube. It follows that if $|A_{i_1,i_2,\ldots,i_{d-1}}| \geq 2$, then A contains an affine d-cube; it remains to show this inequality.

For each k, define $a_k = |A_{i_1,i_2,\ldots,i_k}|$. We have already observed that for A to contain a d-cube, it suffices to have $a_{d-1} \geq 2$. Indeed, it follows from (5) that if $a_{d-2} \geq \sqrt{2n}$ then $a_{d-1} \geq 2$, and so then A contains a d-cube. So, assume for the moment that $a_{d-2} \geq \sqrt{2n}$; then for every $i = 1, 2, \ldots, d-2$, $a_i \geq \sqrt{n}$, and so by Lemma 3.1, and (5),

$$\begin{aligned}
a_1 &\geq \frac{\binom{|A|}{2}}{n-1} \geq \frac{|A|^2}{2(\sqrt{n}+1)^2}, \\
a_2 &\geq \frac{\binom{a_1}{2}}{n-1} \geq \frac{a_1^2}{2(\sqrt{n}+1)^2} \geq \frac{(\frac{|A|^2}{2(\sqrt{n}+1)^2})^2}{2(\sqrt{n}+1)^2} = \frac{|A|^4}{2^3(\sqrt{n}+1)^6}, \\
\vdots & \quad \vdots \\
a_{d-2} &\geq \frac{\binom{a_{d-3}}{2}}{n-1} \geq \frac{|A|^{2^{d-2}}}{2^{2^{d-2}-1}(\sqrt{n}+1)^{2^{d-1}-2}}.
\end{aligned}$$

So to guarantee that A contains a d-cube, it suffices to have

$$\frac{|A|^{2^{d-2}}}{2^{2^{d-2}-1}(\sqrt{n}+1)^{2^{d-1}-2}} \geq \sqrt{2n},$$

and hence it suffices to have

$$|A|^{2^{d-2}} \geq 2^{2^{d-2}-1/2}(\sqrt{n}+1)^{2^{d-1}-1},$$

which we have assumed in (2).

We conclude the proof by observing that the condition $n \geq 2^{d+\frac{d}{2^d-1}}$ guarantees that $|A| \geq 2^d$, enough elements for a d-cube. □

4 The number of monochromatic affine cubes

Let $n \geq R(m,m)$, the Ramsey number for m (under 2-colorings of edges). Then under any 2-coloring of the edges of the complete graph K_n, one is guaranteed at least one monochromatic copy of K_m, but can one ascertain how many monochromatic copies exist? For example, Goodman's [14] well known result counts the number of triangles in a graph and in its complement, and shows that the minimum number of monochromatic triangles is $n(n-1)(n-5)/24$, which is around 1/4 of all, that expected from a random coloring. The similar question can be asked for any Ramsey-type partition theorem and is the subject of recent investigation (see [8], [11], [24], or [33] for examples).

Let $f(n,d,r)$ be the minimum number of monochromatic replete affine d-cubes in any r-colored $[n]$.

Theorem 4.1 *For $d \geq 2$, $r \geq 2$, and n sufficiently large (larger than $h(d,r)$),*

$$\frac{n^{d+1}}{(2r)^{2^d-1}}(1-o(1)) \leq f(n,d,r) \leq \frac{n^{d+1}}{(d+1)!r^{2^d-1}},$$

where $o(1) \to 0$ as $n \to \infty$.

Proof: To prove the lower bound, we use the technique given in the proof of Theorem 2.3. Fix a partition $C_1 \cup C_2 \cup \ldots \cup C_r = [1,n]$. For each $\alpha \in [1,r]$, and any choice of $i_1, i_2, \ldots, i_d \in [1, n-1]$, define $C^\alpha_{i_1} = C_\alpha \cap (C_\alpha - i_1)$, and recursively define

$$C^\alpha_{i_1,i_2,\ldots,i_{k+1}} = C^\alpha_{i_1,i_2,\ldots,i_k} \cap (C^\alpha_{i_1,i_2,\ldots,i_k} - i_{k+1}).$$

If $x \in C^\alpha_{i_1,i_2,\ldots,i_d}$ then $\{x + \sum_{j=1}^d \epsilon_j i_j : \epsilon_j = 0, 1\}$ is a monochromatic affine d-cube contained in $[1, n]$, so the number of monochromatic affine d-cubes is

$$\sum_{\alpha=1}^r \sum_{i_1,i_2,\ldots,i_d} |C^\alpha_{i_1,i_2,\ldots,i_d}|,$$

(where $i_1, i_2, \ldots, i_d$ range over $[1, n-1]$). Using the identity (3) and the inequality $\sum_{i=1}^m |a_i|^k \geq \left(\frac{\sum_{i=1}^m a_i}{m}\right)^k \cdot m$ (for positive integers k,m, and non-negative numbers a_i) repeatedly, we calculate

$$\begin{aligned}
&\sum_{\alpha=1}^r \sum_{i_1,i_2,\ldots,i_d} |C^\alpha_{i_1,i_2,\ldots,i_d}| = \sum_{\alpha=1}^r \sum_{i_1,i_2,\ldots,i_{d-1}} \binom{|C^\alpha_{i_1,i_2,\ldots,i_{d-1}}|}{2} \\
\geq\ & \frac{1}{2} \sum_{\alpha=1}^r \sum_{i_1,i_2,\ldots,i_{d-2}} \left(\frac{\sum_{i_{d-1}} |C^\alpha_{i_1,i_2,\ldots,i_{d-1}}|}{n-1}\right)^2 (n-1)(1-o(1)) \\
=\ & \frac{1}{2(n-1)} \sum_{\alpha=1}^r \sum_{i_1,i_2,\ldots,i_{d-2}} \binom{|C^\alpha_{i_1,i_2,\ldots,i_{d-2}}|}{2}^2 (1-o(1)) \\
\geq\ & \frac{1}{2^{1+2}(n-1)} \sum_{\alpha=1}^r \sum_{i_1,i_2,\ldots,i_{d-3}} \left(\frac{\sum_{i_{d-2}} |C^\alpha_{i_1,i_2,\ldots,i_{d-2}}|}{n-1}\right)^4 (n-1)(1-o(1)) \\
=\ & \frac{1}{2^{1+2}(n-1)^{1+3}} \sum_{\alpha=1}^r \sum_{i_1,i_2,\ldots,i_{d-3}} \binom{|C^\alpha_{i_1,i_2,\ldots,i_{d-3}}|}{2}^4 (1-o(1)) \\
\geq\ & \frac{1}{2^{1+2+4}(n-1)^{1+3}} \sum_{\alpha=1}^r \sum_{i_1,i_2,\ldots,i_{d-4}} \left(\frac{\sum_{i_{d-3}} |C^\alpha_{i_1,i_2,\ldots,i_{d-3}}|}{n-1}\right)^8 (n-1)(1-o(1)) \\
=\ & \frac{1}{2^{1+2+4}(n-1)^{1+3+7}} \sum_{\alpha=1}^r \sum_{i_1,i_2,\ldots,i_{d-4}} \binom{|C^\alpha_{i_1,i_2,\ldots,i_{d-4}}|}{2}^8 (1-o(1)) \\
&\vdots \\
\geq\ & \frac{1}{2^{2^d-1}(n-1)^{2^d-1-d}} \sum_{\alpha=1}^r |C_\alpha|^{2^d} (1-o(1))
\end{aligned}$$

$$\geq \frac{1}{2^{2^d-1}n^{2^d-1-d}} \left(\frac{\sum_{\alpha=1}^r |C_\alpha|}{r}\right)^{2^d} r(1-o(1))$$
$$= \frac{n^{d+1}}{(2r)^{2^d-1}}(1-o(1))$$

where the last line holds since $\sum_{\alpha=1}^r |C_\alpha| = n$. So we have shown that the minimum number of affine d-cubes under any r-coloring of $[n]$ is at least $\frac{n^{d+1}}{(2r)^{2^d-1}}(1-o(1))$, and since almost all affine d-cubes are replete, this gives the lower bound.

If $H(x_0, x_1, \ldots, x_d)$ is a replete affine cube, then the generators $x_1, \ldots, x_d$ are distinct (though this is not a sufficient condition for being replete). For any given x_0, the other x_i's can be chosen only from $[1, n-x_0]$ and hence there are less than

$$\binom{n}{d} + \binom{n-1}{d} + \binom{n-2}{d} + \ldots + \binom{d+1}{d} + 1 = \binom{n+1}{d+1}$$

replete affine d-cubes in $[1,n]$. A random r-coloring of $[1,n]$ yields each cube monochromatic with probability $\frac{r}{r^{2^d}}$, and so we conclude that there is a coloring with no more than $\binom{n+1}{d+1}\frac{1}{r^{2^d-1}}$ monochromatic replete affine d-cubes. Hence, $f(n,d,r) \leq \frac{n^{d+1}}{(d+1)!r^{2^d-1}}$. $\square$

5 Hypergraphs and Theorem 2.1

A *d-uniform hypergraph* is a pair $G = (V, \mathcal{E}) = (V(G), \mathcal{E}(G))$, with vertex set V and hyperedge set $\mathcal{E} \subset [V]^d$. Note that by this definition, each d-set from V may occur only once as a hyperedge. For pairwise disjoint sets $X_1, X_2, \ldots, X_d$, let

$$G = (X_1, X_2, \ldots, X_d, \mathcal{E}(G))$$

denote a d-partite d-uniform hypergraph on vertex set $V(G) = \cup_{i=1}^d X_i$ and edge set $\mathcal{E}(G) \subseteq [V(G)]^d$, where for each $E \in \mathcal{E}(G)$ and each $i = 1, \ldots, d$, $|E \cap X_i| = 1$ holds; the sets $X_1, \ldots, X_d$ will be called *partite* sets.

Let $K^{(d)}(n_1, n_2, \ldots, n_d)$ denote the complete d-partite d-uniform hypergraph on $\sum_{i=1}^d n_i$ vertices, partitioned into sets of sizes $n_1, n_2 \ldots, n_d$, and having $\prod_{i=1}^d n_i$ edges, each edge containing exactly one vertex from each partite set. The complete d-partite d-uniform hypergraph with two vertices in each partite set will be denoted by $K^{(d)}(2,2,\ldots,2)$. For any d-uniform hypergraph H, the maximum number of d-hyperedges in any H-free hypergraph on n vertices is denoted by $\mathrm{ex}(n,H)$. In 1964, Erdős [5] (cf. equation (4.2) in [12]) showed that for each d and $m \geq 2d$,

$$\mathrm{ex}(k, K^{(d)}(2,2,\ldots,2)) \leq k^{d-\frac{1}{2^{d-1}}}. \tag{6}$$

For $d > 2$, there is still an order of magnitude gap between the lower and upper bounds for $\mathrm{ex}(n, K^{(d)}(2,2,\ldots,2))$ (see [21] for discussion). Using these partite hypergraphs, we now give a novel proof of Theorem 2.1 by showing that if $A \subseteq [1,n]$ and $|A| \geq (3d)^d n^{1-\frac{1}{2^d}}$, then A contains an affine d-cube; we make no attempt to optimize constants (since this result is implied by Theorem 2.3).

A hypergraph proof of Theorem 2.1: Fix $d \geq 2$, $m \geq 2d$, and let $n = m^2$. Let $A \subset [1,n]$ with $|A| \geq cn^{1-\frac{1}{2^d}}$, where $c = (3d)^d$. For any $q \in [1,n]$, let $(s(q), t(q)) \in ([0, m-1])^2$ be the unique pair of integers satisfying $q - 1 = s(q)\sqrt{n} + t(q)$. For each $q \in A$, define

$$\mathcal{E}_q = \{(z_1, \ldots, z_d) \in ([-m,m])^d : \sum_{i=1}^{d-1} z_i = s(q), z_d = t(q)\}.$$

If $s \in [0, m-1]$, then are at least m^{d-2} ways of writing $s = \sum_{i=1}^{d-1} z_i$, where each $z_i \in [-m,m]$ and so for each $q \in [1,n]$, $|\mathcal{E}_q| \geq m^{d-2}$.

Now examine the d-partite d-uniform graph $G = (X_1, X_2, \ldots, X_d, \mathcal{E}(G))$ defined by letting each X_i be a distinct copy of the integers $[-m,m]$ and setting $\mathcal{E}(G) = \cup_{q\in A}\mathcal{E}_q$. Then $|\mathcal{E}(G)| \geq |A|m^{d-2} \geq cn^{1-\frac{1}{2^d}}m^{d-2} \geq cm^{d-\frac{1}{2^{d-1}}} \geq (d(2m+1))^{d-\frac{1}{2^{d-1}}}$.

By the result of Erdős given in equation (6) with $|V(G)| = k = d(2m+1)$, there exists a copy of $K^{(d)}(2,\ldots,2)$ in G on, say, vertices y_1, z_1, y_2, z_2, $\ldots$, y_d, z_d, where for each i, $\{y_i, z_i\} \subset X_i$ and $y_i < z_i$. Such a copy corresponds to the affine d-cube $H(x_0, x_1, \ldots, x_d)$, using $x_0 = y_d + m\sum_{i=1}^{d-1} y_i$, for each $i = 1, \ldots, d-1$, $x_i = (z_i - y_i)m$ and $x_d = z_d - y_d$. □

6 Bounds on $h(d,r)$; proof of Theorem 2.5

Proof of upper bound in Theorem 2.5: Fix $d \geq 3$ and $r \geq 2$ and let $n = (2r)^{2^{d-1}}$. Then $n \geq 2^{2^d} \geq (\frac{2^d-2}{\ln 2})^2$, large enough to satisfy the condition in Corollary 2.4. If $[1,n]$ is partitioned into r parts, then by the pigeonhole principle, one part is larger than $n/r \geq 2n^{1-\frac{1}{2^{d-1}}}$, and so Corollary 2.4 guarantees an affine d-cube in that part. Hence $h(d,r) \leq (2r)^{2^{d-1}}$. $\Box$

On the other hand, the lower bound for $h(d,r)$ (in Theorem 2.5) requires more development. We devote the remainder of this section to it. By definition, to prove that $n \leq h(d,r)$, we need to show that there exists an r-coloring of the set $[1,n]$ which prevents monochromatic affine d-cubes. Indeed, our proof relies on finding, for a given d and n, as small as possible value for such an r, a "lower coloring bound". The following simple idea will be useful.

Lemma 6.1 *If a finite collection X of distinct positive integers does not contain any replete affine d-cubes and does not contain any arithmetic progressions of length three, then X does not contain any affine d-cubes.*

Proof: Let X satisfy the assumptions. Suppose $H = H(x_0, x_1, \ldots, x_d) \subset X$ is an affine d-cube, but is not replete. Then there exist $I \subset [1,d]$ and $J \subset [1,d]$, $I \neq J$, so that

$$x_0 + \sum_{i \in I} x_i = x_0 + \sum_{j \in J} x_j,$$

giving

$$\sum_{i \in I \setminus J} x_i = \sum_{j \in J \setminus I} x_j \neq 0.$$

In this case,

$$x_0 + \sum_{i \in I \cap J} x_i, \quad x_0 + \sum_{i \in I} x_i, \quad x_0 + \sum_{i \in I \cup J} x_j,$$

is an arithmetic progression, contrary to our assumption. $\Box$

We will partition $[1,n]$ into cube-free sets in two stages. First, we partition $[1,n]$ into sets none of which contain any three term arithmetic progressions. We then refine this partition into sets not containing any replete affine d-cubes. In this case, each resulting set will not contain any AP_3, nor any replete affine d-cubes, and then we apply Lemma 6.1.

6.1 AP_3-free partitioning

Adapting Behrend's [1] proof (cf. [16], Theorem 6.6, lower bound) gives a partition result.

Theorem 6.2 *For sufficiently large n, there exists a partition $[1, n] = X_1 \cup \ldots \cup X_q$, $q < e^{3\sqrt{\ln n}}$ so that each X_i is AP_3-free and $|X_i| \leq n/e^{\ln 2\sqrt{\ln n}}$.*

Proof: Set $d = \lfloor e^{\sqrt{\ln n}} \rfloor$ (not to be confused with the dimension d of the affine cube as in the rest of this paper), $k = \lceil \frac{\ln(n+1)}{\ln(2d)} \rceil - 1$ and for each $x \in [1, n]$, write

$$x = \sum_{i=0}^{k} x_i (2d)^i,$$

where for each i, $0 \leq x_i \leq 2d - 1$. Note that due to our choice of k, $(2d)^k < n + 1 \leq (2d)^{k+1}$ holds. Put

$$X_{n,d} = \{x \in [n] : 0 \leq x_i \leq d - 1 \text{ for all } i\}.$$

For $I \subset [0, k]$, set

$$y_I = \sum_{i \in I} d(2d)^i$$

and

$$Y_I = y_I + X_{n,d} = \{y_I + x : x \in X_{n,d}\}.$$

This gives $Y_\emptyset = X_{n,d}$ and $|Y_I| = |X_{n,d}|$ for every $I \subseteq [0, k]$. For $x = \sum_{i=0}^{k} x_i (2d)^i$, let

$$I = I(x) = \{i \in [0, k] : x_i \geq d\};$$

then

$$x - y_{I(x)} = \sum_{i \notin I(x)} x_i (2d)^i + \sum_{i \in I(x)} (x_i - d)(2d)^i$$

is an element of $X_{n,d}$, and so $[n] = \bigcup_{I \subseteq [0,k]} Y_I$ is a partition. For an integer s set

$$X_{n,d,s} = \left\{ x \in X_{n,d} : \sum_{i=0}^{k} x_i^2 = s \right\}$$

and for each $I \subseteq [0, k]$, put

$$Y_{I,s} = y_I + X_{n,d,s} = \{y_I + x : x \in X_{n,d,s}\}.$$

Figure 1: Partitioning integers written in base 2d

By definition, $X_{n,d} = \cup_s X_{n,d,s}$ and so also $Y_I = \cup_s Y_{I,s}$ are partitions, where there are $(d-1)^2(k+1)+1$ choices for s (including zero) and 2^{k+1} choices for I. Thus,

$$[1, n] = \bigcup_{I \subseteq [0,k]} \bigcup_{s=0}^{(d-1)^2(k+1)} Y_{I,s}$$

is a partition of $[n]$ into at most $q = 2^{k+1}((d-1)^2(k+1)+1)$ parts (each corresponding to a sphere in $(k+1)$-dimensional Euclidean space).

For the case where $k = 1$ (two coordinates), the different Y_I's can be thought of as translations of a square in a quadrant of the cartesian plane (see Figure 1); the similar notion holds in $k+1$ dimensional Euclidean space. We claim that each part $Y_{I,s}$ is a collection of integers which does not contain an arithmetic progression of length three. Since each $Y_{I,s} = y_I + X_{n,d,s}$ is a translate of $X_{n,d,s}$, it suffices to show that $X_{n,d,s}$ does not contain an arithmetic progression. This has been, however, shown by Behrend [1]. Here, we give the proof for completeness:

Suppose $x = \sum x_i(2d)^i$, $z = \frac{x+y}{2} = \sum z_i(2d)^i$, and $y = \sum y_i(2d)^i$ are distinct elements of $X_{n,d,s}$. For each i, since x_i, y_i, and z_i are all less than d,

$x_i + y_i < 2d$, so no carrying occurs, that is, $\frac{x_i+y_i}{2} = z_i$. However,

$$s = \sum x_i^2 = \sum y_i^2 = \sum \left(\frac{x_i + y_i}{2}\right)^2$$

implies $\sum x_i^2 + \sum y_i^2 = \sum (x_i + y_i)^2/2$, which implies $\sum (x_i - y_i)^2 = 0$ and hence $x = y$, a contradiction.

With the above choice of d (and hence also k) we minimize the order of magnitude of $q = 2^{k+1}((d-1)^2(k+1)+1)$, the number of partition classes. Note that for sufficiently large n, $k < \sqrt{\ln(n+1)}$ and therefore

$$\begin{aligned} q &< 2^{k+1} d^2 (k+1) \\ &\leq e^{\ln 2\sqrt{\ln(n+1)}+1+2\sqrt{\ln n}+\ln\sqrt{\ln(n+1)}} \\ &< e^{3\sqrt{\ln n}} \end{aligned}$$

for sufficiently large n. The last inequality in the statement of the theorem follows from the crude upper bound $|Y_{I,s}| \leq |Y_I| \sim n/2^{k+1}$. $\square$

6.2 Replete-cube-free, density upper bound

The proof of the following result employs the standard deletion technique (cp. [4]).

Lemma 6.3 *For each $d \geq 2$ and every set X of positive integers there exists an $A \subset X$ with*

$$|A| \geq \frac{1}{8} |X|^{1-\frac{d}{2^d-1}}.$$

which does not contain any replete affine d-cubes.

Proof of Lemma 6.3: Fix $d \geq 2$, X and let

$$p = |X|^{-\frac{d}{2^d-1}}.$$

Without loss of generality, we can assume that X is large enough so that $\frac{1}{8}|X|^{1-\frac{d}{2^d-1}} > 2^d - 1$, because if not, then any set A of at most $2^d - 1$ elements would satisfy the lemma.

Let Y be a random subset of X whose elements are chosen independently with probability p.

Since any affine d-cube is determined uniquely by $d+1$ distinct integers in X (the smallest of which plays the role of x_0, the rest the roles of $x_0 + x_1, \ldots, x_0 + x_d$ in the definition of affine d-cube), the expected number of replete affine d-cube's in Y is easily bounded above by

$$\binom{|X|}{d+1} p^{2^d}.$$

Therefore (by Markov's inequality), with probability at least $1/2$, the number of replete affine d-cubes in Y does not exceed

$$2\binom{|X|}{d+1} p^{2^d} < \frac{1}{3}|X|p. \tag{7}$$

On the other hand, the number of elements in any random subset Y is a binomially distributed random variable with expectation $|X|p$. Given the size of $|X|p$, we have $\left\lfloor \frac{1}{2}|X|p \right\rfloor < |X|p$, and a simple argument (see [7] p. 151, for example) using the fact that the sequence $\{\binom{|X|}{j} p^j (1-p)^{|X|-j}\}_{j=0}^{|X|}$ is increasing from $j=0$ to $(|X|+1)p$ and decreasing afterwards, we conclude that

$$\mathrm{Prob}(|Y| \geq \left\lfloor \frac{1}{2}|X|p \right\rfloor) > \frac{1}{2}. \tag{8}$$

Hence there exists an instance of $A^* \subset X$ satisfying both of the above events; fix such an A^*. Due to equations (7) and (8), deleting an element from each replete affine d-cube in A^*, we get a set A with no replete affine d-cube such that

$$|A| \geq |A^*| - 2\binom{|X|}{d+1} > \left\lfloor \frac{1}{2}|X|p \right\rfloor - \frac{1}{3}|X|p > \frac{1}{8}|X|p,$$

where the last inequality follows from the restriction on $|X|p$. □

6.3 Partitioning into replete-cube-free sets

To prove a lower bound for $h(d,r)$, we show a way to partition any set into subsets, each of which does not contain a replete affine d-cube; a greedy coloring algorithm is used.

Lemma 6.4 *For each $d \geq 2$, there exists a constant $c = c(d)$ so that for any set of positive integers X, there exists a partition $X = A_0 \cup A_1 \cup \ldots \cup A_r$ into $r + 1 \leq c|X|^{\frac{d}{2^d-1}}$ colors so that no color class contains a replete affine d-cube.*

Proof of Lemma 6.4: Set $c = \frac{12(2^d-1)}{d \ln 2}$; we will give an $r + 1$-coloring $X = A_0 \cup A_1 \cup \ldots \cup A_r$ with no replete affine d-cube in any one color class A_i. By Lemma 6.3, there is a set $A_1 \subset X$ containing no replete affine d-cube and

$$\frac{1}{8}\left(\frac{|X|}{2}\right)^{1-\frac{d}{2^d-1}} \leq \frac{1}{8}|X|^{1-\frac{d}{2^d-1}} \leq |A_1| \leq |X|/2.$$

Since $|X \backslash A_1| \geq |X|/2$, by Lemma 6.3 (applied to sets of size $|X|/2$), we can choose $A_2 \subseteq X \backslash A_1$ containing no replete affine d-cube and

$$\frac{1}{8}\left(\frac{|X|}{2}\right)^{1-\frac{d}{2^d-1}} \leq |A_2| \leq \frac{|X|}{2}.$$

Continue this process, by Lemma 6.3 recursively choose (pairwise disjoint) sets

$$A_1, A_2, \ldots, A_{r_1},$$

where for each $j = 1, \ldots, r_1$, $A_j \subset X$ is free of replete affine d-cubes,

$$A_j \subset X \backslash \bigcup_{i=1}^{j-1} A_i, \quad \text{and} \quad |A_j| > \frac{1}{8}\left(\frac{|X|}{2}\right)^{1-\frac{d}{2^d-1}} \tag{9}$$

with r_1 the least so that

$$|X \backslash \bigcup_{i=1}^{r_1} A_i| < \frac{|X|}{2}.$$

Without loss of generality, we can also assume that

$$|X \backslash \bigcup_{i=1}^{r_1} A_i| \geq \frac{|X|}{4} \tag{10}$$

since we can limit our greed by possibly taking A_{r_1} smaller than maximal size.

Combining (9) and (10), with Lemma 6.3, we infer that

$$r_1 \le \frac{\frac{3}{4}|X|}{\frac{1}{8}\left(\frac{|X|}{2}\right)^{1-\frac{d}{2^d-1}}} = 12\left(\frac{|X|}{2}\right)^{\frac{d}{2^d-1}}.$$

Again, by Lemma 6.3 (with $|X|/4$ as the set size), choose successively $A_{r_1+1}, A_{r_1+2}, \ldots, A_{r_1+r_2}$, where for each $j = r_1+1, \ldots, r_1+r_2$, the relation (9) holds and

$$\frac{1}{8}\left(\frac{|X|}{4}\right)^{1-\frac{d}{2^d-1}} \le |A_j| \le \frac{|X|}{4},$$

with r_2 the smallest so that

$$\left|X \backslash \bigcup_{i=1}^{r_1+r_2} A_i\right| < \frac{|X|}{4}$$

while

$$\frac{|X|}{8} < \left|X \backslash \bigcup_{i=1}^{r_1+r_2} A_i\right|.$$

In this case,

$$r_2 \le \frac{\frac{3}{8}|X|}{\frac{1}{8}\left(\frac{|X|}{4}\right)^{1-\frac{d}{2^d-1}}} = 12\left(\frac{|X|}{4}\right)^{\frac{d}{2^d-1}}.$$

Continuing this greedy process using $1 + r_1 + r_2 + \cdots r_k$ colors, where $k = \lfloor \log_2 |X| \rfloor - d + 1$, we color all but at most $2^d - 1$ elements of X; these remaining we call the color class A_0. Thus using $1 + r_1 + r_2 + \cdots r_k$ colors one can color X with no color class containing a replete affine d-cube. Since

$$\begin{aligned}
\sum_{i=1}^{k} r_i &\le 12\sum_{i=1}^{\infty}\left(\frac{|X|}{2^i}\right)^{\frac{d}{2^d-1}} \\
&= \frac{12}{2^{\frac{d}{2^d-1}} - 1}|X|^{\frac{d}{2^d-1}} \\
&< 12\frac{2^d-1}{d\ln 2}|X|^{\frac{d}{2^d-1}}
\end{aligned}$$

we are done. □

6.4 Final step in proof of lower bound for $h(d,r)$

We now prove for $d \geq 2$, and $r \geq 2$ that $r^{\frac{2^d-1}{d}(1-o(1))} \leq h(d,r)$, where $o(1)$ tends to zero as r increases.

Proof of lower bound in Theorem 2.5: Fix n large (large enough to apply Theorem 6.2). It suffices to provide a partition of $[1,n]$ into at most $n^{\frac{d+o(1)}{2^d-1}}$ parts, each part affine d-cube-free. By Theorem 6.2, let $[1,n] = X_1 \cup \ldots \cup X_q$ be a partition into at most $q = e^{3\sqrt{\ln n}}$ parts, each X_i not containing any three term arithmetic progression, and each $|X_i| \leq n/2^{\sqrt{\ln n}}$. For each $i = 1, \ldots q$, by Lemma 6.4, partition $X_i = A_{i,1} \cup \ldots \cup A_{i,k(i)}$,

$$k(i) \leq \frac{12}{2^{\frac{d}{2^d-1}}} |X_i|^{\frac{d}{2^d-1}} < \frac{12}{2^{\frac{d}{2^d-1}}} \left(n/2^{\sqrt{\ln n}}\right)^{\frac{d}{2^d-1}}$$

into parts, none of which contains a replete affine d-cube. For each $i = 1, \ldots, q$ and for each $j = 1, \ldots, k(i)$, $A_{i,j}$ contains neither a three term arithmetic progression nor a replete affine d-cube, so by Lemma 6.1, each $A_{i,j}$ is affine d-cube-free. The total number of partitions is at most

$$\begin{aligned} q \cdot \max k(i) &\leq e^{3\sqrt{\ln n}} \cdot \frac{12}{2^{\frac{d}{2^d-1}} - 1} (n/2^{\sqrt{\ln n}})^{\frac{d}{2^d-1}} \\ &= n^{\frac{d+o(1)}{2^d-1}}, \end{aligned}$$

For large n. □

We conclude by noting that we have proved Theorem 2.5 for $d \geq 2$, however this is only an improvement over known results for $d \geq 3$ (cf [2]).

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