On Deuber's partition theorem for (m, p, c)-sets

Dedicated to the memory of Walter Deuber [1942–1999]

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Abstract

In 1973, Deuber published his famous proof of Rado's conjecture regarding partition regular sets. In his proof, he invented structures called (m, p, c)-sets and gave a partition theorem for them based on repeated applications of van der Waerden's theorem on arithmetic progressions. In this paper, we give the complete proof of Deuber's, however with the more recent parameter set proof of his partition result for (m, p, c)-sets. We then adapt this parameter set proof to show that for any k, m, p, c, every K_k-free graph on the positive integers contains an (m, p, c)-set, each of whose rows are independent sets.

1 Introduction

Deuber's proof of Rado's conjecture appeared first in German [3] and various forms have since appeared in surveys (see, e.g.,[4], [16], [18]). The main tool in Deuber's proof is a structure called an (m, p, c)-set. We give a proof of Deuber's theorem to detail the connection between (m, p, c)-sets and partition regular systems. Some parts of the proof given here can be found in the above sources, but this proof follows a slightly different outline given by Deuber [5].

We also examine a parameter set proof for Deuber's partition result on (m, p, c)-sets. Although this proof has been outlined to various degrees in the literature, we will need to closely examine the technique for later use, and so we include the proof in complete detail for reference.

In trying to answer questions regarding independent sets in triangle-free graphs, (m, p, c)-sets arise in a very natural way. As a step in answering such questions, we prove that for any k, m, p, c, every K_k-free graph on the positive integers contains an (m, p, c)-set, each of whose rows are independent sets.

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In the next section, we describe Rado's conjecture regarding partition regular sets and Deuber's solution. In Section 3, we give the notation and major theorems for parameter sets. The parameter set proof for Deuber's partition theorem for (m, p, c)-sets appears in Section 4. In Section 5 we review some of the recent research done regarding arithmetic structure of independent sets in K_k-free graphs on natural numbers. Finally, in Section 6, we prove the statement in the abstract regarding independent rows of (m, p, c)-sets.

We use $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ to denote the natural numbers (not including zero), the integers, and the rationals, respectively. We also use the notation $[a, b] = \{z \in Z : a \leq z \leq b\}$ and occasionally abbreviate $[1, n]$ by $[n]$. A partition of a set X into r parts can be interpreted as a colouring $\Delta : X \longrightarrow [r]$, where a subset of any one colour class $\Delta^{-1}(j)$ is called monochromatic. Both partition and colouring conventions will be used in this paper.

2 Rado's conjecture and Deuber's solution

Definition 2.1 *Let M be an m by n matrix with integer entries. The linear system $M\mathbf{x} = \mathbf{0}$ is partition regular (over $\mathbb{N}$) if for any finite partition $\mathbb{N} = C_1 \cup \cdots \cup C_r$, there exists i and $\mathbf{y} \in (C_i)^n$ so that $M\mathbf{y} = \mathbf{0}$. The matrix M is partition regular if and only if the linear system $M\mathbf{x} = \mathbf{0}$ is partition regular.*

If the matrix M in the above definition were to have rational coefficients, multiplying by an appropriately large number gives an equivalent system with integer coefficients.

A solution to the equation $x + y - 2z = 0$ is a 3-term arithmetic progression of the form $\{x, (x + y)/2, y\}$ and so the equation is partition regular by van der Waerden's theorem [24]. Corresponding to a triple $x, y, x + y$ is the equation $x + y - z = 0$, and so is also partition regular by Schur's theorem [23]. On the other hand, the system $x + y = 3z$ is not partition regular (see, e.g., [9], [16], or [18] for details and more examples).

In his thesis, Rado [20] gave a characterization of partition regular systems in terms of something called the "columns property"—which we now define.

Definition 2.2 *Let $A = [\mathbf{a_1} \ldots \mathbf{a}_n]$ be a matrix with integer entries and column vectors $\mathbf{a}_i$. We say that A has the columns property if there exists a partition $[n] = I_1 \cup \cdots \cup I_m$ so that $\sum_{i \in I_1} \mathbf{a}_i = \mathbf{0}$, and for each $j = 1, 2, \ldots, m-1$, there exist rational numbers $\alpha_{i,j}$ so that*

$$\sum_{i \in I_1 \cup \ldots \cup I_j} \alpha_{i,j} \mathbf{a}_i = \sum_{i \in I_{j+1}} \mathbf{a}_i.$$

Theorem 2.3 (Rado [20]) *If A is a matrix with integer coefficients, the system $A\mathbf{x} = \mathbf{0}$ is partition regular if and only if A has the columns property.*

Rado extended his results in [21] to include other cases, for example, to systems $A\mathbf{x} = \mathbf{b}$ where entries from A are algebraic numbers, entries in $\mathbf{b}$ are complex and solutions are complex. In [16] is a short proof of how partition regularity over $\mathbb{N}$ is equivalent to partition regularity over $\mathbb{Z}\backslash\{0\}$ or $\mathbb{Q}\backslash\{0\}$, however we continue to use only solutions in $\mathbb{N}$. See, for example, [1] and [2] for other extensions of Rado's theorem.

Definition 2.4 *A set $X \subset \mathbb{N}$ is called partition regular (or large) if and only if for every partition regular matrix P, X contains a solution of P, that is, there exists $\mathbf{x} \in X^n$ so that $P\mathbf{x} = \mathbf{0}$.*

Rado conjectured that if a large set was partitioned into finitely many sets then one of these sets was again large. This conjecture remained open for forty years until Deuber [3] proved it in his PhD thesis. His main tool was a structure he invented called an "(m,p,c)-set"; we will outline Deuber's proof of the Rado conjecture, but we invite the interested reader to see the thorough works [4], [16] and [18] for more details.

We remind the reader of the notation $[-p, p] = \{z \in Z : -p \leq z \leq p\}$ which appears in the following definition.

Definition 2.5 *A set of integers M is an (m, p, c)-set if each element of M is positive, and there exists positive integers $x_0, x_1, \ldots, x_m$ so that M is a union $M = R_0(M) \cup R_1(M) \cup \cdots \cup R_m(M)$, where*

$$\begin{aligned}
R_0(M) &= \{cx_0 + \lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_m x_m : \lambda_1, \ldots, \lambda_m \in [-p, p]\}, \\
R_1(M) &= \{cx_1 + \lambda_2 x_2 + \ldots + \lambda_m x_m : \lambda_2, \ldots, \lambda_m \in [-p, p]\}, \\
&\vdots \\
R_{m-1}(M) &= \{cx_{m-1} + \lambda_m x_m : \lambda_m \in [-p, p]\}, \\
R_m(M) &= \{cx_m\}.
\end{aligned}$$

In this case we write $M = (x_0, x_1, \ldots, x_m)_{p,c}$ and we say that $R_k(M)$ is the $(k+1)$-th row of M.

Deuber's original definition had the generators starting with x_1, and so, for example, his notion of an $(1, p, c)$ set was a singleton $\{cx_1\}$, whereas we use generators starting x_0, and so the singleton $\{cx_0\}$ is a $(0, p, c)$-set. From the definition of an (m, p, c)-set with generators $x_0, x_1, \ldots, x_m$, the x_i's must decrease rapidly in size in order that entire set consists of non-negative integers, and for any m, p, c, an (m, p, c)-set exists by first choosing x_m arbitrarily, then x_{m-1} large enough, and so on. The generators of an (m, p, c)-set can be chosen so that any element of the set appears in exactly one row and is determined by precisely one linear combination therein.

The key connection between solutions to partition regular systems and (m, p, c)-sets is that they are "cofinal":

Theorem 2.6 (Deuber) (i) *For every partition regular matrix A, there exists m, p, c so that every (m, p, c)-set contains a solution of $A\mathbf{x} = \mathbf{0}$, and* **(ii)** *for every m, p, c, there exists a partition regular matrix A (usually huge) to that every solution of $A\mathbf{x} = \mathbf{0}$ contains an (m, p, c)-set.*

Proof of (i): By Rado's theorem, A has the columns property, that is, there exists a partition $[n] = I_1 \cup \cdots \cup I_m$ so that $\sum_{i \in I_1} \mathbf{a}_i = \mathbf{0}$, and for each $j = 1, 2, \ldots, m-1$, there exist rationals $\alpha_{i,j}$ so that $\displaystyle\sum_{i \in I_1 \cup \cdots \cup I_j} \alpha_{i,j} \mathbf{a}_i = \sum_{i \in I_{j+1}} \mathbf{a}_i$. Let c be a common multiple of the denominators of the $\alpha_{i,j}$'s, and let $p = \max_{i,j} |c\alpha_{i,j}|$. We claim that m, p, c are as desired. For each $k = 1, \ldots, m$, let A_k be the submatrix of A with columns indexed by $I_1 \cup \cdots \cup I_k$. We will prove by induction on k that every (k, p, c)-set contains a solution of $A_k\mathbf{x} = \mathbf{0}$.

For $k = 0$, the claim holds since $\sum_{i \in I_1} \mathbf{a}_i = \mathbf{0}$ and so any x with all entries identical is a solution.

Suppose that the claim is true for some $k \geq 0$, and examine a $(k+1, p, c)$-set $(x_0, x_1, \ldots, x_{k+1})_{p,c}$. By induction hypothesis, for each $i \in I_1 \cup \cdots \cup I_k$ there is $y_i \in (x_0, x_1, \ldots, x_k)_{p,c}$, so that

$$\sum_{i \in I_1 \cup \cdots \cup I_k} \mathbf{a}_i y_i = 0. \tag{1}$$

By the columns property of A,

$$\sum_{i \in I_1 \cup \cdots \cup I_k} \alpha_{i,k} \mathbf{a}_i = \sum_{i \in I_{k+1}} \mathbf{a}_i.$$

Putting $\beta_i = -c\alpha_{i,k}$ and multiplying by x_{k+1} gives

$$\sum_{i \in I_1 \cup \cdots \cup I_k} \mathbf{a}_i \beta_i x_{k+1} + \sum_{i \in I_{k+1}} \mathbf{a}_i c x_{k+1} = \mathbf{0}. \tag{2}$$

Adding equations (1) and (2) yields

$$\sum_{i \in I_1 \cup \cdots \cup I_k} \mathbf{a}_i (y_i + \beta_i x_{k+1}) + \sum_{i \in I_{k+1}} \mathbf{a}_i c x_{k+1} = \mathbf{0}.$$

Each $y_i \in (x_0, x_1, \ldots, x_k)_{p,c}$ and since $\beta_{i,j} \in [-p, p]$ we have each $y_i + \beta_i x_{k+1} \in (x_0, x_1, \ldots, x_{k+1})_{p,c}$. Also $cx_{k+1} \in (x_0, x_1, \ldots, x_{k+1})_{p,c}$. So we have exhibited a solution of $A_{k+1}\mathbf{x} = \mathbf{0}$ contained in $(x_0, x_1, \ldots, x_{k+1})_{p,c}$, completing the inductive step and hence the proof of (i).

Proof of (ii): The proof is by induction on m. For $m = 0$, the system $cy_1 - cy_2 - y_3 = 0$ is partition regular and its solution contains an element of the form cb, and $\{cb\}$ is a $(0, p, c)$-set.

Now supppose that $A\mathbf{y} = \mathbf{0}$ is a partition regular system whose every solution contains an (m, p, c)-set. Let $A = (a_{ij})$ be $r \times n$ with integer entries, and let $[n] = I_1 \cup \cdots \cup I_\ell$ be the partition guaranteed by the columns property. Examine the $(r + (2p+1)n) \times (2n+1)$ matrix

$$A' = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & 0 & \dots & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & & & \vdots & \vdots \\ a_{r1} & a_{r2} & \dots & a_{rn} & 0 & \dots & \dots & 0 & 0 \\ & & & & & & & & \\ c & 0 & \dots & 0 & -c & 0 & \dots & 0 & p \\ c & 0 & \dots & 0 & -c & 0 & \dots & 0 & p-1 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ c & 0 & \dots & 0 & -c & 0 & \dots & 0 & -p \\ & & & & & & & & \\ 0 & c & \dots & 0 & 0 & -c & \dots & 0 & p \\ 0 & c & \dots & 0 & 0 & -c & \dots & 0 & p-1 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & c & \dots & 0 & 0 & -c & \dots & 0 & -p \\ & & & & & & & & \\ & & \vdots & & & & \vdots & & \\ & & & & & & & & \\ 0 & 0 & \dots & c & 0 & 0 & \dots & -c & p \\ 0 & 0 & \dots & c & 0 & 0 & \dots & -c & p-1 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & c & 0 & 0 & \dots & -c & -p \end{pmatrix}.$$

For each $j = 1, \dots, \ell$, put $I'_j = I_j \cup \{i + n : i \in I_j\}$ and put $I'_{\ell+1} = \{2n+1\}$. With the partition $[2n+1] = I'_1 \cup \cdots \cup I'_{\ell+1}$, A' satisfies the columns property and so is partition regular.

Let $(y'_1, \dots, y'_{2n+1}) \in \mathbb{N}^{2n+1}$ be a solution to $A'\mathbf{y} = \mathbf{0}$ and put $Y = \{y'_1, \dots, y'_{2n+1}\}$. Then by induction hypothesis, $\{y'_1, \dots, y'_n\} \subset Y$ contains an (m, p, c)-set $X = (x_0, \dots, x_m)_{p,c}$ (looking at the first r equations in $A\mathbf{y} = \mathbf{0}$). Setting $y'_{2n+1} = x_{m+1}$, it is not too difficult to verify that the $(m+1, p, c)$-set

$$(x_0, \dots, x_m, x_{m+1})_{p,c} = \{x + \lambda x_{m+1} : x \in X, \lambda \in [-p, p]\} \cup \{cx_{m+1}\}$$

is contained in Y. $\square$

Deuber's classic characterization of partition regular sets now follows directly from definitions and Theorem 2.6.

Corollary 2.7 (Deuber) *$X \subset \mathbb{N}$ is partition regular if and only if for every m, p, c, there is an (m, p, c)-set contained in X.*

Proof: Let X be partition regular, and let m, p, c be given. By (ii), find P so that every solution of $P\mathbf{x} = \mathbf{0}$ contains an (m, p, c)-set. By Definition 2.4, there exists $\mathbf{y} \in X^n$ so that $P\mathbf{y} = \mathbf{0}$, and so $\mathbf{y}$ 'contains' an (m, p, c) set. Now suppose that for every m, p, c, X contains an (m, p, c)-set. Let P be partition regular, and by (i), fix m, p, c so that every (m, p, c)-set contains a solution of $P\mathbf{x} = \mathbf{0}$. By assumption, X contains an (m, p, c)-set and hence a solution of $P\mathbf{x} = \mathbf{0}$. □

Partitioning large sets is thus tantamount to partitioning (m, p, c)-sets, and Deuber found a partition theorem for these, completing the last major link in the proof of Rado's conjecture.

Theorem 2.8 (Deuber) *For every m, p, c, r, there exists n, q, d so that for every (n, q, d)-set X and for every finite partition $X = X_1 \cup \cdots \cup X_r$, there exists i and an (m, p, c)-set Y so that $Y \subset X_i$.*

We defer the proof of Theorem 2.8 to the next section; we will not follow the original but instead give a parameter set version. We first conclude how Theorem 2.8 proves Rado's conjecture.

Theorem 2.9 (Deuber) *For every large X and every finite partition $X = X_1 \cup \cdots \cup X_r$, some X_i is large.*

Proof: Without loss, assume that $r = 2$. In hopes of a contradiction, assume that the theorem is false, that is, there exists a partition $X = X_1 \cup X_2$ so that neither X_1 nor X_2 is large. By Corollary 2.7, there exist $m_1, p_1, c_1, m_2, p_2, c_2$ so that for every (m_1, p_1, c_1)-set M_1 and (m_2, p_2, c_2)-set M_2 both $M_1 \not\subseteq X_1$ and $M_2 \not\subseteq X_2$. Let m, p, c be large enough so that any (m, p, c)-set contains both an (m_1, p_1, c_1)-set and an (m_2, p_2, c_2)-set (we leave this as an exercise to find such m, p, c). By Theorem 2.8 there exists n, q, d and so that for every partition of an (n, q, d)-set $N = N_1 \cup N_2$ one of N_1 or N_2 contains an (m, p, c)-set. Since X is large, it contains an (n, q, d)-set N'. Putting $X_1' = N \cap X_1$ and $X_2' = N \cap X_2$, one of X_1' or X_2' contains a (m, p, c)-set M'. Supposing X_1' contains M', this contradicts $M_1 \not\subseteq X_1$. □

Added remarks: If for each m, p, c an (m, p, c)-set is fixed, the set of sums using at most one element from each (m, p, c)-set was found to be partition regular in [13]. One of Deuber's students, Meike Schröder [22], used a generalized columns property and (m, p, c)-sets to characterize partition regular systems of linear inequalities. Also, Prömel (another student of Deuber) has written a very poignant note [17] in memory of Deuber.

In the next section, we give the necessary information about parameter sets to prove Theorem 2.8 in Section 4 and an adaptation thereof in Section 6.

3 Parameter words

See [19] for a recent survey of results, applications, and notation for parameter words. Let A be a finite alphabet and $\xi_1, \xi_2, \ldots, \xi_m$ be symbols not in A, called *parameters*. As usual, we use $A^n = \{f : n \to A\}$. For $0 \leq m \leq n$, define the set of *m-parameter words* of length n over A by

$$[A]\binom{n}{m} = \left\{ \begin{array}{ll} f : n \to (A\cup & \{\xi_1, \xi_2, \ldots, \xi_m\}) : \forall j \leq m, f^{-1}(\xi_j) \neq \emptyset, \\ & \text{and, } \forall i < j, \min f^{-1}(\xi_i) < \min f^{-1}(\xi_j) \end{array} \right\}.$$

So $[A]\binom{n}{m}$ can be viewed as a set of ordered n-tuples containing each of $\xi_1, \ldots, \xi_m$ at least once, and if $i < j$, the first occurrence of ξ_i must precede the first occurrence of ξ_j. Observe that $A^n = [A]\binom{n}{0}$. For $f \in [A]\binom{n}{m}$ and $g \in [A]\binom{m}{k}$ we define the composition $f \circ g \in [A]\binom{n}{k}$ by

$$f \circ g = \begin{cases} f(i) & \text{if } f(i) \in A, \\ g(j) & \text{if } f(i) = \xi_j. \end{cases}$$

It is straightforward to check that composition of parameter words is associative. The shorthand notation $f \circ [A]\binom{m}{k} = \{f \circ g : g \in [A]\binom{m}{k}\}$ is often useful.

For $f \in [A]\binom{n}{m}$, define the *space* of f, $\mathrm{sp}(f) = f \circ [A]\binom{m}{0}$, to be the set of (0-parameter) words from $[A]\binom{n}{0}$ which are formed by replacing all occurrences of any one parameter with the same element from A throughout the word. The space of a parameter word is often referred to as a *parameter set.* An *m-dimensional (combinatorial) subspace* of A^n (or simply, m-space) is the space of some word in $[A]\binom{n}{m}$. If $f \in [A]\binom{n}{1}$ then we say $\mathrm{sp}(f)$ is a *combinatorial line* in A^n. For example, for some alphabet A, if $f \in [A]\binom{4}{2}$ is given by $f = (a, \xi_1, \xi_2, \xi_1)$, then $\mathrm{sp}(f) = \{(a, x_1, x_2, x_1) : x_1, x_2 \in A\}$ is a 2-dimensional subspace of A^4, and for $g = (\xi_1, b)$, $\mathrm{sp}(f \circ g) = \{(a, x, b, x) : x \in A\}$ is a one-dimensional subspace of $\mathrm{sp}(f)$.

There are two main theorems regarding parameter sets which we require.

Theorem 3.1 (Hales-Jewett [12]) *For each finite alphabet A and positive integers m and r, there exists $n = HJ(|A|, m, r)$ so that for any r-colouring $\Delta : [A]\binom{n}{0} \longrightarrow [r]$ there exists $f \in [A]\binom{n}{m}$ so that $f \circ [A]\binom{m}{0}$ is monochromatic.*

The Hales-Jewett theorem was generalized from colouring of points (0-spaces) to colouring of k-spaces.

Theorem 3.2 (Graham-Rothschild [11]) *Let $m \geq k \geq 0$, $r \geq 1$ and a finite alphabet A be given. Then there exists $n = GR(|A|, k, m, r)$ so that for every r-colouring $\Delta : [A]\binom{n}{k} \longrightarrow [r]$, there exists $f \in [A]\binom{n}{m}$ so that $f \circ [A]\binom{m}{k}$ is monochromatic.*

4 Proof of partition theorem for (m, p, c)-sets

Leeb [14] first suggested a parameter set proof (which repeatedly invokes the Hales-Jewett theorem) for Deuber's partition theorem for (m, p, c)-sets (Theorem 2.8). Prömel [16] developed Leeb's idea and wrote a very elegant parameter set proof for Deuber's theorem. For reference, we repeat it here (with kind permission). The proof is non-trivial, so we give it complete with all details; the technique will be repeated in the next section.

We repeat the statement of Deuber's partition theorem for (m, p, c)-sets, however with colouring notation:

Theorem 2.8 *For every m, p, c, and r, there exists n, q, d so that for any (n, q, d)-set X and any colouring $\Delta : X \longrightarrow [r]$, there exists an (m, p, c)-set Y which is monochromatic.*

Proof: Throughout the proof, r is fixed (in fact, it suffices to prove the theorem for $r = 2$, but the extra generality comes at no extra cost). Assume, also with loss of generality, that $p > c$, since if p is smaller, we carry out the proof for a larger p', then restrict the alphabet of coefficients for the found (m, p', c)-set to find an (m, p, c)-set contained within.

For each $k = 0, 1, \ldots, rm$, let $q_k = c^{2^{rm-k}-1}p$, $d_k = c^{2^{rm-k}}$, and $A_k = [-q_k, q_k]$. Note that $d_{k-1} = d_k^2$ and that $q_{k-1} = q_k d_k$. Now let $rm+1 = n_{rm} < n_{rm-1} < \ldots < n_1 < n_0$, satisfy the recursion

$$n_k - k = \mathrm{HJ}(|A_{k+1}|, n_{k+1} - k, r)$$

for each $k = 0, 1, \ldots, rm - 1$. Put $n = n_0$, $q = q_0$, $d = d_0$, let $X = N_0$ be an (n, q, d)-set, and fix an r-colouring $\Delta : N_0 \longrightarrow [r]$.

The first idea in the proof is to examine the colouring Δ restricted to the first row of $X = N_0$, and in this row, find a monochromatic first row of some (n_1, q_1, d_1)-set $N_1 \subset N_0$. We then look at the second row of N_1, and find a monochromatic second row of some (n_2, q_2, d_2)-set $N_2 \subset N_1$. The first row of N_2 will be contained in the first row of N_1, and hence is also monochromatic (though perhaps of a different colour than the second row). We iterate this procedure recursively until we find a sequence $N_1, N_2, \ldots, N_{rm}$ where $X = N_0 \supset N_1 \supset \ldots \supset N_{rm}$ and each N_k is an (n_k, q_k, d_k)-set so that for each $i = 0, \ldots, k-1$, $R_i(N_k)$ is monochromatic. We now give the details to the recursion.

Recursion step: For some $k \geq 0$, assume we have found an (n_k, q_k, d_k)-set

$$N_k = N_k(w_0, w_1, \ldots, w_{n_k})_{q_k,d_k} \subset N_0$$

where for each $i < k$, $R_i(N_k)$ is monochromatic with respect to Δ (note that for $k = 0$, this is a vacuous assumption). Furthermore assume that for each $i = 0, 1, \ldots, n_k$, we have $w_i = \sum_{j \in I_i} \beta_{i,j} x_j$ where each $\beta_{i,j} \in [-q_0, q_0]$, where the I_j's are disjoint, and $\min(I_j) < \min(I_{j'})$ whenever $j < j'$ (note that in the case $k = 0$, each $I_j = \{j\}$, and so this assumption is again trivial).

The colouring Δ restricted to $R_k(N_k)$ induces another colouring $\Delta_{k+1} : [A_{k+1}]\binom{n_k-k}{0} \longrightarrow [r]$ of each $f = (f(k+1), \ldots, f(n_k)) \in [A_{k+1}]\binom{n_k-k}{0}$ defined by

$$\Delta_{k+1}(f) = \Delta(d_k w_k + d_{k+1} f(k+1) w_{k+1} + \ldots + d_{k+1} f(n_k) w_{n_k}),$$

where for each $j = k+1, \ldots, n_k$, $f(j) \in [-q_{k+1}, q_{k+1}] = A_{k+1}$, thus $d_{k+1} f(j) \in [-q_k, q_k] = A_k$, and so Δ_{k+1} is a partial colouring of the $(k+1)$-th row of N_k. By the choice of n_k, there exists $h = (h(k+1), \ldots, h(n_k)) \in [A_{k+1}]\binom{n_k-k}{n_{k+1}-k}$, so that $\mathrm{sp}(h)$ is monochromatic. Fix such an h with parameters $\lambda_{k+1}, \ldots, \lambda_{n_{k+1}}$; name the set of fixed coordinates $C_k = \{i : h(i) \in A_{k+1}\}$, and the sets of moving coordinates by $C_{k+1} = \{i : h(i) = \lambda_{k+1}\}$, $\ldots$, $C_{n_{k+1}} = \{i : h(i) = \lambda_{n_{k+1}}\}$. By the notation used for parameter sets, we have $k < \min(C_{k+1}) < \min(C_{k+2}) < \ldots < \min(C_{n_{k+1}})$ and that all the C_i's are pairwise disjoint.

Translating this, there exists $S_k \subset R_k(N_k)$, of the form

$$S_k = \left\{ \begin{array}{r} d_k w_k + d_{k+1} \sum_{i \in C_k} h(i) w_i + \lambda_{k+1} \sum_{i \in C_{k+1}} d_{k+1} w_i + \ldots + \lambda_{n_{k+1}} \sum_{i \in C_{n_{k+1}}} d_{k+1} w_i \\ : \lambda_{k+1}, \ldots, \lambda_{n_{k+1}} \in A_{k+1} \end{array} \right\}$$

$$= \left\{ \begin{array}{r} d_{k+1} \left(d_{k+1} w_k + \sum_{i \in C_k} h(i) w_i \right) + \lambda_{k+1} \sum_{i \in C_{k+1}} d_{k+1} w_i + \ldots + \lambda_{n_{k+1}} \sum_{i \in C_{n_{k+1}}} d_{k+1} w_i \\ : \lambda_{k+1}, \ldots, \lambda_{n_{k+1}} \in A_{k+1} \end{array} \right\}$$

which is monochromatic.

Set $z_k = d_{k+1} w_k + \sum_{i \in C_k} h(i) w_i$ and for each $j = k+1, \ldots, n_{k+1}$, put $z_j = \sum_{i \in C_j} d_{k+1} w_i$. Define

$$N_{k+1} = N_{k+1}(d_{k+1} w_0, d_{k+1} w_1, \ldots, d_{k+1} w_{k-1}, z_k, \ldots, z_{n_{k+1}})_{q_{k+1}, d_{k+1}},$$

which we claim is an $(n_{k+1}, q_{k+1}, d_{k+1})$-set. To see that N_{k+1} is indeed an $(n_{k+1}, q_{k+1}, d_{k+1})$-set, we need to show that N_{k+1} consists of non-negative integers. For this it suffices to show that $N_{k+1} \subset N_k$.

Claim: $N_{k+1} \subset N_k$.

Proof of Claim: By construction, $R_k(N_{k+1}) = S_k \subset R_k(N_k)$ and so $R_k(N_{k+1})$

is a (monochromatic) subset of $N_k \subset N_0$. For each $j = 0, \ldots, k-1$,

$$\begin{aligned}
& R_j(N_{k+1}) \\
&\quad = \quad R_j\left(N_{k+1}(d_{k+1}w_0, d_{k+1}w_1, \ldots, d_{k+1}w_{k-1}, z_k, \ldots, z_{n_{k+1}})_{q_{k+1},d_{k+1}}\right) \\
&\quad = \quad \left\{ d_{k+1}(d_{k+1}w_j) + \sum_{i=j+1}^{k-1} \lambda_i d_{k+1} w_i + \sum_{i=k}^{n_{k+1}} \lambda_i z_i : \lambda_{j+1}, \ldots, \lambda_{n_{k+1}} \in A_{k+1} \right\} \\
&\quad \subset \quad \left\{ d_k w_j + \sum_{i=j+1}^{k-1} \lambda_i^{'} w_i + \sum_{i=k}^{n_{k+1}} \lambda_i^{'} z_i : \lambda_{j+1}^{'}, \ldots, \lambda_{nk+1}^{'} \in A_k \right\} \\
&\quad = \quad R_j\left(N_k(w_0, w_1, \ldots, w_{n_k})_{q_k,d_k}\right),
\end{aligned}$$

since for each $i = j+1, \ldots, k-1$, $\lambda_i \in A_{k+1}$ implies that $\lambda_i d_{k+1} = \lambda_i^{'} \in A_k$.

To complete the proof of the claim that $N_{k+1} \subset N_k$, it remains to see that for any $j > 0$, we have $R_{k+j}(N_{k+1}) \subset N_k$. To check this, fix $j > 0$ and let $x \in R_{k+j}(N_k)$ be arbitrary, given by

$$x = d_{k+1} z_{k+j} + \xi_{k+j+1} z_{k+j+1} + \ldots + \xi_{n_{k+1}} z_{n_{k+1}}$$

where each $\xi_i \in A_{k+1}$ is fixed. Then

$$x \quad = \quad d_{k+1} \sum_{i \in C_{k+j}} d_{k+1} w_i \quad + \quad \xi_{k+j+1} d_{k+1} \sum_{i \in C_{k+j+1}} w_i \quad + \ldots + \quad \xi_{n_{k+1}} d_{k+1} \sum_{i \in C_{n_k}} w_i$$

which is an element of $R_{\min(C_{k+j})}(N_k) \subset N_k$, because $\min(C_{k+j})$ is indeed the smallest element of $\cup_{i=k+j}^{n_{k+1}} C_i$. This concludes the check that $R_{k+j}(N_{k+1}) \subset N_k$, and hence the proof of the claim that $N_{k+1} \subset N_k$.

Claim: Each of the rows $R_0(N_{k+1}), \ldots, R_k(N_{k+1})$ are monochromatic.
Proof of Claim: We have shown that for each $j = 0, \ldots, k-1$, $R_j(N_{k+1}) \subset R_j(N_k)$ holds and since each for $j < k$, $R_j(N_k)$ is monochromatic by assumption, so is each $R_j(N_{k+1})$. By construction, $S_k = R_k(N_{k+1})$ is monochromatic, finishing the proof of the claim.
End of recursion step.

Observe that $q_{rm} = p$, $d_{rm} = c$. Using the above recursion, find successively $N_1 \supset N_2 \supset \ldots \supset N_{rm}$, where,

$$N_{rm} = N_{rm}(y_0, y_1, \ldots, y_{rm})_{p,c} \subset N_0$$

is an (rm, p, c)-set with each of the rows $R_0(N_{rm}), R_1(N_{rm}), \ldots, R_{rm-1}(N_{rm})$ monochromatic under Δ. Since $R_{rm}(N_{rm}) = \{cy_{rm}\}$ is a single element, it is also monochromatic. Hence all $rm+1$ rows of N_{rm} are monochromatic. By the pigeonhole principle, there exists a subfamily of $m+1$ rows $R_{i_0}(N_{rm}), R_{i_1}(N_{rm})$,

..., $R_{i_m}(N_{rm})$, all of which receive the same colour. In this case, since the leading generator in each $R_i(N_{rm})$ is y_i, then

$$M = M(y_{i_0}, y_{i_1}, \ldots, y_{i_m})_{p,c} \subset N_{rm}$$

is the desired monochromatic (m, p, c)-set. $\square$

5 Independent sets in K_k-free graphs

For a set S let $[S]^2$ denote the set of unordered distinct pairs of elements of S. Let $G = (V(G), E(G))$ be a graph on vertex set $V(G)$ with edges $E(G) \subset [V(G)]^2$. A subset $X \subset V(G)$ is said to be independent in G if $[X]^2 \cap E(G) = \emptyset$. The complete graph on n vertices is denoted by K_n.

Erdős [7] and Hajnal (see [8]) inspired questions about arithmetic structure on independent sets in K_k-free graphs on $\mathbb{N}$. For example, in a triangle-free graph on $\mathbb{N}$, can one always find an independent set of the form $x, y, x + y$? Luczak, Rodl and Schoen [15] proved that in any K_k free graph on $\mathbb{N}$, arbitrarily large finite sum-sets could be found independent. In [6] certain infinite analogues were discussed. In [10], arbitrarily large multiple arithmetic progressions could be found independent and arithmetic progressions together with their difference, also independent. Many, including Deuber, Leader, Prömel, Rodl, and this author have been actively pursuing the question: If G is a K_k-free graph on $\mathbb{N}$, can one solve any partition regular system of equations in an independent set, that is, are there arbitrarily large independent (m, p, c)-sets? In the next section appears only one step to answering this question in the affirmative. We now look at some of the techniques in [10], since one of these is a notion central to the proof in the next section.

The following guarantees independent lines in a Hales-Jewett cube whose elements are vertices of a K_k-free graph.

Theorem 5.1 ([10]) *Given k and alphabet $A = \{a_1, a_2, \ldots, a_l\}$, with $l \geq 2$ letters, there exists n so that for every K_k-free graph $G = (A^n, E(G))$, there exists $h \in [A]\binom{n}{1}$ so that $sp(h)$ is independent in G.*

Just as van der Waerden's theorem follows directly from the Hales-Jewett theorem, the following is a direct consequence of Theorem 5.1.

Corollary 5.2 ([10]) *Given k and ℓ, if G is a K_k-free graph on the positive integers, then there exists a ℓ-term arithmetic progression which is independent in G.*

Generalizing Theorem 5.1 from lines to spaces was done by replacing the alphabet A with A^p, however we give the direct proof here for completeness.

Theorem 5.3 ([10]) *Given k, p, and alphabet A with $|A| = l \geq 2$ letters, there exists $n = IS(|A|, k, p)$ so that for every K_k-free graph G on vertex set A^n, there exists $h \in [A]\binom{n}{p}$ so that $sp(h) = h \circ [A]\binom{n}{0}$ is independent in G.*

Proof: Fix $m \geq k-1$ and put $n = \mathrm{GR}(|A|, p, mp, \binom{l^p}{2} + 1)$. Suppose that G is a graph on A^n which is K_k-free. Let $C = \{\{w, z\} : w \in A^p, z \in A^p, w \neq z\}$ and fix some linear order $\prec$ on these $\binom{l^p}{2}$ pairs. Define a colouring $\Delta : [A]\binom{n}{p} \longrightarrow C \cup \{0\}$ of each $h \in [A]\binom{n}{p}$ according to where edges first occur in $sp(h)$ as follows:

$$\Delta(h) = \begin{cases} \{w, z\} & \text{if } \{w, z\} \text{ is least in } (C, \prec) \text{ so that } (h \circ w, h \circ z) \in E(G), \\ 0 & \text{if } E(G) \cap [\mathrm{sp}(h)]^2 = \emptyset. \end{cases}$$

So $\Delta(h) = 0$ means that no edge occurs in the graph induced by $\mathrm{sp}(h)$. Under this colouring, guaranteed by the choice of n, fix $f \in [A]\binom{n}{mp}$ and $c \in C \cup \{0\}$ so that for each $g \in [A]\binom{mp}{k}$, $\Delta(f \circ g) = c$. We want to show that $c = 0$.

Seeking a contradiction, suppose that $c \neq 0$. Then $c = \{w, z\}$ for some fixed words $w, z \in A^p$, that is, for every $h \in f \circ [A]\binom{mp}{p}$, $\Delta(h) = \{w, z\}$. We now need to describe some particular parameter words $g_{i,j}$, and to do this simply, we give two local definitions. Let $\xi \in [\emptyset]\binom{p}{p}$ be an abbreviation for the word $(\xi_1, \ldots, \xi_p)$. We also use the notation xy to mean simple concatenation of two parameter words, x and y; for example, $ww = (w(1), \ldots, w(p), w(1), \ldots, w(p))$ and $\xi\xi = (\xi_1, \ldots, \xi_p, \xi_1, \ldots, \xi_p)$. Now for each $0 \leq i \leq j \leq m$, examine the word

$$g_{ij} = \underbrace{ww \cdots w}_{i \text{ copies}} \underbrace{\xi\xi \cdots \xi}_{j-i} \underbrace{zz \cdots z}_{m-j}.$$

Notice that when $i \neq j$, $g_{i,j} \in [A]\binom{mp}{p}$, and for each i, $g_{i,i} \in [A]\binom{mp}{0}$. Put $h_{i,j} = f \circ g_{i,j}$. We claim that the $m+1$ vertices of the form $h_{i,i}$, $i = 0, 1, \ldots, m$ determine a complete graph. To see this, observe that for every $i < j$,

$$h_{i,i} = h_{i,j} \circ z \in \mathrm{sp}(h_{i,j})$$

$$h_{j,j} = h_{i,j} \circ w \in \mathrm{sp}(h_{i,j}),$$

and since $\Delta(h_{i,j}) = \{w, z\}$, $(h_{i,i}, h_{j,j}) \in E(G)$. Thus we obtain a complete graph on $m+1$ vertices, a contradiction since $m + 1 \geq k$.

Hence $c = 0$, and so there exists $g \in [A]\binom{mp}{p}$ so that for $h = f \circ g$, $\Delta(h) = 0$, that is, $\mathrm{sp}(h) = h \circ [A]\binom{p}{0}$ is independent. In fact, for the case $p = 1$, for all $g \in [A]\binom{m}{1}$, $\mathrm{sp}(f \circ g)$ is an independent set. $\square$

For integers s and ℓ, a s-fold arithmetic progression of length l is a set of integers of the form $\{a_0 + \lambda_1 a_1 + \lambda_2 a_2 + \ldots + \lambda_s a_s : \lambda_1, \ldots, \lambda_s \in [0, l-1]\}$. Just as Corollary 5.2 was an easy consequence of Theorem 5.1, so too the next result follows from Theorem 5.3.

Corollary 5.4 ([10]) *For every k, $s \geq 1$, and $l \geq 2$, there exists an n so that for any K_k-free graph G on $[1, l^n]$, there is an s-fold arithmetic progression of length l which forms an independent set in G.*

6 Independent rows of (m, p, c)-sets

Theorem 6.1 *For every k, m, p, c there exists n, q, d so that for any (n, q, d)-set $N = N(x_0, \ldots, x_n)_{q,d}$ and any K_k-free graph G on N, there exists an (m, p, c)-set $M = M(y_0, \ldots, y_m)_{p,c}$ so that*

(1) each row of M is an independent set in G, and

(2) there are disjoint sets $J_0, J_1, \ldots, J_m$ with $\min(J_0) < \min(J_1) < \cdots < \min(J_m)$ and constants $\beta_{i,j}$ so that $y_0 = \sum_{j \in J_0} \beta_{0,j} x_j, \ldots, y_m = \sum_{j \in J_m} \beta_{n,j} x_j$; so for $i < j$, $R_i(M)$ is contained in an earlier row of N than is $R_j(M)$.

Proof: The proof is similar to that of the parameter set proof for Deuber's partition theorem for (m, p, c)-sets, except that rather than using the Hales-Jewett theorem at each step, we invoke Theorem 5.3.

Assume, without loss of generality, that $p > c$. For each $s = 0, 1, \ldots, m$, define $q_s = c^{2^{m-s}-1} p$, $d_s = c^{2^{m-s}}$ and let $A_s = [-q_s, q_s]$. Note that for each $s \geq 1$, $d_{s-1} = d_s^2$, $q_{s-1} = q_s d_s$, and $A_s \subset A_{s-1}$. Using the notation in Theorem 5.3, let $m + 1 = n_m < n_{m-1} < \ldots < n_1 < n_0$, satisfy the recursion

$$n_s - s = \mathrm{IS}(|A_{s+1}|, k, n_{s+1} - s)$$

for each $s = 0, 1, \ldots, m - 1$. Put $n = n_0$, $q = q_0$, $d = d_0$ and let $N = N_0 = N_0(x_0, x_1, \ldots, x_{n_0})_{q,d}$ be an (n, q, d)-set, and let G be a K_k-free graph with vertex set N.

Recursion step: For some $s \geq 0$, assume we have found an (n_s, q_s, d_s)-set

$$N_s = N_s(w_0, w_1, \ldots, w_{n_s})_{q_s, d_s} \subset N_0$$

where for each $i < s$, $R_i(N_s)$ is an independent set (note that for $s = 0$, this is a vacuous assumption). Furthermore, assume that for each $i = 0, 1, \ldots, n_s$, we have $w_i = \sum_{j \in I_i} \beta_{i,j} x_j$ where each $\beta_{i,j} \in [-q_0, q_0]$, then the I_j's are disjoint, and $\min(I_j) < \min(I_{j'})$ whenever $j < j'$ (note that in the case $s = 0$, each $w_j = x_j$ where $I_j = \{j\}$, and so this assumption is again trivial).

Examine the function $\alpha_{s+1} : [A_{s+1}]\binom{n_s - s}{0} \longrightarrow R_s(N_s)$ defined by

$$\alpha_{s+1}(f(s+1), \ldots, f(n_s)) = d_s w_s + d_{s+1} f(s+1) w_{s+1} + \ldots + d_{s+1} f(n_s) w_{n_s},$$

where for each $j = s + 1, \ldots, n_s$, $f(j) \in [-q_{s+1}, q_{s+1}] = A_{s+1}$, and thus $d_{s+1} f(j) \in [-q_s, q_s] = A_s$. By choice of n_s, there exists $h \in [A_{s+1}]\binom{n_s - s}{n_{s+1} - s}$, so that $\mathrm{sp}(h)$ is independent. Fix such an h with parameters $\lambda_{s+1}, \ldots, \lambda_{n_{s+1}}$ and put $C_s = \{i : h(i) \in A_{s+1}\}$, $C_{s+1} = \{i : h(i) = \lambda_{s+1}\}$, ..., $C_{n_{s+1}} = \{i : h(i) = \lambda_{n_{s+1}}\}$. Note that $\min(C_{s+1}) < \min(C_{s+2}) < \ldots < \min(C_{n_{s+1}})$ and that the C_i's are pairwise disjoint. This means that there exists $S_s \subset R_s(N_s)$, of the form

$$S_s = \left\{ \begin{array}{r} d_s w_s + d_{s+1}\sum_{i\in C_s} h(i)w_i + \lambda_{s+1}\sum_{i\in C_{s+1}} d_{s+1}w_i + \ldots + \lambda_{n_{s+1}}\sum_{i\in C_{n_{s+1}}} d_{s+1}w_i \\ : \lambda_{s+1},\ldots,\lambda_{n_{s+1}} \in A_{s+1} \end{array} \right\}$$

$$= \left\{ \begin{array}{r} d_{s+1}\left(d_{s+1}w_s + \sum_{i\in C_s} h(i)w_i\right) + \lambda_{s+1}\sum_{i\in C_{s+1}} d_{s+1}w_i + \ldots + \lambda_{n_{s+1}}\sum_{i\in C_{n_{s+1}}} d_{s+1}w_i \\ : \lambda_{s+1},\ldots,\lambda_{n_{s+1}} \in A_{s+1} \end{array} \right\}$$

which is independent.

Set

$$z_s = d_{s+1}w_s + \sum_{i\in C_s} h(i)w_i$$

and for each $j = s+1,\ldots,n_{s+1}$, put $z_j = \sum_{i\in C_j} d_{s+1}w_i$. Define

$$N_{s+1} = N_{s+1}(d_{s+1}w_0, d_{s+1}w_1, \ldots, d_{s+1}w_{s-1}, z_s, z_{s+1}, \ldots, z_{n_{s+1}})_{q_{s+1},d_{s+1}},$$

To see that N_{s+1} is indeed an $(n_{s+1}, q_{s+1}, d_{s+1})$-set, we need to show that N_{s+1} consists of non-negative integers, for which it suffices to show that $N_{s+1} \subset N_s$. By construction, $R_s(N_{s+1}) = S_s \subset R_s(N_s)$ and so $R_s(N_{s+1})$ is a (independent) subset of N_s. It is also easy to see that for each $j = 0,\ldots,s-1$, $R_j(N_{s+1}) \subseteq R_j(N_s)$. Now for some $j \geq 1$, let $x \in R_{s+j}(N_{s+1})$ be arbitrary and given by

$$x = d_{s+1}z_{s+j} + \eta_{s+j+1}z_{s+j+1} + \ldots + \eta_{n_{s+1}}z_{n_{s+1}}$$

where each $\eta_i \in A_{s+1}$ is fixed. Then

$$x = d_{s+1}\sum_{i\in C_{s+j}} d_{s+1}w_i + \eta_{s+j+1}d_{s+1}\sum_{i\in C_{s+j+1}} w_i + \ldots + \eta_{n_{s+1}}d_{s+1}\sum_{i\in C_{n_s}} w_i$$

and since $\min(C_{s+j})$ is indeed the smallest element of $\cup_{i=s+j}^{n_{s+1}} C_i$, we have

$$x \in R_{\min(C_{s+j})}(N_s) \subset N_s.$$

Hence $R_{s+j}(N_{s+1}) \subseteq N_s$, and so $N_{s+1} \subset N_s$.

We have noted that for each $j = 0,\ldots,s-1$, $R_j(N_{s+1}) \subset R_j(N_s)$ holds, and since each for $j < s$, $R_j(N_s)$ is an independent set by assumption, so is each $R_j(N_{s+1})$. By construction, $S_s = R_s(N_{s+1})$ is an independent set, and so each of the rows $R_0(N_{s+1}),\ldots,R_s(N_{s+1})$ are independent sets. Lastly, since the sets of coordinates C_i were disjoint and the sets I_i were disjoint, the generators of

N_{s+1} are linear combinations over disjoint collections of x_j's. This completes the recursion step.

Observe that for $s = m$, we have $q_s = p$, $d_s = c$. Using the above recursion, find successively $N_1 \supset N_2 \supset \ldots \supset N_m = M$, where $M = (z_0, z_1, \ldots, z_m)_{p,c} \subset N_0$ is an (m, p, c)-set and each of the first m rows $R_0(M), R_1(M), \ldots, R_{m-1}(M)$ are independent sets. We also observe that $R_m(M)$ consists of a single element, and so is also an independent set. Hence all $m + 1$ rows of M are independent sets. □

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