An alternate proof of Szemerédi's cube lemma using extremal hypergraphs

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Abstract

A collection H of integers is called an *affine d-cube* if there exist $d+1$ positive integers $x_0, x_1, \ldots, x_d$ so that

$$H = \left\{ x_0 + \sum_{i \in I} x_i : I \subseteq \{1, 2, \ldots, d\} \right\}.$$

In 1969, Szemerédi found a density result for affine cubes, namely, that for any positive integer d, there exists a constant c so that if $A \subseteq \{1, 2, \ldots, n\}$ and $|A| \geq cn^{1-\frac{1}{2^d}}$, then A contains an affine d-cube. Using extremal hypergraphs, we offer an entirely different proof of this fact (though with worse constant) which also yields a slightly stronger statement.

1 Introduction

For any positive integer m we use the notation $[m] = [1, m] = \{1, 2, \ldots, m\}$.

Definition 1.1 *A collection H of integers is called a d-*dimensional affine cube*, or more simply, an* affine d-cube *if there exist $d+1$ positive integers $x_0, x_1, \ldots, x_d$ so that*

$$H = \left\{ x_0 + \sum_{i \in I} x_i : I \subseteq [d] \right\}. \tag{1}$$

If all sums in (1) are distinct, then $|H| = 2^d$, and thus H is saturated, or replete.

*Partially supported by DFB-SFB 343, Diskrete Structuren in der Mathematik, Bielefeld

For a set X we use the standard notations $\mathcal{P}(X) = \{Y : Y \subseteq X\}$ and $[X]^s = \{S \subset X : |S| = s\}$. It will often be convenient to use $X = [n]$. An arithmetic progression of length k will be referred to as an AP_k.

If an affine d-cube H is generated by $x_0, x_1, \ldots, x_d$, then we write $H = H(x_0, x_1, \ldots, x_d)$. For example, $H(1, 1, 1) = \{1, 2, 3\}$, while a replete affine 2-cube is $H(1, 3, 9) = \{1, 4, 10, 13\}$. Note that $H(x_0, x_1, \ldots, x_d)$ may differ from, say, $H(x_1, x_0, \ldots, x_d)$.

In 1892, Hilbert [11] proved a non-trivial Ramsey-type theorem for affine cubes. The following is a finite version thereof, following from Hilbert's original result by compactness.

Theorem 1.2 (Hilbert [11]) *For every r, d, there exists a least number $h(d, r)$ so that for every coloring*

$$\chi : [h(d, r)] \to [r],$$

there exists an affine d-cube monochromatic under χ.

The original statement of the theorem asserted that if one colored the positive integers with finitely many colors, then one color class contained a (monochromatic) d-cube. Theorem 1.2 follows from van der Waerden's Theorem [17] since an AP_{d+1} given by $\{a, a+k, a+2k, \ldots, a+dk\}$ is the affine d-cube $H(a, k, k, \ldots, k)$. See [1] for further references.

In 1969, Szemerédi [15] proved that if a set A of positive integers has positive upper density, (i.e., $\overline{\lim_{n\to\infty}} |A \cap [1, n]| > 0$) then A contains an AP_4. In that same paper[15] (Lemma $p(\delta, l)$, p. 93), Szemerédi gave a density version of Theorem 1.2; Szemerédi's so-called "cube lemma" can also be found in [8]. Explicit bounds which follow directly from Szemeré'di's argument are also mentioned in [6].

Theorem 1.3 (Szemerédi [15]) *For each d, there exists a constant c so that for sufficiently large n, if $A \subset [n]$ and*

$$|A| \geq cn^{1-\frac{1}{2^d}}, \tag{2}$$

then A contains an affine d-cube.

Szemerédi's 1975 paper [16] proving a density result for arbitrarily long arithmetic progressions has occasionally been errantly attributed as the source of the cube lemma. Since the 1975 paper is so complicated, such claims were hard to verify. Perhaps as a result, typographical misprints such as the "2" in the exponent of (2) printed as "e" have survived—but with positive effect. One such misprint led the present authors to find the following following slight improvement, but with a more complicated proof:

Theorem 1.4 *For each integer $d \geq 2$ there exists an n_0 so that for every $n \geq n_0$, if $A \subset [1, n]$ satisfies*

$$|A| \geq (1+\delta)2^{d-3/2}n^{1-1/2^d} \tag{3}$$

then A contains $(1-o(1))n^2/2^{d-1}$ replete affine d-cubes.

We make a few comments on some subtle differences between our density result and Szemerédi's. The results given in this paper guarantee not only one d-cube as does Szemerèdi's, but on the order of n^2 of them. It is not clear what the precise order of magnitude should be, since by a result in [4] (Theorem 2, with $k = 2^{d+1}$ and $l = 2^{d+1} - (d+1)$), for a fixed $\epsilon \in (0, 1]$ one can conclude that if $A \subset [1, n]$ satisfies $|A| \geq \epsilon n$, then A contains on the order of n^{d+1} affine d-cubes, that is, a positive fraction of all such cubes. We also observe that our results guarantee *replete* affine cubes.

With Theorem 1.3 in hand, we observe that the upper bound for $h(d, r)$ appearing in [1] is now immediately improved by applying the pigeonhole principle. The details have been noted in [9] and [10].

2 The density result; Proof of Theorem 1.4

The basic idea used here for proving Theorem 1.4 is to first give an extremal result for a very special class of partite hypergraphs, then to interpret hyperedges thereof as numbers.

2.1 Extremal results for d-partite hypergraphs

A *d-uniform hypergraph* is a pair $G = (V, \mathcal{E}) = (V(G), \mathcal{E}(G))$, with vertex set V and hyperedge set $\mathcal{E} \subset [V]^d$. Note that by this definition, each d-set from V may occur only once as a hyperedge. For pairwise disjoint sets $X_1, X_2, \ldots, X_d$, let

$$G = (X_1, X_2, \ldots, X_d, \mathcal{E}(G))$$

denote a d-partite d-uniform hypergraph on vertex set $V(G) = \cup_{i=1}^d X_i$ and edge set $\mathcal{E}(G) \subseteq [V(G)]^d$, where for each $E \in \mathcal{E}(G)$ and each $i = 1, \ldots, d$, $|E \cap X_i| = 1$ holds; the sets $X_1, \ldots, X_d$ will be called *partite* sets.

Let $K^{(d)}(n_1, n_2, \ldots, n_d)$ denote the complete d-partite d-uniform hypergraph on $\sum_{i=1}^{d} n_i$ vertices, partitioned into sets of sizes $n_1, n_2 \ldots, n_d$, and having $\prod_{i=1}^{d} n_i$ edges, each edge containing exactly one vertex from each partite set. [The "(d)" in the notation is not redundant; it depicts the number of vertices per hyperedge.] The complete d-partite d-uniform hypergraph with two vertices in each partite set will be denoted by $K^{(d)}(2, 2, \ldots, 2)$. For any d-uniform hypergraph H, the maximum number of d-hyperedges in any H-free hypergraph on n vertices is denoted by $\mathrm{ex}(n, H)$.

In 1964 Erdős [2] (cf. equation (4.2) in [5]) showed that there exists a universal constant $c < 1$ so that for each d and sufficiently large n,

$$\mathrm{ex}(n, K^{(d)}(2, 2, \ldots, 2)) \leq cn^{d - \frac{1}{2^{d-1}}}$$

holds; for $d > 2$, at present there is still a wide gap between the lower and upper bounds for $\mathrm{ex}(n, K^{(d)}(2, 2, \ldots, 2))$ (see [9] for discussion).

Instead of considering the maximum number of of d-hyperedges in *any* $K^{(d)}(2, 2, \ldots, 2)$-free hypergraph G, we will obtain a best possible (up to constant a multiple) extremal result when such G's are chosen only from a very special class of d-uniform hypergraphs, namely those which are d-partite and have a particular shape.

Definition 2.1 *For positive integers $d \geq 2$ and a, let $\mathcal{G}(d, a)$ be the class of d-partite d-uniform hypergraphs $G = (X_1, X_2, \ldots, X_d, \mathcal{E}(G))$ which satisfy $|X_1| = a$ and for each $i = 2, \ldots, d$, $|X_i| = a^{2^{i-2}}$ (hence $|X_2| = a$ as well). Define $p(1, a) = \binom{a}{2}$, and for $d \geq 2$, define*

$$p(d, a) = \binom{a}{2} \prod_{i=2}^{d} \binom{a^{2^{i-2}}}{2},$$

the number of ways to pick two vertices from each partite set in a graph from $\mathcal{G}(d, a)$.

Lemma 2.2 *Let $\delta > 0$ be a real number and $a \geq \max\{2, 1/(8\delta^2)\}$. For each integer $d \geq 2$ and any $G \in \mathcal{G}(d, a)$, if*

$$|\mathcal{E}(G)| \geq (1 + \delta) 2^{d-3/2} \cdot a^{(2^d - 1)/2}$$

then G contains $p(d-1, a) = (1 - o(1))(a^{2^{d-1}}/2^{d-1})$ (as $a \to \infty$) copies of $K^{(d)}(2, \ldots, 2)$. Up to a multiplicative constant, this result is sharp, that is, there exists a constant c so that for every $d \geq 2$, and sufficiently large a there exists a $G \in \mathcal{G}(d, a)$ with $|\mathcal{E}(G)| = ca^{(2^d - 1)/2}$ which is $K^{(d)}(2, \ldots, 2)$-free.

It may be interesting to note that a conclusion similar to Lemma 2.2 follows from [7] (Lemma 5.6), however such is not quite suitable for our purpose.

Proof of Lemma 2.2: Fix $\delta > 0$ and let $G = (X_1, X_2, \mathcal{E}(G)) \in \mathcal{G}(2, a)$ for some fixed $a \geq \max\{2, 1/(8\delta^2)\}$. The proof is by induction on d.

Let $d = 2$, and so $|X_1| = |X_2| = a$. Assume that $|\mathcal{E}(G)| \geq (1+\delta)\sqrt{2}a^{3/2}$. For any $x \in V(G) = X_1 \cup X_2$, $\deg(x)$ denotes the degree of x in G; for $i, j \in V(G)$, let $\deg(i, j)$ denote the pairwise degree of i and j, that is, the number of common neighbors to i and j. The following counting argument uses well known techniques (cf. [2]); similar counting will be used in the inductive step. For Jensen's inequality for convex functions see, for example, [13]. The number of copies of $K^{(2)}(2, 2) \cong C_4$ in G is

$$\begin{aligned}
\sum_{\{i,j\}\in[X_1]^2} \binom{\deg(i,j)}{2} &\geq \binom{a}{2}\binom{\sum_{[X_1]^2}\deg(i,j)/\binom{|X_1|}{2}}{2} \quad \text{(by Jensen's inequality)},\\
&= \binom{a}{2}\binom{\frac{1}{\binom{a}{2}}\sum_{x\in X_2}\binom{\deg(x)}{2}}{2} \quad \text{(counting from } X_2\text{)}\\
&\geq \binom{a}{2}\binom{\frac{a}{\binom{a}{2}}\binom{|\mathcal{E}(G)|/|X_2|}{2}}{2} \quad \text{(again by convexity)},\\
&= \binom{a}{2}\binom{\frac{2}{a-1}\binom{(1+\delta)\sqrt{2a}}{2}}{2}\\
&\geq \binom{a}{2},
\end{aligned}$$

where the last line follows because $a \geq \max\{2, 1/(8\delta^2)\}$.

Now assume that the theorem is true for some $d \geq 2$ with $\delta > 0$ and sufficiently large a; we will show that the theorem holds for $d + 1$ and the same δ.

Let $G = (X_1, \ldots, X_d, X_{d+1}, \mathcal{E}(G)) \in \mathcal{G}(d + 1, a)$ with

$$|\mathcal{E}(G)| \geq (1 + \delta)2^{d-1/2} \cdot a^{2^d-1/2}.$$

We need to show that G contains

$$p(d, a) = \binom{a}{2}\binom{a}{2}\binom{a^2}{2}\cdots\binom{a^{2^{d-2}}}{2}$$

copies of $K^{(d+1)}(2,2,\ldots,2)$. To simplify the calculations, let us introduce some notation.

Let $\mathcal{H}$ be the family of $2d$-sets formed by taking two vertices from each of $X_1,\ldots,X_d$ (not X_{d+1}); note that $|\mathcal{H}| = p(d,a)$. For each $H \in \mathcal{H}$, let

$$d(H) = |\{x \in X_{d+1} : H \cup \{x\} \text{ induces a copy of } K^{(d+1)}(2,2\ldots,2,1)\}|,$$

and for each $x \in X_{d+1}$, define

$$h(x) = |\{H \in \mathcal{H} : H \cup \{x\} \text{ induces a copy of } K^{(d+1)}(2,2\ldots,2,1)\}|.$$

As with ordinary graphs, for $x \in V(G)$, let $\deg(x) = |\{E \in \mathcal{E}(G) : x \in E\}|$.

Before we start the next sequence of inequalities, let us make an observation justifying the third line in the sequence. For any fixed $x \in X_{d+1}$ and a large enough, partitioning those edges containing x into sets just large enough to apply the induction hypothesis, yielding $p(d-1,a)$ copies of $K^{(d)}(2,\ldots,2)$ for each such set, shows

$$h(x) \geq \left\lfloor \frac{\deg(x)}{(1+\delta)2^{d-3/2}a^{(2^d-1)/2}} \right\rfloor \cdot p(d-1,a). \tag{4}$$

The number of copies of $K^{(d+1)}(2,2,\ldots,2)$ in G is (where now the third line follows from equation (4))

$$\begin{aligned}
\sum_{H\in\mathcal{H}} \binom{d(H)}{2} &\geq |\mathcal{H}| \binom{\frac{1}{|\mathcal{H}|}\sum_{H\in\mathcal{H}} d(H)}{2} \quad \text{(by Jensen's inequality)} \\
&= p(d,a) \binom{\frac{1}{p(d,a)}\sum_{x\in X_{d+1}} h(x)}{2} \quad \text{(counting from } X_{d+1}) \\
&\geq p(d,a) \binom{\frac{1}{p(d,a)}\sum_{x\in X_{d+1}} \left\lfloor \frac{\deg(x)}{(1+\delta)2^{d-3/2}\cdot a^{2^{d-1}-1/2}} \right\rfloor p(d-1,a)}{2} \\
&\geq p(d,a) \binom{\frac{p(d-1,a)}{p(d,a)}\sum_{x\in X_{d+1}} \left[\frac{\deg(x)}{(1+\delta)2^{d-3/2}\cdot a^{2^{d-1}-1/2}} - 1 \right]}{2}
\end{aligned}$$

$$
\begin{aligned}
&= p(d,a)\left(\frac{\frac{1}{\binom{a^{2^{d-2}}}{2}}\left[\frac{\sum_{x\in X_{d+1}}\deg(x)}{(1+\delta)2^{d-3/2}\cdot a^{2^{d-1}-1/2}} - |X_{d+1}|\right]}{2}\right)\\
&= p(d,a)\left(\frac{\frac{1}{\binom{a^{2^{d-2}}}{2}}\left[\frac{|\mathcal{E}(G)|}{(1+\delta)2^{d-3/2}\cdot a^{2^{d-1}-1/2}} - a^{2^{d-1}}\right]}{2}\right)\\
&\geq p(d,a)\left(\frac{\frac{1}{\binom{a^{2^{d-2}}}{2}}\left[\frac{(1+\delta)2^{d-1/2}\cdot a^{2^{d}-1/2}}{(1+\delta)2^{d-3/2}\cdot a^{2^{d-1}-1/2}} - a^{2^{d-1}}\right]}{2}\right)\\
&= p(d,a)\left(\frac{\frac{1}{\binom{a^{2^{d-2}}}{2}}(2a^{2^{d-1}} - a^{2^{d-1}})}{2}\right)\\
&\geq p(d,a).
\end{aligned}
$$

We now show that the result is sharp by employing a well known construction. Fix a finite projective plane of order q with points P and lines $\mathcal{L}$. Let $a = q^2+q+1$ and form the equibipartite graph $G' = (P, \mathcal{L}, \mathcal{E}(G'))$ defined by $\{p, L\} \in \mathcal{E}(G')$ if and only if the point p is incident with line L. In this case, $G' \in \mathcal{G}(2,a)$ and is $K_{2,2}$-free with $|\mathcal{E}(G')| = (1+o(1))a^{3/2}$ edges. (Only a slight modification is needed when a is not of the form q^2+q+1.)

Create $G \in \mathcal{G}(d,a)$ by embedding G on the two smallest partite sets, and extending edges of G' to the remaining $d-2$ partite sets in all possible ways. The resulting d-partite d-uniform hypergraph is $K^{(d)}(2,2,\ldots,2)$-free and contains $ca^{(2^d-1)/2}$ edges for a suitable constant c. □

Using $a = n/2$, the base step of the proof of Lemma 2.2 immediately gives the following, the first part of which is likely folklore.

Corollary 2.3 *For any $\delta > 0$, if G is an equibipartite graph on n vertices and*

$$|\mathcal{E}(G)| \geq \frac{1+\delta}{2}\cdot n^{3/2}$$

then G contains $\binom{n/2}{2}$ copies of $C_4 \cong K_{2,2}$.

The projective plane construction given in the proof of Lemma 2.2 gives that $(1+\delta)(n/2)^{3/2}$ edges may be required for the existence of a single $K_{2,2}$

in an equibipartite graph. It would be of interest to narrow this gap of $\sqrt{2}$ in the multiplicative constant.

If in the statement of Corollary 2.3 we do not insist that G be equibipartite, then, as is found by many constructions, (for example, see, [12], Problem 10.36, [14], or [3] for more references) $n^{3/2}(1-o(1))/2$ are necessary for the appearance of a $K_{2,2}$.

2.2 Hypergraphs to integers

Now we demonstrate a bijection between edges in a complete hypergraph $G \in \mathcal{G}(d,a)$ (recall Definition 2.1) and the elements of an initial interval of positive integers.

Lemma 2.4 *For integers $a \geq 2$, $d \geq 2$, and any integer $x \in [1, a^{2^{d-1}}]$, there is a unique d-tuple of non-negative integers, $\boldsymbol{\alpha}(x) = \langle \alpha_d, \alpha_{d-1}, \ldots, \alpha_1 \rangle$, where $0 \leq \alpha_1 \leq a-1$ and for $j = 2, \ldots, d$, $0 \leq \alpha_j \leq a^{2^{j-2}} - 1$ so that*

$$x = 1 + \alpha_1 + \sum_{i=2}^{d} \alpha_i a^{2^{i-2}}.$$

Proof of Lemma 2.4: The number of such d-tuples is $a \cdot a \cdot a^2 \cdot a^4 \cdots a^{2^{d-2}} = a^{2^{d-1}}$. To see that the representation is unique, suppose that

$$1 + \alpha_1 + \sum_{i=2}^{d} \alpha_i a^{2^{i-2}} = 1 + \beta_1 + \sum_{i=2}^{d} \beta_i a^{2^{i-2}}.$$

Then

$$\begin{aligned}
(\beta_d - \alpha_d) a^{2^{d-2}} &= \alpha_1 - \beta_1 + \sum_{i=2}^{d-1} (\alpha_i - \beta_i) a^{2^{i-2}} \\
&\leq |\alpha_1 - \beta_1| + \sum_{i=2}^{d-1} |(\alpha_i - \beta_i)| a^{2^{i-2}} \\
&\leq a - 1 + \sum_{i=2}^{d-1} (a^{2^{i-2}} - 1) a^{2^{i-2}} \\
&= a^{2^{d-2}} - 1,
\end{aligned}$$

and so $\beta_d = \alpha_d$. Proceeding by downward induction shows that $\beta_j = \alpha_j$ for each $j = d-1, \ldots, 1$. $\square$

Lemma 2.5 *For any integer $d \geq 2$, real number $\delta > 0$, and for $a \geq \max\{2, 1/(8\delta^2)\}$, if $A \subseteq [1, a^{2^{d-1}}]$ satisfies*

$$|A| \geq (1+\delta)2^{d-3/2} \cdot a^{2^{d-1}-1/2}, \tag{5}$$

then A contains $p(d-1, a) \sim a^{2^d}/2^{d-1}$ replete affine d-cubes.

Proof of Lemma 2.5: Let a be large enough so that Lemma 2.2 holds and let A satisfy (5). Construct

$$G = (X_1, \ldots, X_d, \mathcal{E}(G)) \in \mathcal{G}(d, a)$$

as follows. Let $X_1 = [0, a-1]$, and for each $j = 2, 3, \ldots, d$, let X_j be a copy of $[0, a^{2^{j-2}} - 1]$ (where $X_1, X_2, \ldots, X_d$ are pairwise disjoint). To each $x \in A$, assign the d-tuple $\boldsymbol{\alpha}(x)$ as in Lemma 2.4, and let $\mathcal{E}(G) = \{\boldsymbol{\alpha}(x) : x \in A\}$. Since $|\mathcal{E}(G)| = |A| \geq (1+\delta)2^{d-3/2}a^{2^{d-1}-1/2}$, by Lemma 2.2, G contains $p(d-1, a)$ copies of $K^{(d)}(2, 2, \ldots, 2)$. We claim that each copy of $K^{(d)}(2, 2, \ldots, 2)$ corresponds to a replete affine d-cube in A.

Fix a copy of $K^{(d)}(2, 2, \ldots, 2)$ in G on vertices $\alpha_1, \beta_1, \ldots, \alpha_d, \beta_d$, (where for each i, $\alpha_i \in X_i$ and $\beta_i \in X_i$), and without loss, let $\alpha_i < \beta_i$ for each i. Put

$$x_0 = 1 + \alpha_1 + \sum_{i=2}^{d} \alpha_i a^{2^{i-2}},$$

$x_1 = \beta_1 - \alpha_1$ and for each $j = 2, \ldots, d$, put $x_j = (\beta_j - \alpha_j)a^{2^{j-2}}$. The set (see Figure 1)

$$H = \{x_0 + \sum_{j \in J} x_j : J \subseteq [1, d]\} \subset A$$

is an affine d-cube with, for example, largest element $x_0 + \sum_{i=1}^{d} x_i = 1 + \beta_1 + \sum_{i=2}^{d} \beta_i a^{2^{i-2}}$. $\Box$

2.3 Last part of proof of Theorem 1.4

We can now give an upper bound for the density of a set not containing any affine d-cubes.

Proof of Theorem 1.4: Fix $d \geq 2$, $\delta > 0$ and let $a \geq \max\{2, 1/(8\delta^2)\}$ be large enough so that Lemma 2.2 holds. Let

$$a^{2^{d-1}} \leq n < (a+1)^{2^{d-1}}.$$

Figure 1: Hyperedges as numbers

Then

$$n^{1-1/2^d} = (1 + o(1))a^{2^{d-1}-1/2}$$

(as $n \to \infty$). Now Lemma 2.5 applies yielding $p(d-1, a) \sim a^{2^d}/2^{d-1} \sim n^2/2^{d-1}$ replete affine d-cubes. For large n, the $(1+o(1))$ factor is absorbed by the $(1+\delta)$. □

Though Lemma 2.2 gives a sharp result, the extremal graph given in the proof thereof does not necessarily prevent any affine d-cube (for example, examine the collection of edges formed by fixing a vertex in each partite set but one—the integers thereby defined may certainly contain an affine d-cube).

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