Extremal problems for sets forming Boolean algebras and complete partite hypergraphs

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Abstract

Three classes of finite structures are related by extremal properties: complete d-partite d-uniform hypergraphs, d-dimensional affine cubes of integers, and families of 2^d sets forming a d-dimensional Boolean algebra. We review extremal results for each of these classes and derive new ones for Boolean algebras and hypergraphs, several obtained by employing relationships between the three classes. Related partition or coloring problems are also studied for Boolean algebras. Density results are given for Boolean algebras of sets all of whose atoms are the same size.

1 Introduction

In this section we state definitions and results for Boolean algebras of sets. We defer the proofs of these main results until Sections 4 and 5. In Section 2 we present needed density theorems for certain partite hypergraphs, while improving slightly some known bounds. Section 3 contains facts about certain families of integers, also required for proofs regarding Boolean algebras.

For a set X, $\mathcal{P}(X) = \{Y : Y \subseteq X\}$ denotes the power set of X.

Definition 1.1 *A collection $\mathcal{B} \subseteq \mathcal{P}(X)$ forms a* d-dimensional Boolean algebra *if and only if there exist pairwise disjoint sets $X_0, X_1, \ldots, X_d \in \mathcal{P}(X)$, all non-empty with perhaps the exception of X_0, so that*

$$\mathcal{B} = \left\{ X_0 \cup \bigcup_{i \in I} X_i : I \subseteq [1, d] \right\}.$$

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In general, we shall restrict ourselves to the case where X is finite. It will often be convenient to use the notation $[n] = [1,n] = \{1,2,\ldots,n\}$ and use $X = [n]$.

Definition 1.2 *Given an n-element set X and a positive integer d, define $b(n,d)$ to be the maximum size of a family $\mathcal{F} \subset \mathcal{P}(X)$ which does not contain a d-dimensional Boolean algebra.*

Note that a 1-dimensional Boolean algebra is simply a pair of sets, one contained in the other and so, by Sperner's theorem (see [40] for one of many proofs),

$$b(n,1) = \binom{n}{\lfloor n/2 \rfloor} \sim (\sqrt{2/\pi}) n^{-1/2} \cdot 2^n.$$

Erdős and Kleitman [16] found that there exist constants c_1 and c_2 so that for n sufficiently large,

$$c_1 n^{-1/4} \cdot 2^n \leq b(n,2) \leq c_2 n^{-1/4} \cdot 2^n.$$

Voigt [46] asked about a general bound for $b(n,d)$. Such a question turns out to be quite difficult, and bounds for general d are far apart. In Theorems 4.2 and 4.3 we show that for each $d \geq 1$ there exists a positive constant c so that for n sufficiently large,

$$n^{-\frac{d}{2^{d+1}-2}(1-o(1))} \cdot 2^n \leq b(n,d) \leq c n^{-1/2^d} \cdot 2^n.$$

Definition 1.3 *Given an n-element set X and positive integer d, define $r(d,n)$ to be the largest integer so that for every partition $\mathcal{P}(X) = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \cdots \cup \mathcal{F}_{r(d,n)}$ into $r(d,n)$ color classes, one class contains a d-dimensional Boolean algebra.*

We observe that $r(1,n) = n$ and in Theorem 4.6, we show that

$$\frac{3}{4}(1-o(1)) n^{1/2} \leq r(2,n) \leq (1+o(1)) n^{1/2}.$$

For $d > 2$, Theorem 4.7 gives bounds of the form

$$c n^{1/2^d} \leq r(d,n) \leq n^{\frac{d}{2^d-1}(1+o(1))}.$$

Definition 1.4 *A d-dimensional Boolean algebra $\mathcal{B} = \{B_0 \cup \bigcup_{i \in I} B_i : I \subseteq [1,d]\}$ is said to be* uniform *if $|B_1| = |B_2| = \ldots = |B_d|$. Define $b_u(n,d)$ to be the maximum size of a family in $\mathcal{P}([n])$ which does not contain a uniform d-dimensional Boolean algebra.*

In Theorem 5.1, we show that for any $\epsilon > 0$ and n large enough,

$$\frac{1}{n^{o(1)}} 2^n \leq b_u(n,d) \leq \epsilon \cdot 2^n.$$

2 Density results for complete d-partite hypergraphs

2.1 Notation

Here, and throughout this paper, we use the standard notation $[X]^s = \{S \subset X : |S| = s\}$. A *simple d-uniform hypergraph* is a pair $G = (V, \mathcal{E}) = (V(G), \mathcal{E}(G))$, with vertex set V and set $\mathcal{E} \subseteq [V]^d$ of hyperedges, (also called d-hyperedges or d-edges). An ordinary graph is a 2-uniform hypergraph, with hyperedges called, simply, edges. A d-uniform hypergraph $G = (V, \mathcal{E})$ is called *k-partite* if there exists a partition $V = X_1 \cup \cdots \cup X_k$ of the vertex set into *partite sets* so that for each $E \in \mathcal{E}$ and each X_i, $|E \cap X_i| \leq 1$ (and so each hyperedge in d-partite d-uniform hypergraph has precisely one vertex from each partite set). A d-partite hypergraph $(V, \mathcal{E})$ with partite sets $X_1, X_2, \ldots, X_d$, is denoted $(X_1, X_2, \ldots, X_d, \mathcal{E})$. The complete d-partite d-uniform hypergraph with two vertices in each partite set and having 2^d hyperedges is denoted by $K^{(d)}(2, 2, \ldots, 2)$. The graph C_4, also denoted $K^{(2)}(2, 2)$ or $K_{2,2}$, is such an example.

For any d-uniform hypergraph H, we let $\mathrm{ex}(n, H)$ denote the maximum number of d-hyperedges in a hypergraph on n vertices which does not contain a copy of H. See, for example, [6], [19], [20], [24], or [43] for further references on extremal numbers.

2.2 Known bounds for ex$(n, K^{(d)}(2, 2, \ldots, 2))$

2.2.1 Upper bounds

Upper bounds for $\mathrm{ex}(n, K_{2,2})$ have been well studied (see for example, [8], [13], [15], [17], [38]). As in, for example, [38], counting pairs of vertices in neighborhoods yields

$$\mathrm{ex}(n, K_{2,2}) \leq (1 + \sqrt{4n-3})n/4 = (1 + (o))\frac{1}{2}n^{3/2}. \tag{1}$$

The same technique gives a similar bound for bipartite graphs.

Lemma 2.1 *If G is an equibipartite graph on n vertices ($n/2$ in each partite set), and $|\mathcal{E}(G)| > \frac{n}{4}(1 + \sqrt{2n-3})$, then G contains a $K_{2,2}$.*

To give the lower bound on $r(2, n)$ in Theorem 4.6, we employ a result closely related to Lemma 2.1, one for ordered graphs originally found by Koubek and Rödl [37]. For completeness, we include a simplified version of their proof. We consider graphs on the vertex set $\{0, 1, \ldots, n-1\}$ with directed edges (i, j) where $i < j$. Let $\vec{C_4}$ denote the particular ordered 4-cycle which has each of the first two vertices connected to each of the last two. Let $p(n)$ denote the maximum number of edges in an ordered graph on n vertices which does not contain a copy of $\vec{C_4}$.

Theorem 2.2 $p(n) \le (1+o(1))\frac{2}{3}n^{3/2}$.

Proof: Fix a graph G on vertex set $\{0, 1, \ldots n-1\}$ with edge set $\mathcal{E}$ and suppose that G contains no $\vec{C}_4$. Let $1 < m < n$ and put $m = \alpha n$. For each $j = 1, \ldots, n-1$, put $y_j = |\{i < m : (i,j) \in \mathcal{E}\}|$. If $\sum_{j=1}^{n-1} \binom{y_j}{2} > \binom{m}{2}$ were to hold, then for some $a < b < m$, there exist c, d with $a < b < c < d$ inducing a $\vec{C}_4$, and so

$$\sum_{j=1}^{n-1} \binom{y_j}{2} \le \binom{m}{2},$$

from which we infer

$$\sum_{j=1}^{n-1} (y_j - 1)^2 < m^2. \tag{2}$$

Thus the total number of edges is

$$\begin{aligned}
\sum_{j=1}^{n-1} y_j &< n + \sum (y_j - 1) \\
&< n + (n \sum (y_j - 1)^2)^{1/2} \\
&< n + (nm^2)^{1/2} \qquad \text{(by (2))} \\
&= n + \alpha n^{3/2}.
\end{aligned}$$

Since $p((1-\alpha)n)$ is an upper bound on the number of edges (i,j) with $i \ge m$,

$$p(n) < n + \alpha n^{3/2} + p((1-\alpha)n). \tag{3}$$

Iterating (3) $t-1$ more times,

$$\begin{aligned}
p(n) &< n \sum_{i=0}^{t-1} (1-\alpha)^i + \alpha n^{3/2} \sum_{i=0}^{t-1} ((1-\alpha)^{3/2}) + p((1-\alpha)^t n) \\
&= n \left(\frac{1 - (1-\alpha)^t}{\alpha} \right) + \alpha n^{3/2} \left(\frac{1 - (1-\alpha)^{3t/2}}{1 - (1-\alpha)^{3/2}} \right) + p((1-\alpha)^t n).
\end{aligned}$$

Letting t tend to infinity, $(1-\alpha)^t n$ is eventually less than 2, and $p(0) = p(1) = 0$; that is, $p((1-\alpha)^t n)$ eventually vanishes and so

$$\begin{aligned}
p(n) &< \frac{n}{\alpha} + \frac{\alpha n^{3/2}}{1 - (1-\alpha)^{3/2}} \\
&\le \frac{n}{\alpha} + \frac{\alpha n^{3/2}}{1 - (1 - \frac{3}{2}\alpha + \frac{3}{4}\alpha^2)} \qquad \text{(by series expansion)} \\
&= \frac{n}{\alpha} + \frac{2}{3} \left(\frac{n^{3/2}}{1 - \alpha/2} \right) \\
&< \frac{n}{\alpha} + \frac{2}{3}(1+\alpha) n^{3/2}.
\end{aligned}$$

Since α was arbitrary, $p(n) \le (1+o(1))\frac{2}{3}n^{3/2}$. $\square$

In [15], Erdős gave an upper bound on $K^{(d)}(l, l, \ldots, l))$ for general d and l (see also [24], equation (4.2)). We require only the case $l = 2$; in the original, the proof yields a constant $c < 1$, and for simplicity, we omit it.

Theorem 2.3 (Erdős [15]) *For each d and n sufficiently large,*

$$ex(n, K^{(d)}(2, 2, \ldots, 2)) \le n^{d - \frac{1}{2^{d-1}}}.$$

This result is central in our finding an upper bound for $b(n, d)$.

2.2.2 Lower bounds

In the special case where q is a prime power and $n = q^2 + q + 1$, Reiman [41] showed

$$\mathrm{ex}(n, K_{2,2}) > (\sqrt{4n-3} - 1)(n-1)/4 = (1 - o(1))\frac{1}{2}n^{3/2},$$

verifying that the upper bound (1) is asymptotically correct. Füredi [25] has since shown that Reiman's construction (see Problem 10.36 in [39]) is optimal, giving $\mathrm{ex}(n, K_{2,2})$ precisely for those certain n. Unfortunately, for $d \ge 3$, known upper and lower bounds for $\mathrm{ex}(n, K^{(d)}(2, \ldots, 2))$ are still very far apart.

In [15], it was stated that there is a universal constant C so that for any integers $l > 1$ and $d > 1$ and n sufficiently large, $\mathrm{ex}(n; K^{(d)}(l, \ldots, l)) \ge n^{d - C/l^{d-1}}$. Unfortunately, the proof for this claim has not been found (see also [24], p. 259). By the *probabilistic deletion method* (see [14], [20]), for every $d \ge 2$ there is a constant $c = c(d)$ so that for n sufficiently large,

$$\mathrm{ex}(n, K^{(d)}(2, 2, \ldots, 2)) > cn^{d - \frac{d}{2^d - 1}}. \tag{4}$$

For $d = 2$, (4) yields only $\mathrm{ex}(n, K_{2,2}) > cn^{4/3}$, still far from known constructive lower bounds.

2.3 New lower bounds on ex$(n, K^{(d)}(2, 2, \ldots, 2))$

Our aim in this section is to lower the exponent in (4); this can be achieved for many values of $d \ge 3$ by employing a modification of the probabilistic deletion method which uses affine subspaces.

We begin by examining the case $d = 3$. From (4) and Theorem 2.3, one has

$$cn^{18/7} < \mathrm{ex}(n, K^{(3)}(2, 2, 2)) \le n^{11/4}$$

(for n large enough).

Theorem 2.4 *For n sufficiently large,*

$$ex(n, K^{(3)}(2,2,2)) > \frac{n^{13/5}}{4 \cdot 3^{8/5}}(1 - o(1)).$$

In the proof of Theorem 2.4 (and subsequently, Theorem 2.5) we will freely use the following well known fact (for example, see [10], p. 137). The number of r-dimensional affine subspaces contained in an s-dimensional vector space on ℓ^s points is

$$\frac{\ell^s}{\ell^r}\begin{bmatrix} s \\ r \end{bmatrix}_\ell = \frac{\ell^s(\ell^s-1)(\ell^s-\ell)\cdots(\ell^s-\ell^{r-1})}{\ell^r(\ell^r-1)(l^r-\ell)\cdots(\ell^r-\ell^{r-1})} = (1+o(1))\ell^{(s-r)(r+1)}.$$

where $o(1) \to 0$ as $\ell \to \infty$. For the sake of clarity, we do not write $\lceil\cdot\rceil$ or $\lfloor\cdot\rfloor$ and we make no effort to optimize constants in the following proof.

Proof of Theorem 2.4: Let $\ell \le (n/3)^{1/5}$ be a prime satisfying $l^5 = (n/3)(1 - o(1))$. The existence of such an ℓ is guaranteed by the prime number theorem (for example, see [4], Exercise 13, p. 102) for any n large enough (where $o(1) \to 0$ as $n \to \infty$). Set $m = \ell^5$ and let V be a 5-dimensional vector space over $\mathrm{GF}(\ell)$. Let X, Y be pairwise disjoint vertex sets each of cardinality m and disjoint from V. We will construct a 3-uniform 3-partite hypergraph $G' = (X, Y, V, \mathcal{E}')$ on $3m \sim n$ vertices with no copies of $K^{(3)}(2,2,2)$ by first, naming a collection of triples $\hat{\mathcal{E}} \subset X \times Y \times V$, then deleting some triples to form $\mathcal{E}'$.

Let $\mathcal{L}$ denote the set of lines (1-dimensional affine subspaces) and $\mathcal{R}$ be the collection of 3-dimensional affine subspaces in V. Then $|\mathcal{L}| = (1+o(1))\ell^8$, each $R \in \mathcal{R}$ contains ℓ^3 vertices and $(1+o(1))\ell^4$ lines, and $|\mathcal{R}| = (1+o(1))\ell^8$.

For each $x \in X$, $y \in Y$ independently select at random $R_{xy} \in \mathcal{R}$ and put

$$\mathcal{E} = \{(x,y,z) : x \in X, y \in Y, z \in R_{xy}\}.$$

Note that $|\mathcal{E}| = m^2\ell^3 = \ell^{13}$ for any member of our random space.

Let $Q = Q(\mathcal{E})$ be a random variable counting the number of quintuples (x_1, x_2, y_1, y_2, L) such that $x_1, x_2 \in X$, $y_1, y_2 \in Y$, $L \in \mathcal{L}$ and

$$L \subseteq R_{x_1y_1} \cap R_{x_1y_2} \cap R_{x_2y_1} \cap R_{x_2y_2}. \tag{5}$$

Since for a fixed line $L \in \mathcal{L}$ and $x \in X$, $y \in Y$, the probability that L is contained in R_{xy} is $(1-o(1))\ell^{-4}$ (the number of lines in R_{xy} divided by the total number of lines), the expected number of lines L satisfying (5) is $[(1 - o(1))\ell^{-4}]^4(1+o(1))\ell^8 = (1-o(1))\ell^{-8}$. Summing over all $x_1, x_2 \in X$, $y_1, y_2 \in Y$, we infer that the expected number of quintuples satisfying (5) equals

$$\mathrm{E}(Q) = \binom{m}{2}^2(1-o(1))\ell^{-8} = (1-o(1))\frac{\ell^{12}}{4}.$$

Fix $\hat{G} = (X, Y, V, \hat{\mathcal{E}})$ with $Q(\hat{\mathcal{E}}) \le \mathrm{E}(Q)$. For every x_1, x_2, y_1, y_2, L satisfying (5), fix one pair x_i, y_j, $i \in \{1,2\}$, $j \in \{1,2\}$, say, x_1, y_1 and delete all hyperedges of the form $(x_1, y_1, z) : z \in L$. This way, we delete ℓ hyperedges for each quintuple satisfying (5) and thus $Q(\hat{\mathcal{E}}) \cdot \ell \leq \left(\frac{1}{4} - o(1)\right) \ell^{13}$ hyperedges altogether. Deleting these hyperedges from $\hat{G}$ we obtain a 3-uniform hypergraph $G' = (X, Y, V, \mathcal{E}')$ with $(\frac{3}{4} - o(1))\ell^{13}$ hyperedges. To prove the theorem, it remains only to show that G' contains no $K^{(3)}(2,2,2)$.

Indeed, suppose that $\{x_1, x_2, y_1, y_2, z_1, z_2\}$ is the vertex set of a copy of $K^{(3)}(2,2,2)$ contained in G'; then (5) holds for the line L containing z_1 and z_2. However, $\hat{\mathcal{E}} \backslash \mathcal{E}'$ contains all triples of the form (x_1, y_1, z), $z \in L$, and hence neither (x_1, y_1, z_1) nor (x_1, y_1, z_2) are in $\mathcal{E}'$, contradicting our assumption. $\square$

For larger d, if one attempts to similarly improve on (4) by extending the affine space technique of Theorem 2.4, a certain condition on d must be met. We first examine this condition, give the theorem, then after the proof, briefly mention why this condition was necessary.

For $d \geq 3$ define $s = s(d)$ to be the smallest positive integer s (if it exists) so that $\frac{sd-1}{2^d-1}$ is an integer. By the Chinese remainder theorem, $s(d)$ exists precisely when d and $2^d - 1$ are relatively prime, and this holds, for example, when d is a prime or a power of 2, and does not hold when, for example, $d = 6$.

Theorem 2.5 *If $d \geq 3$ is so that $s = s(d)$ exists then there is a constant $c = c(d)$ and $n_0 = n_0(d)$ so that for all $n \geq n_0$,*

$$ex(n, K^{(d)}(2, 2, \ldots, 2)) > cn^{d - \frac{d-1/s}{2^d-1}},$$

Proof: Fix $d \geq 3$ and $s = s(d)$ and put

$$r = \frac{s(d+1-2^d) - 1}{1 - 2^d} = s - \frac{sd-1}{2^d - 1}.$$

We imitate the proof of Theorem 2.4 (where $d = 3$, $s = 5$ and $r = 3$) and so only outline the calculations.

Choose a prime $\ell \leq (n/d)^{1/s}$ satisfying $\ell^s = (1 - o(1))n/d$, and let V be a s-dimensional vector space over $\mathrm{GF}(\ell)$ on $m = \ell^s$ points. Let $\mathcal{R}$ be the collection of r-dimensional affine subspaces of V and let $\mathcal{L}$ be the lines.

Let $G = (X_1, X_2, \ldots, X_{d-1}, V, \mathcal{E})$ be a random d-partite d-uniform hypergraph defined as follows. Each of $X_1, \ldots, X_{d-1}$ and V are pairwise disjoint m-vertex sets. The hyperedge set $\mathcal{E}$ is defined by, for each $x_1 \in X_1, \ldots, x_{d-1} \in X_{d-1}$ fixing at random $R_{x_1,\ldots,x_{d-1}} \in \mathcal{R}$, and then setting

$$\mathcal{E} = \{(x_1, \ldots, x_{d-1}, v) : x_1 \in X_1, \ldots, x_{d-1} \in X_{d-1}, v \in R_{x_1,\ldots,x_{d-1}}\}.$$

The random G has $|\mathcal{E}| = m^{d-1}\ell^r = \ell^{s(d-1)+r}$ hyperedges.

Let $Q = Q(\mathcal{E})$ be the random variable counting the number of $(2d-1)$-tuples $(x_1, \overline{x}_1, x_2, \overline{x}_2, \ldots, x_{d-1}, \overline{x}_{d-1}, L)$ (where each $x_i, \overline{x}_i \in X_i$ and $L \in \mathcal{L}$) such that

$$L \subseteq \bigcap \{R_{z_1, z_2 \ldots, z_{d-1}} : (z_1, \ldots, z_{d-1}) \in \{x_1, \overline{x}_1\} \times \cdots \times \{x_{d-1} \overline{x}_{d-1}\}\} \qquad (6)$$

As before, the expected number of $(2d-1)$-tuples satisfying (6) is

$$\begin{aligned}
\mathrm{E}(Q) &= \binom{m}{2}^{d-1}\left(\frac{\#\text{ of lines in an } R}{|\mathcal{L}|}\right)^{2^{d-1}}|\mathcal{L}| \\
&\sim \frac{1}{2^{d-1}}\ell^{2(d-1)s}\left(\frac{\ell^{2r-2}}{\ell^{2s-2}}\right)^{2^{d-1}}\ell^{2s-2} \\
&= \frac{1}{2^{d-1}}\ell^{2sd-2-(s-r)2^d} \\
&= \frac{1}{2^{d-1}}\ell^{2sd-1-\frac{sd-1}{2^d-1}2^d-1} \\
&= \frac{1}{2^{d-1}}\ell^{sd-\frac{sd-1}{2^d-1}-1}.
\end{aligned}$$

Fix a hypergraph $\hat{G} = (X_1, X_2, \dots, X_{d-1}, V, \hat{\mathcal{E}})$ with $Q(\hat{\mathcal{E}}) \le \mathrm{E}(Q)$. For each $(2d-1)$-tuple $(x_1, \overline{x}_1, \dots, x_{d-1}, \overline{x}_{d-1}, L)$ satisfying (6), fix say, $x_1, x_2, \dots, x_{d-1}$, and for each $v \in L$ delete the d-hyperedge $(x_1, \dots, x_{d-1}, v)$ from $\hat{\mathcal{E}}$. Let $G' = (X_1, \dots, X_{d-1}, V, \mathcal{E}')$ be the hypergraph which remains after deletion of these edges. Since at most

$$Q(\hat{\mathcal{E}}) \cdot \ell \le \frac{1}{2^{d-1}}\ell^{sd-\frac{sd-1}{2^d-1}}$$

d-hyperedges have been deleted, we have

$$\begin{aligned}
|\mathcal{E}'| &\ge \frac{2^{d-1}-1}{2^{d-1}}\ell^{sd-\frac{sd-1}{2^d-1}} \\
&= cn^{d-\frac{d-1/s}{2^d-1}}
\end{aligned}$$

remaining edges, where c is a constant depending only on d. As in the proof of Theorem 2.4, the hypergraph G' contains no $K^{(d)}(2, \dots, 2)$. □

In the statement of Theorem 2.5, we required that $s(d)$ exists, or equivalently, that $\gcd(d, 2^d-1) = 1$. We now outline why this condition was necessary. For the deletion aspect of the proof to work, we needed $E(Q) \cdot \ell \le |\hat{\mathcal{E}}|$, that is, $2sd - 1 - (s-r)2^d \le s(d-1) + r$, giving $r \le \frac{s(2^d-d-1)+1}{2^d-1}$. On the other hand, for this technique to yield an improvement over the exponent in (4), we require $r > \frac{s(2^d-d-1)}{2^d-1}$. Combining these inequalities, we get $0 < r(2^d-1) - s(2^d-d-1) \le 1$, and so $r(2^d-1) - s(2^d-d-1) = 1$. Thus $2^d - 1$ and $2^d - d - 1$ are relatively prime, hence so are d and $2^d - 1$.

For example, when $d = 4$, Theorem 2.5 applies with $s = 4$ (and $r = 3$); together with the upper bound given by Theorem 2.3, we get a constant c so that

$$cn^{15/4} \le \mathrm{ex}(n, K^{(4)}(2, 2, 2, 2)) \le n^{31/8}$$

holds for all sufficiently large n.

3 Integers and cubes

3.1 Sidon sets

A *Sidon set* is a collection of integers whose pairwise sums $a+b$, $(a \neq b)$ are all distinct; these are also referred to as B_2-sets. In the proof of the upper bound in Theorem 4.6, we use the following result due to Singer [44] to produce a partition into Sidon sets. See also [21] for a simple construction of a single Sidon set, but with different bounds.

Theorem 3.1 (Singer) *Let m be a prime power. There exist $m+1$ integers*

$$0 \leq x_1 < x_2 < \ldots < x_{m+1} \leq m^2 + m$$

so that the m^2+m differences $x_i - x_j, 1 \leq i \neq j \leq m+1$, are distinct modulo m^2+m+1.

For example, with $m = 4$, the integers 0, 1, 6, 8, and 18 have distinct differences (modulo 21).

3.2 Affine cubes

Definition 3.2 *For a non-negative integer x_0 and positive integers $x_1, \ldots, x_d$, the family*

$$H(x_0, x_1, \ldots, x_d) = \left\{ x_0 + \sum_{i \in I} x_i : I \subseteq [1, d] \right\}$$

is called a d-dimensional affine cube, or simply, an affine d-cube.

Very closely related to Boolean algebras, we will be concerned with both coloring and density results for affine d-cubes. The first result in this direction is perhaps the first non-trivial result in Ramsey theory, published in 1892.

Theorem 3.3 (Hilbert [36]) *For every r, d, there is a least number $h(d,r)$ so that for every coloring $\chi : [h(d,r)] \to [r]$, there exists an affine d-cube monochromatic under χ.*

In [9] it was shown that $h(2,r) = (1+o(1))r^2$. Also in [9], it was noted that there exist constants c_1 and c_2 so that

$$r^{c_1 d} \leq h(d,r) \leq r^{c_2^d}, \tag{7}$$

where $c_2 \sim 2.6$ follows from Hilbert's proof (using Fibonacci numbers). We note some improvements to this in Theorem 3.8.

Theorem 3.4 (Behrend [5]) *There exists a constant c so that for m sufficiently large, there exists $B \subset [m]$ not containing any arithmetic progressions of length three, and satisfying*

$$|B| \geq m e^{-c\sqrt{\ln m}} = m^{1-o(1)}.$$

By definition, for any affine d-cube H, $|H| \le 2^d$ trivially holds; we say that an affine d-cube H is *replete* if $|H| = 2^d$, that is, if all the sums defining H are distinct. We now list some results from [33] and [34], for later use in Section 4.

Lemma 3.5 *For each d, and every set X of positive integers, there exists $A \subset X$ not containing any replete affine d-cubes and*

$$|A| \ge \frac{1}{8}|X|^{1-\frac{d}{2^d-1}}.$$

The following lemma is a combination of Theorem 3.4 and a special case of Lemma 3.5, stated separately for later use in the proof of Theorem 4.2. The notation $[a, b]$ indicates a closed interval of integers.

Lemma 3.6 *For every d there is a constant c so that for every k and every m, there is a set $S \subset [k+1, k+m]$ containing no replete affine d-cubes nor containing any arithmetic progression of length 3, yet has at least*

$$|S| \ge cm^{1-\frac{d}{2^d-1}(1-o(1))}$$

elements.

Proof: By Theorem 3.4, let $B \subset [1, m]$ containing no arithmetic progression, and with

$$|B| \ge me^{-\sqrt{\ln m}} = m^{1-o(1)}.$$

The translation of B, $B_k = \{b + k : b \in B\}$ also has no arithmetic progression. With B_k playing the role of X, Lemma 3.5 yields S as desired. $\Box$

Szemerédi [45] (Lemma $p(\delta, l)$, p. 93) gave a density version of Hilbert's theorem. Two more proofs of Szemerédi's "cube lemma" were given in [39] (problem 14.12), one of which was modified to give the following.

Theorem 3.7 *For each d there exists a constant c so that for n sufficiently large, if $A \subseteq [1, n]$ satisfies $|A| \ge 2n^{1-\frac{1}{2^{d-1}}}$, then A contains an affine d-cube.*

Finding non-trivial lower bounds for density results seems to be quite hard. An upper bound for the number $h(d, r)$ is the trivial one obtained by an associated density result, say Theorem 3.7.

Theorem 3.8 *For each $d \ge 2$, $r \ge 2$,*

$$r^{\frac{2^d-1}{d}(1-o(1))} \le h(d, r) \le (2r)^{2^{d-1}},$$

where $o(1)$ tends to 0 as r increases.

4 Boolean Algebras

4.1 Lower bound for $b(n,d)$; a density result

To streamline the proof of the theorem, we provide the following simple estimate regarding the number of subsets of a set which are close to the average size.

Lemma 4.1 *For n sufficiently large and each i satisfying $-\sqrt{n}/2 \le i \le \sqrt{n}/2$,*

$$\binom{n}{n/2+i} \ge \frac{1}{\sqrt{e}}\binom{n}{n/2}.$$

Proof: It suffices to show the result for $i = \sqrt{n}/2$, which we shall assume is an integer (as well as $n/2$).

$$\begin{aligned}
\frac{\binom{n}{n/2-\sqrt{n}/2}}{\binom{n}{n/2}} &= \prod_{j=0}^{\sqrt{n}/2-1} \frac{n/2-j}{n/2+\sqrt{n}/2-j} \\
&= \prod_{j=0}^{\sqrt{n}/2-1} \left(1 - \frac{\sqrt{n}}{n+\sqrt{n}-2j}\right) \\
&\ge \left(1-\frac{1}{\sqrt{n}}\right)^{\sqrt{n}/2} \\
&\ge \left(e^{\frac{-1/\sqrt{n}}{1-1/\sqrt{n}}}\right)^{\sqrt{n}/2} \\
&> e^{-1/2},
\end{aligned}$$

where the penultimate inequality follows from $1-x \ge e^{-x/(1-x)}$. $\square$

Theorem 4.2 *For each $d > 2$ and n sufficiently large,*

$$2^n n^{-\frac{d}{2^{d+1}-2}(1-o(1))} \le b(n,d).$$

Proof: Fix n and let X be a set of n elements. We will construct a large family $\mathcal{F}$ of subsets of X which contains no d-dimensional algebra.

Applying Lemma 3.6 with $m = \sqrt{n}$ and $k = \frac{n}{2} - \frac{\sqrt{n}}{2} - 1$, let

$$S \subset \left[\frac{n}{2} - \frac{\sqrt{n}}{2}, \frac{n}{2} + \frac{\sqrt{n}}{2}\right]$$

be a collection of

$$|S| = n^{\frac{1}{2}-\frac{d}{2^{d+1}-2}(1-o(1))}$$

integers that contains no replete affine d-cube and no arithmetic progression of length three. Define

$$\mathcal{F} = \{Y \subset X : |Y| \in S\}.$$

Calculating the size of $\mathcal{F}$,

$$\begin{aligned}
|\mathcal{F}| &= \sum_{s \in S} \binom{n}{s} \\
&> |S| e^{-1/2} \binom{n}{n/2} \qquad \text{(by Lemma 4.1)} \\
&\sim |S| \frac{1}{\sqrt{\pi e n}} 2^n \\
&= 2^n n^{-\frac{d}{2^{d+1}-2}(1-o(1))}.
\end{aligned}$$

So $\mathcal{F}$ contains the desired number of elements. It remains to show that $\mathcal{F}$ does not contain a d-dimensional algebra.

Suppose, in hopes of a contradiction, that there exist pairwise disjoint subsets of X, say, $B_0, B_1, \ldots, B_d$ so that the family

$$\mathcal{B} = \left\{ B_0 \cup \bigcup_{i \in I} B_i : I \subseteq [1, d] \right\}$$

is contained entirely in $\mathcal{F}$. If all of the sets in $\mathcal{B}$ are different sizes, then the set

$$\{|B| : B \in \mathcal{B}\} = \left\{ |B_0| + \sum_{i \in I} |B_i| : I \subseteq [1, d] \right\} \subset S$$

is a replete affine d-cube, a contradiction.

So there must be two elements of $\mathcal{B}$ with the same size. Suppose that $C, D \in \mathcal{B}$ satisfy $|C \cap D| = a$ and $|C| = |D| = a + b$. Since $\mathcal{B}$ is a Boolean algebra, the sets $C \cap D$, C, and $C \cup D$ are contained in $\mathcal{B}$, but in this case, the respective sizes (which are members of S) a, $a + b$, $a + 2b$ form an arithmetic progression, another contradiction.

We conclude that $\mathcal{F}$ does not contain any d-dimensional Boolean algebras. □

4.2 Upper bound for $b(n, d)$

The proof of the following density result is based on the proof of a similar statement in [42].

Theorem 4.3 *For each $d \geq 1$ there exists a constant c so that*

$$b(n, d) \leq c n^{-1/2^d} \cdot 2^n.$$

First we give a preparatory discussion of chains in Boolean lattices, then give the proof of Theorem 4.3 which relies both on these notions and a result from Section 2 on hypergraphs.

Let Y be a set of t vertices. A collection $\mathcal{C} \subseteq \mathcal{P}(Y)$ of subsets of Y is a *chain* if and only if for every $A, B \in \mathcal{C}$, either $A \subset B$ or $B \subset A$. A chain $\mathcal{C} \subseteq \mathcal{P}(Y)$ is *symmetric* if for every $C \in \mathcal{C}$ there exists $C' \in \mathcal{C}$ so that $\{|C|, |C'|\} = \{\lceil t/2 \rceil + i, \lfloor t/2 \rfloor - i\}$ for some $i \geq 0$. A chain is *convex* if whenever $A \subset B \subset C$ and both A and C are in the chain, then so is B.

There are a number of methods by which a t-dimensional Boolean lattice can be partitioned into $\binom{t}{\lfloor t/2 \rfloor})\}$ disjoint symmetric convex chains (one is inductive, likely due to de Bruin; also see [1], [2], pp. 436, 439, [31], or [32], p. 30).

Let $\boldsymbol{\mathcal{C}} = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_{\binom{t}{\lfloor t/2 \rfloor}}\}$ be a decomposition of $\mathcal{P}(Y)$ into disjoint symmetric convex chains, and let $\boldsymbol{\mathcal{C}}^{>2i} \subset \boldsymbol{\mathcal{C}}$ denote the subcollection of those chains having length greater than $2i$. Since each chain $\mathcal{C} \in \boldsymbol{\mathcal{C}}^{>2i}$ contains a different set with $\lfloor t/2 \rfloor - i$ vertices, it follows that

$$|\boldsymbol{\mathcal{C}}^{>2i}| = \binom{t}{\lfloor t/2 \rfloor - i}.$$

For any permutation $\pi : Y \to Y$ of the vertices of Y and for any chain $C \in \boldsymbol{\mathcal{C}}$, the collection

$$\pi(\mathcal{C}) = \{\pi(C) : C \in \mathcal{C}\}$$

is also a chain, so

$$\pi(\boldsymbol{\mathcal{C}}) = \{\pi(\mathcal{C}) : \mathcal{C} \in \boldsymbol{\mathcal{C}}\}$$

is also a symmetric chain decomposition of $\mathcal{P}(Y)$, with $\pi(\boldsymbol{\mathcal{C}}^{>2i}) \subset \pi(\boldsymbol{\mathcal{C}})$.

Lemma 4.4 *Let Y be a set of t elements. Fix $D \subset Y$ and let*

$$\boldsymbol{\mathcal{C}} = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_{\binom{t}{\lfloor t/2 \rfloor}}\}$$

be a fixed decomposition of the power set $\mathcal{P}(Y)$ into disjoint symmetric convex chains. If $\pi : Y \to Y$ is a permutation chosen randomly from the set of $t!$ permutations of Y, then

$$\textit{prob}(D \in \pi(\mathcal{C}) \textit{ for some } \mathcal{C} \in \boldsymbol{\mathcal{C}}^{>2i}) > \left(1 - \frac{2i+2}{t}\right)^i.$$

Proof: If $|D| \leq \lfloor t/2 \rfloor - i$ or $|D| \geq \lfloor t/2 \rfloor + i$, then since $\pi(\cup \boldsymbol{\mathcal{C}}^{>2i})$ contains all sets of these sizes, then $D \in \pi(\mathcal{C})$ for some $C \in \boldsymbol{\mathcal{C}}^{>2i}$, that is, the probability is 1.

Now fix $D \subset Y$ with $\lfloor t/2 \rfloor - i < |D| < \lfloor t/2 \rfloor + i$. Set $\mathcal{S} = \cup \boldsymbol{\mathcal{C}}^{>2i} \cap [Y]^{|D|}$. Let $\pi : Y \to Y$ be a random permutation. For a fixed $S \subset Y$ chosen with $|S| = |D|$,

$$\text{Prob}(\pi^{-1}(D) = S) = \frac{|D|!(t - |D|)!}{t!} = \frac{1}{\binom{t}{|D|}}.$$

Hence,

$$\begin{aligned}
\mathrm{Prob}(D \in \pi(\mathcal{S})) &= \mathrm{Prob}(\pi^{-1}(D) \in \mathcal{S}) \\
&= \sum_{S \in \mathcal{S}} \mathrm{Prob}(\pi^{-1}(D) = S) \\
&= \frac{|\mathcal{S}|}{\binom{t}{|D|}} \\
&= \frac{|\mathcal{C}^{>2i}|}{\binom{t}{|D|}} \\
&\geq \frac{\binom{t}{\lfloor t/2 \rfloor - i}}{\binom{t}{\lfloor t/2 \rfloor}} \\
&= \prod_{j=0}^{i-1} \frac{\lfloor t/2 \rfloor - j}{\lceil t/2 \rceil + i - j} \\
&\geq \prod_{j=0}^{i-1} \frac{t/2 - 1/2 - j}{t/2 + 1/2 + i - j} \\
&\geq \prod_{j=0}^{i-1} \left(1 - \frac{i+1}{t/2 + 1/2 + i - j}\right) \\
&> \left(1 - \frac{2i+2}{t}\right)^i . \ \Box
\end{aligned}$$

The following fact follows from a simple averaging argument; we omit the proof.

Lemma 4.5 *Let $H = (V_1, \ldots, V_d, \mathcal{E}(H))$ be a given d-partite d-uniform hypergraph and let $v \leq \min_{1 \leq i \leq d}\{|V_i|\}$. For each $i = 1, \ldots, d$, there exist vertex sets $W_i \subseteq V_i$, $|W_i| = v$, so that the the subgraph H' induced by $\cup_{i=1}^{d} W_i$ has edge density at least that of H, that is,*

$$\frac{|\mathcal{E}(H')|}{v^d} \geq \frac{|\mathcal{E}(H)|}{|V_1| \cdot \ldots \cdot |V_d|}.$$

We are now prepared to prove an upper bound for $b(n, d)$.

Proof of Theorem 4.3: Let X be a set of n elements and fix a positive integer d. Let

$$c = 10^d 2^{-1/2^{d-1}} d^{d-1/2^d}. \tag{8}$$

and $\mathcal{F} \subset \mathcal{P}(X)$ satisfy

$$|\mathcal{F}| \geq cn^{-1/2^d} 2^n. \tag{9}$$

We will show that $\mathcal{F}$ contains a Boolean algebra of dimension d.

Partition $X = X_1 \cup X_2 \cup \ldots \cup X_d$ into d sets, each with size $\lfloor n/d \rfloor \leq |X_j| \leq \lceil n/d \rceil$. For each $j = 1, \ldots d$, fix $\boldsymbol{\mathcal{C}}_j$, a symmetric chain decomposition of $\mathcal{P}(X_j)$; for i to be determined later, let $\boldsymbol{\mathcal{C}}_j^{>2i} \subseteq \boldsymbol{\mathcal{C}}_j$ be the subcollection of those chains longer than $2i$. For each $j = 1, \ldots, d$ let $\pi_j : X_j \to X_j$ denote a permutation of X_j chosen randomly from the collection of all $|X_j|!$ permutations on X_j (the permutations $\pi_1, \pi_2, \ldots, \pi_d$ are chosen independently). Let $\mathcal{F}_{\pi_1,\ldots,\pi_d} \subset \mathcal{F}$ be a random subset of $\mathcal{F}$ defined by

$$\mathcal{F}_{\pi_1,\ldots,\pi_d} = \{F \in \mathcal{F} : \forall j = 1, \ldots, d, \exists \mathcal{D}(j) \in \pi_j(\boldsymbol{\mathcal{C}}_j^{>2i}) \text{ with } F \cap X_j \in \mathcal{D}(j)\}. \tag{10}$$

By Lemma 4.4, for any $F \in \mathcal{F}$,

$$\text{Prob}\left(F \in \mathcal{F}_{\pi_1,\ldots,\pi_d}\right) > \prod_{j=1}^{d} \left(1 - \frac{2i+2}{|X_j|}\right)^i. \tag{11}$$

Fix $i = \lfloor \sqrt{n/d} \rfloor$, sufficient for our purpose in what follows. Then for sufficiently large n, as $|X_j| \geq \lfloor n/d \rfloor$, the right hand side of (11) can be further bounded from below by

$$\left(1 - \frac{2\sqrt{\lfloor n/d \rfloor} + 2}{\lfloor n/d \rfloor}\right)^{\lfloor \sqrt{n/d} \rfloor d} > \left(1 - \frac{2.1}{\sqrt{n/d}}\right)^{(\sqrt{n/d})d} > \left(\frac{1}{e^{2.1}}\right)^d > (.1)^d.$$

Hence the expected number of sets in $\mathcal{F}_{\pi_1,\ldots,\pi_d}$ is

$$\mathrm{E}(|\mathcal{F}_{\pi_1,\ldots,\pi_d}|) > (.1)^d |\mathcal{F}|. \tag{12}$$

Fix a choice of $\hat{\pi}_1, \ldots, \hat{\pi}_d$ for which (12) is realized. For each $j = 1, \ldots, d$, set $\boldsymbol{\mathcal{D}}_j = \hat{\pi}_j(\boldsymbol{\mathcal{C}}_j^{>2i})$, the family of disjoint chains in X_j longer than $2i$, and write

$$\boldsymbol{\mathcal{D}}_j = \left\{\mathcal{D}_{j,k_j} : 1 \leq k_j \leq \binom{|X_j|}{\lfloor |X_j|/2 \rfloor - i}\right\}.$$

Put $\mathcal{G} = \mathcal{F}_{\hat{\pi}_1,\ldots,\hat{\pi}_d}$. Note that by (10) and (12),

$$\mathcal{G} = \{F \in \mathcal{F} : \forall j = 1, \ldots, d, \exists \mathcal{D}(j) \in \boldsymbol{\mathcal{D}}_j \text{ with } F \cap X_j \in \mathcal{D}(j)\},$$

and

$$|\mathcal{G}| > (.1)^d |\mathcal{F}|. \tag{13}$$

For each choice of $k_1, \ldots, k_d$ (the k_i's not necessarily distinct), define the set system

$$\mathcal{D}_{1,k_1} \otimes \cdots \otimes \mathcal{D}_{d,k_d} = \{\cup_{j=1}^d D_{j,k_j} : D_{j,k_j} \in \mathcal{D}_{j,k_j}\},$$

and also define

$$\mathcal{D} = \bigcup_{k_1,,\ldots,k_d} (\mathcal{D}_{1,k_1} \otimes \cdots \otimes \mathcal{D}_{d,k_d}),$$

where now we have $\mathcal{G} = \mathcal{F} \cap \mathcal{D}$. Also for each $j = 1, \ldots, d$, set $s_j = |\cup \boldsymbol{\mathcal{D}}_j|$ the number of sets in chains in $\boldsymbol{\mathcal{D}}_j$. Furthermore, put

$$\mathcal{G}_{k_1,\ldots,k_d} = \mathcal{F} \cap (\mathcal{D}_{1,k_1} \otimes \cdots \otimes \mathcal{D}_{d,k_d}).$$

We observe that by (10) and (13),

$$|\mathcal{G}| = \left| \bigcup_{k_1,\ldots,k_d} \mathcal{G}_{k_1,\ldots,k_d} \right| = |\mathcal{F} \cap \mathcal{D}| > (.1)^d |\mathcal{F}|.$$

Since

$$\sum_{k_1,\ldots,k_d} |\mathcal{D}_{1,k_1}| \cdots |\mathcal{D}_{d,k_d}| = s_1 \cdots s_d < 2^n,$$

we infer that there is a choice of $\hat{k}_1, \ldots, \hat{k}_d$ so that

$$\frac{|\mathcal{G}_{\hat{k}_1,\ldots,\hat{k}_d}|}{|\mathcal{D}_{1,\hat{k}_1}| \cdots |\mathcal{D}_{d,\hat{k}_d}|} \geq \frac{|\mathcal{G}|}{s_1 \cdots s_d} > \frac{(.1)^d |\mathcal{F}|}{2^n}. \tag{14}$$

Using (9), we obtain from (14),

$$|\mathcal{G}_{\hat{k}_1,\ldots,\hat{k}_d}| > c(.1)^d \cdot n^{-1/2^d} |\mathcal{D}_{1,\hat{k}_1}| \cdots |\mathcal{D}_{d,\hat{k}_d}|. \tag{15}$$

By Lemma 4.5, for each $j = 1, \ldots, d$, choose $\mathcal{D}^*_{j,\hat{k}_j} \subset \mathcal{D}_{j,\hat{k}_j}$ with $|\mathcal{D}^*_{j,\hat{k}_j}| = 2\lfloor \sqrt{n/d} \rfloor$ so that for

$$\mathcal{G}^*_{\hat{k}_1,\ldots,\hat{k}_d} = \mathcal{F} \cap (\mathcal{D}^*_{1,\hat{k}_1} \otimes \cdots \otimes \mathcal{D}^*_{d,\hat{k}_d}),$$

the corresponding inequality to (15) holds, namely,

$$\begin{aligned} |\mathcal{G}^*_{\hat{k}_1,\ldots,\hat{k}_d}| &> c(.1)^d \cdot n^{-1/2^d} |\mathcal{D}^*_{1,\hat{k}_1}| \cdots |\mathcal{D}^*_{d,\hat{k}_d}| \\ &= c(.1)^d \cdot n^{-1/2^d} \cdot (2\lfloor \sqrt{n/d} \rfloor)^d. \end{aligned}$$

For $m = d \cdot 2\lfloor \sqrt{n/d} \rfloor$, then $n \geq (m/2)^2 \cdot 1/d$, and hence

$$\begin{aligned} |\mathcal{G}^*_{\hat{k}_1,\ldots,\hat{k}_d}| &> c(.1)^d \left(\left(\frac{m}{2}\right)^2 \frac{1}{d} \right)^{-1/2^d} \left(\frac{m}{d}\right)^d \\ &= c(.1)^d 2^{1/2^{d-1}} d^{-d+2^{-d}} m^{d-1/2^{d-1}}. \end{aligned} \tag{16}$$

By the choice of c, (8) and (16) yield

$$|\mathcal{G}^*_{\hat{k}_1,\ldots,\hat{k}_d}| > m^{d-1/2^{d-1}}. \tag{17}$$

For each $j = 1, \ldots, d$ consider $Y_j = \mathcal{D}^*_{j,\hat{k}_j}$ as a vertex set, vertices being subsets of X_j in the chain $\mathcal{D}^*_{j,\hat{k}_j}$. Using the d-partite d-uniform hypergraph

$$\mathcal{H} = (Y_1, \ldots, Y_d, \mathcal{G}^*_{\hat{k}_1,\ldots,\hat{k}_d}),$$

then Theorem 2.3 (with $d \cdot 2\lfloor\sqrt{n/d}\rfloor$ as the number of vertices) and (17) imply that there is a copy of $K^{(d)}(2, 2, \ldots, 2)$ in $\mathcal{H}$. That is, for each $j = 1, \ldots, d$, there are $A^0_j, A^1_j \in \mathcal{D}^*_{j,\hat{k}_j}$ with $A^0_j \neq A^1_j$, and, say, $A^0_j \subset A^1_j$, so that for any choice of $(\delta_1, \ldots, \delta_d) \in \{0, 1\}^d$,

$$A^{\delta_1}_1 \cup \ldots \cup A^{\delta_d}_d \in \mathcal{G}^*_{\hat{k}_1,\ldots,\hat{k}_d} \subset \mathcal{F}.$$

In this case,

$$\{A^{\delta_1}_1 \cup \ldots \cup A^{\delta_d}_d : (\delta_1, \ldots, \delta_d) \in \{0, 1\}^d\}$$

is the desired d-dimensional Boolean algebra (see [30], Lemma 5.7) (with meet $(A^0_1 \cup \cdots \cup A^0_d)$ and join $(A^1_1 \cup \cdots \cup A^1_d)$) completing the proof. □

4.3 Bounds on $r(d, n)$; partition results

An easy proof by induction yields that for any positive integer n, $r(1, n) = n$. We now examine the case $d = 2$.

Theorem 4.6 *For n sufficiently large,*

$$(1 - o(1))\frac{3}{4}n^{1/2} \leq r(2, n) \leq (1 + o(1))n^{1/2}.$$

Proof: We first show the lower bound. Let $\epsilon > 0$, and fix a coloring

$$\mathcal{P}([n]) = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \cdots \cup \mathcal{F}_r,$$

where $r \leq \frac{3\sqrt{n}}{4(1+\epsilon)}$. By the pigeon-hole principle, there is one family $\mathcal{F}_k$ containing at least

$$\frac{\binom{n}{2}}{\frac{3\sqrt{n}}{4(1+\epsilon)}} \sim (1 + \epsilon)\frac{2}{3}n^{3/2}$$

intervals $[i, j] \subset [n]$. By Theorem 2.2, the graph $([n], \mathcal{F}_k)$ induced by these intervals contains a $\vec{C_4}$ on, say, vertices $a < b < c < d$. The intervals $[a, c]$, $[b, c]$, $[b, d]$, $[a, d]$ form a monochromatic 2-dimensional Boolean algebra (where $X_0 = [b, c]$, $X_1 = [a, b]$, and $X_2 = [c, d]$), proving the lower bound.

To see the upper bound, it suffices to give a $(1+o(1))n^{1/2}$-coloring of $\mathcal{P}([n])$ which multicolors every 2-dimensional Boolean algebra; this will be done in a manner similar to that used in [9] and in [11] (or summarized in [29]). Let m be a prime power and let $0 \le x_1 < x_2 \ldots < x_{m+1} < m^2+m+1$ be as in (Singer's) Theorem 3.1. For each $j = 1, \ldots, m+1$, define

$$Y_j = \{x_i - x_j \pmod{m^2+m+1} : 1 \le i \le m+1, i \ne j\} \subset [1, m^2+m],$$

and put $Y_0 = \{0\}$. A simple calculation shows that if $\{a, b, c, d\} \subset Y_j$ for some j, and $a + b = c + d$, then $\{a, b\} = \{c, d\}$. Furthermore, the Y_j's partition the set $[0, m^2+m]$ and for each $j \ne 0$, $|Y_j| = m$.

For each $j = 0, 1, 2, \ldots, m+1$, define

$$\mathcal{S}_j = \{X \subset [1, m^2+m] : |X| \in Y_j\}.$$

This defines a decomposition of the power set of $[1, m^2+m]$ into $m+2$ classes. If for some j, there were sets $A, B, C, D \in \mathcal{S}_j$ with $|A| + |B| = |C| + |D|$, then $\{|A|, |B|\} = \{|C|, |D|\}$, and so these four sets do not form a 2-dimensional Boolean algebra (see [16]).

Now for a given n, let $m = m(n)$ be the smallest prime power so that $n \le m^2+m$. Since the ratio between consecutive prime powers tends to one, (as $n \to \infty$) the minimum number of color classes required to prevent a monochromatic 2-dimensional Boolean algebra is at most $m + 2 = (1 + o(1))\sqrt{m^2+m} = (1 + o(1))\sqrt{n}$. $\square$

In the proof of the lower bound $r(2, n) \ge (1 - o(1))\frac{3}{4}\sqrt{n}$, the colors of *only* $\binom{n}{2}$ sets (intervals) mattered, not the entire power set, so one might suspect that the lower bound can be improved. We note that the above idea extends a proof technique using only $n^2/4$ sets, an argument following from ideas in [16], and [18] (as mentioned in [3]) which yields $r(2, n) \ge (1 - o(1))\sqrt{n/2}$; instead of using all intervals and Theorem 2.2, use only those containing $n/2$ and $n/2 + 1$, and then apply Lemma 2.1 to the corresponding bipartite graph.

For general d, upper and lower bounds on $r(d, n)$ are still far apart.

Theorem 4.7 *For $d > 2$, there exists a constant c_1 so that*

$$c_1 n^{1/2^d} \le r(d, n) \le n^{\frac{d}{2^d-1}(1+o(1))},$$

where $o(1)$ tends to 0 as n tends to infinity.

Proof: To prove the lower bound, let c be the constant from Theorem 4.3 and put $c_1 = 1/c$. If we color $\mathcal{P}([n])$ with fewer than $c_1 n^{1/2^d}$ colors, then one color class contains $cn^{-1/2^d}2^n$ elements. By Theorem 4.3 one class contains a d-dimensional Boolean algebra.

For a number r, to prove that $r(d, n) < r$, it suffices to produce a partition $\mathcal{P}([n]) = \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_r$ so that each $\mathcal{F}_i$ contains no d-dimensional Boolean algebra of sets. It follows from the proof of the lower bound in Theorem 3.8 that there exists a partition $[n] = S_1 \cup S_2 \cup \cdots \cup S_r$, with $r = n^{\frac{d}{2^d-1}(1+o(1))}$, each class containing no replete affine d-cube, nor any arithmetic progressions. For each

$i = 1, \ldots, d$, put $\mathcal{F}_i = \{F \subset [n] : |F| \in S_i\}$. As in the proof of Theorem 4.2, it is now not difficult to see that each $\mathcal{F}_i$ does not contain any d-dimensional Boolean algebras. $\square$

5 Uniform Boolean algebras

Recall that a d-dimensional Boolean algebra $\mathcal{B} = \{B_0 \cup \bigcup_{i \in I} B_i : I \subseteq [1, d]\}$ is called *uniform* if $B_1, \ldots, B_d$ are all the same size, and $b_u(n, d)$ is the maximum size of a family in $\mathcal{P}([n])$ which does not contain a uniform d-dimensional Boolean algebra.

Theorem 5.1 *For each d, and any $\epsilon > 0$, there exists n_0 so that for every $n \geq n_0$,*

$$\frac{1}{n^{o(1)}} 2^n \leq b_u(n, d) \leq \epsilon \cdot 2^n.$$

In Theorem 4.3 we showed that if a subset of $\mathcal{P}([n])$ is chosen with $cn^{-1/2^d} 2^n$ elements, then this subset contains a d-dimensional Boolean algebra. If the factor $cn^{-1/2^d}$ is weakened to some fixed $\epsilon > 0$, then, for all n large enough, the d-dimensional Boolean algebra guaranteed by Theorem 4.3 can be taken to be uniform. The main tool used to see this is a density version of the Hales-Jewett theorem, which we now briefly describe.

Let $A = \{a_1, a_2, \ldots, a_t\}$ be an alphabet of t distinct letters. Let $A^m = \{f : [m] \to A\}$ denote the set of words (also called points) $f = (f(1), f(2), \ldots, f(m))$ of length m formed by letters from A. A *combinatorial line* in A^m is a collection of words $\{g_1, \ldots, g_t\} \subset A^m$ so that there exists a partition of the coordinates $[m] = F \cup M$ [*F-ixed* and *M-oving*] so that all g_i's agree on the fixed coordinates, and vary over the alphabet on the moving coordinates, that is, for every g_p, g_q

$$\begin{aligned} g_p(i) &= g_q(i) \quad \text{for each } i \in F, \text{ and} \\ g_p(j) &= a_p \quad \text{for each } j \in M. \end{aligned}$$

A density version of the Hales-Jewett theorem [35] was proved by Furstenberg and Katznelson [28] (or see [27] for survey paper):

Theorem 5.2 *For any $\epsilon > 0$ and any alphabet A, $|A| = t$, there exists m_0 so that for $m \geq m_0$, if $\mathcal{S} \subset A^m$ satisfies $|\mathcal{S}| \geq \epsilon t^m$, then $\mathcal{S}$ contains a combinatorial line.*

Proof of upper bound in Theorem 5.1: Put $A = \mathcal{P}([d])$, $t = 2^d = |A|$, and without loss, assume that $m = n/d$ is an integer. Any word from A^m has the form $f = (S_1, \ldots, S_m)$, where for each $i = 1, \ldots, m$, $S_i \subset [d]$. We will use special notation to describe subsets of $[n] = [md]$. For each $i = 1, \ldots, m$, let

$[d]_i = \{1_i, 2_i, \ldots, d_i\}$ be a copy of $[d]$ and write $[n] = [md] = \cup_{i=1}^m [d]_i$, the union of m disjoint copies of $[d]$. Consider the bijection $\psi : A^m \to \mathcal{P}(md)$ defined by

$$\psi((S_1, \ldots, S_m)) = \cup_{i=1}^m \{s_i : s \in S_i\}.$$

For example, with $d = 2$, $m = 6$ and $f = (\emptyset, \{2\}, \{1, 2\}, \emptyset, \{2\}, \{1\})$, we have

$$\psi(f) = \{2_2, 1_3, 2_3, 2_5, 1_6\}.$$

Now fix $\epsilon > 0$ and d and let m be so large that Theorem 5.2 applies, and let $\mathcal{L} = \{f_1, f_2, \ldots, f_t\}$ be a combinatorial line in A^m with fixed coordinates $F \subset [m]$ and moving coordinates $M \subset [m]$. We claim that the family $\psi(\mathcal{L}) = \{\psi(f_j) : j = 1, \ldots, t\}$ is a d-dimensional uniform Boolean algebra.

Let B_0 be the union of those subsets of $[d]_i$'s determined by the fixed coordinates; to be precise, for each $i = 1, \ldots, m$, put $f_1(i) = S_i$ and

$$B_0 = \bigcup_{i \in F} \phi(f_1(i)) = \bigcup_{i \in F} \{s_i : s \in S_i\}.$$

Thus, B_0 can be interpreted as $\psi(f_1)$ provided f_1 is chosen so that $f_1(j) = \emptyset$ (or $\emptyset_j$) for each $j \in M$. For each $j = 1, 2, \ldots, d$, put $B_j = \{j_i : i \in M\}$. Clearly $|B_1| = |B_2| = \ldots = |B_d| = |M|$, and all the B_j's are disjoint. Now, since for any set $J \subset [d]$, there is a word $f \in \mathcal{L}$ so that for every $i \in M$, $f(i) = J$, we see that for each $J \subset [1, d]$,

$$\left(B_0 \cup \bigcup_{j \in J} B_j \right) \in \psi(\mathcal{L}). \quad \Box$$

Proof of lower bound in Theorem 5.1: Essentially, one duplicates the proof of Theorem 4.2, except without mention of the "Hilbert set".

Let $S \subset [\frac{n}{2} - \frac{\sqrt{n}}{2}, \frac{n}{2} + \frac{\sqrt{n}}{2}]$ which contains no arithmetic progression of length 3 and is as large as possible. By Behrend's theorem (Theorem 3.4, using $m = \sqrt{n}$, and then translating the set B by $\frac{n}{2} - \frac{\sqrt{n}}{2} - 1$), we can have $|S| = (n^{1/2})^{1-o(1)} = n^{1/2-o(1)}$. Defining $\mathcal{S} = \{X \subset [n] : |X| \in S\}$,

$$|\mathcal{S}| \geq \binom{n}{n/2 - \sqrt{n}} |S| \sim c \frac{2^n}{\sqrt{n}} |S| = \frac{2^n}{n^{o(1)}}.$$

It is now not difficult to see that $\mathcal{S}$ contains no d-dimensional uniform Boolean algebra. $\Box$

6 Conclusion

It might be reasonable to look for a relationship between lower bounds for $\mathrm{ex}(n, K^{(d)}(2, 2, \ldots, 2))$ and $b(n, d)$—after all, upper bounds are analogous, and one is used to prove the other (see the proof of Theorem 4.3). Efforts to find the putative correspondence have failed as of yet. If one can improve the known upper bound on $\mathrm{ex}(n, K^{(d)}(2,2,\ldots,2))$ for some $d > 2$, then one immediately improves other results in this paper (e.g., Theorem 4.3).

One could ask extremal questions for families of sets forbidding only certain types of substructures in a Boolean algebra, as in union-free families; instead of investigating these here, we refer the reader to [3] and [22] for an introduction and further references. We may consider the work here as one kind of extension of Sperner's Lemma; many other interesting extensions of Sperner's Lemma have been made in similar directions, for example, [12], [23], and [26]. Another perspective may be taken from the point of hypercubes and extremal questions thereof (e.g., see [7]).

To summarize, we list some of the bounds mentioned in this paper. For $d = 2$:

$$\frac{1-o(1)}{2n^{1/2}} \le \frac{\mathrm{ex}(n, K_{2,2})}{n^2} \le \frac{1+o(1)}{2n^{1/2}};$$

$$(1-o(1))r^2 \le h(2,r) \le (1+o(1))r^2;$$

$$(1-o(1))\frac{3}{4}n^{1/2} \le r(2,n) \le (1+o(1))n^{1/2};$$

$$c_1 n^{-1/4} \le \frac{b(n,2)}{2^n} \le c_2 n^{-1/4}.$$

For $d \ge 3$:

$$\frac{c}{n^{\frac{d}{2^d-1}}} \le \frac{\mathrm{ex}(n, K^{(d)}(2,2,\ldots,2))}{n^d} \le \frac{1}{n^{\frac{1}{2^{d-1}}}};$$

$$r^{\frac{2^d-1}{d}(1-o(1))} \le h(d,r) \le (2r)^{2^{d-1}};$$

$$cn^{\frac{1}{2^d}} \le r(d,n) \le n^{\frac{d}{2^d-1}(1+o(1))};$$

$$\frac{1}{n^{\frac{d}{2^{d+1}-2}(1-o(1))}} \le \frac{b(n,d)}{2^n} \le \frac{c}{n^{1/2^d}}.$$

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