Evaluation of Credit Value Adjustment with a Random Recovery Rate via a Structural Default Model

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Credit value adjustment

Consider a financial derivative instrument held between two parties, the institution and its counterparty.

$$\text{Risky value} = \text{risk-free value} - \text{CVA}$$

$$\text{CVA} \approx (1 - \text{R}) \sum_{k=1}^{K} \text{DF}(t_k) \times \text{EE}(t_k) \times \text{PD}(t_{k-1}, t_k)$$

- $(1 - \text{R})$: the *loss given default*
- $\text{DF}(t_k)$: the risk-free *discount factor* for time t_k
- $\text{EE}(t_k)$: the *expected risk exposure* for the institution at t_k
- $\text{PD}(t_{i-1}, t_i)$: the default probability between dates t_{i-1} and t_i

See, for example, Grogery (2012, Chapter 12) and Feng and Volkmer (2012).

Denoting by τ the default time of the counterparty, we propose

$$\text{CVA} \approx \sum_{k=1}^{K} \text{DF}(t_k) \times \text{EE}(t_k) \times \underbrace{\mathbb{E}\left[(1 - \text{R}_\tau)1_{\{t_{k-1}<\tau\le t_k\}}\right]}_{\text{main focus}}. \tag{1}$$

Assumptions:

- The institution itself cannot default.
- Risk-free valuation can be performed.
- The credit exposure is independent of default probability and loss given default.

The Lévy first-passage model

Under the risk-neutral setting:

- The counterparty's asset process $V_t, t \geq 0$, follows

$$V_t = V_0 \, e^{Z_t},$$

  where Z_t is a spectrally negative Lévy process.

- $\mathbb{E}(V_t) = V_0 \, e^{rt}$ with r the constant interest rate.

- For a threshold level $L < V_0$, default time is defined as

$$\tau = \inf \{t : V_t \leq L\} = \inf \{t : \ln(V_0/L) + Z_t \leq 0\} .$$

Denote

$$X_t = \ln(V_0/L) + Z_t.$$

- τ is the ruin time of process X_t starting from $\ln(V_0/L)$.
- $-X_\tau$ specifies the default severity.
- Assume $\mathrm{R}_\tau = R(-X_\tau)$, where $R(\cdot) \in [0,1]$ is loss settlement function which is non-increasing defined on $[0,\infty)$. For instance,

$$\mathrm{R}_\tau = V_\tau/V_0 = (L/V_0)\, e^{X_\tau} \in (0, L/V_0).$$

The idea of introducing a loss settlement function can also be found in Tang and Yuan (2013) and van Damme (2011).

Some remarks of the model:

- According to the empirical study by Carr *et al.* (2002), risk-neutral processes for equity prices should be processes of infinite activity and finite variation.
- A firm's asset value is exposed to shocks (represented by negative jumps), which is the main concern in risk management practice.
- The process Z_t should be such that the expectation in CVA (1) can be efficiently calculated with enough accuracy.

Meromorphic Lévy processes

We assume

$$Z_t = \mu t - S_t$$

with $\mu > 0$ and S_t a pure-jump meromorphic process with only upward jumps.

Lévy measure of Z_t: $\Pi(\mathrm{d}x) = \pi(x)\mathrm{d}x$, where

$$\pi(x) = 1_{\{x<0\}} \sum_{m=1}^{\infty} b_m e^{\rho_m x}, \qquad b_m > 0, \;\; \rho_m > 0.$$

Laplace exponent of Z_t:

$$\begin{aligned} \psi(s) := \ln \mathbb{E}(e^{sZ_1}) &= \mu s + \int_{-\infty}^{0} (e^{sx} - 1)\, \Pi(\mathrm{d}x) \\ &= \mu s + \sum_{m=1}^{\infty} b_m \left(\frac{1}{\rho_m + s} - \frac{1}{\rho_m} \right). \end{aligned}$$

Examples:

- Z_t is an *θ-process* if

$$b_m = \frac{2}{\pi} c\beta m^2, \quad \rho_m = \beta(\alpha + m^2), \qquad c, \alpha, \beta > 0.$$

Now

$$\psi(s) = \mu s - c\sqrt{\alpha + s/\beta} \coth\left(\pi\sqrt{\alpha + s/\beta}\right) + c\sqrt{\alpha} \coth(\pi\sqrt{\alpha}).$$

- Z_t is a β-process with $\lambda \in (1, 2)$ if

$$b_m = c\beta \binom{m + \lambda - 2}{m - 1}, \quad \rho_m = \beta(\alpha + m), \qquad c, \alpha, \beta > 0.$$

Now

$$\psi(s) = \mu s + c\mathrm{B}(1 + \alpha + s/\beta, 1 - \lambda) - c\mathrm{B}(1 + \alpha, 1 - \lambda).$$

See Kuznetsov (2010a, b) for properties of these processes.

Generalized expected discounted penalty function

Consider the process

$$X_t = x + Z_t, \quad \text{with } x \geq 0.$$

Definition 1

The generalized expected discounted penalty function (EDPF) of X_t is

$$\phi(x;r) := \mathbb{E}\left[e^{-r\tau} w(-X_\tau, X_{\tau-}, \underline{X}_{\tau-}) 1_{\{\tau<\infty\}} \middle| X_0 = x \right],$$

and the generalized finite-time EDPF of X is

$$\phi_t(x;r) := \mathbb{E}\left[e^{-r\tau} w(-X_\tau, X_{\tau-}, \underline{X}_{\tau-}) 1_{\{\tau<t\}} \middle| X_0 = x \right],$$

with $r \geq 0$ and w a bounded measurable function on $\mathbb{R}_+^3 = [0,\infty)^3$.

Biffis and Morales (2010) and Kuznetsov and Morales (2014) have introduced the generalized EDPF into actuarial literature.

Let us go back to the main focus in CVA (1).

$$\mathbb{E}\left[(1-\mathrm{R}_\tau)1_{\{t_{k-1}<\tau\le t_k\}}\right] = \mathbb{E}\left(1_{\{t_{k-1}<\tau\le t_k\}}\right) - \mathbb{E}\left(\mathrm{R}_\tau 1_{\{t_{k-1}<\tau\le t_k\}}\right)$$

$$= \mathbb{Q}\left(\tau \le t_k\right) - \mathbb{Q}\left(\tau \le t_{k-1}\right) - (L/V_0)\left[\mathbb{E}\left(e^{X_\tau}1_{\{\tau\le t_k\}}\right) - \mathbb{E}\left(e^{X_\tau}1_{\{\tau\le t_{k-1}\}}\right)\right].$$

In terms of generalized EDPF,

$$\mathbb{Q}(\tau \le t) = \phi_t(x;0) \qquad \text{with } w(-X_\tau, X_{\tau-}, \underline{X}_{\tau-}) = 1$$

and

$$\mathbb{E}\left(e^{X_\tau}1_{\{\tau\le t\}}\right) = \phi_t(x;0) \qquad \text{with } w(-X_\tau, X_{\tau-}, \underline{X}_{\tau-}) = e^{X_\tau}.$$

Note that the Laplace transform of ϕ_t becomes ϕ:

$$\int_0^\infty e^{-qt}\phi_t(x;r)\mathrm{d}t = \frac{\phi(x;r+q)}{q}.$$

$$\Downarrow$$

$$\int_0^\infty e^{-qt}\mathbb{Q}\left(\tau \le t\right)\mathrm{d}t = q^{-1}\sum_{n\ge 1} c_n e^{-\zeta_n x}$$

and

$$\int_0^\infty e^{-qt}\mathbb{E}\left(e^{X_\tau}1_{\{\tau<t\}}\right)\mathrm{d}t = \frac{\Phi(q)}{q^2}\sum_{m,n\ge 1}\frac{c_n\zeta_n b_m e^{-\zeta_n x}}{(\Phi(q)+\rho_m)(\rho_m-\zeta_n)(\rho_m+1)},$$

where

- $\{-\zeta_n : n = 1,2,\ldots\}$ are negative simple roots of $\psi(z) = q$,
- $\Phi(q)$ is the unique positive root of $\psi(z) = q$,
- $c_n = (1/\zeta_n + 1/\Phi(q))\, q/\psi'(-\zeta_n)$.

Computing the finite-time EDPF

By an inverse Laplace transform based on the Gaver-Stehfest algorithm, the function $\phi_t(x;q)$ is approximated by

$$\phi_t^{GS}(x;q;M) = \sum_{n=1}^{2M} \frac{a_n}{n} \phi(x;q+nt^{-1}\ln 2)$$

with $M \in \mathbb{N}$ large and

$$a_n = (-1)^{M+n} \sum_{j=\lfloor (n+1)/2 \rfloor}^{n \wedge M} \frac{j^{M+1}}{M!} \binom{M}{j} \binom{2j}{j} \binom{j}{n-j}.$$

Numerical experiments

Using the method of valuation in terms of bond prices, which has been discussed in Hull (2015), we study the following interest rate swap.

Table: Hypothetical interest rate swap

Effective date	18-Mar-2015
Termination date	18-Mar-2018
Notional principal	USD 100 million
Payment dates	From 18-Sep-2015 to and including 18-Mar-2018
Fixed-rate payer	Institution
Fixed rate	0.6376 per annum
Floating-rate payer	Counterparty
Floating rate	USD 6-month LIBOR

[a] The fixed rate can be determined by setting the present value of the fixed rate payments equal to the present value of the floating rate payments.
[b] All rates are quoted nominal continuously.

Table: Forward LIBOR rates

Date	Futures price	Futures rate	Convexity adjustment	Forward rate
18-Mar-2015	99.7316	0.2684	0.00000000	0.2684
18-Jun-2015	99.6100	0.3900	0.00053824	0.3895
18-Sep-2015	99.4100	0.5900	0.00161472	0.5884
18-Dec-2015	99.2000	0.8000	0.00322944	0.7968
18-Mar-2016	98.9800	1.0200	0.00538240	1.0146
18-Jun-2016	98.7500	1.2500	0.00807360	1.2419
18-Sep-2016	98.5300	1.4700	0.01130304	1.4587
18-Dec-2016	98.3350	1.6650	0.01507072	1.6499
18-Mar-2017	98.1800	1.8200	0.01937664	1.8006
18-Jun-2017	98.0400	1.9600	0.02422080	1.9358
18-Sep-2017	97.9250	2.0750	0.02960320	2.0454
18-Dec-2017	97.8200	2.1800	0.03552384	2.1445
18-Mar-2018	97.7400	2.2600	0.04198272	2.2180

[a] The futures prices and the volatility of the short rate (0.928%) are retrieved from the Bloomberg.

Table: LIBOR zero rates

Date	6 m	12 m	18 m	24 m	30 m	36 m
18-Mar-2015	0.3289	0.5108	0.7166	0.9260	1.1145	1.2779
18-Sep-2015	0.6826	0.9104	1.1251	1.3108	1.4677	-
18-Mar-2016	1.1283	1.3413	1.5169	1.6614	-	-
18-Sep-2016	1.5543	1.7113	1.8392	-	-	-
18-Mar-2017	1.8682	1.9816	-	-	-	-
18-Sep-2017	2.0949	-	-	-	-	-

Table: Value of the interest rate swap

Date	B_{fixed}	B_{float}	EE
18-Mar-2015	100.0000	100.0000	0.0000
18-Sep-2015	97.9622	100.0000	2.0378
18-Mar-2016	97.9832	100.0000	2.0168
18-Sep-2016	98.2188	100.0000	1.7812
18-Mar-2017	98.6663	100.0000	1.3337
18-Sep-2017	99.2735	100.0000	0.7265

If the counterparty is **American Express Co**:

	1-y	2-y	3-y	5-y	7-y	10-y
Market CDS	9.032	17.501	23.032	40.751	56.684	68.666
Calibrated CDS	7.734	17.473	27.587	44.643	56.671	67.669

[a] The calibrated process is a θ-process with $c = 3.521$, $\alpha = 1.24$, $\beta = 4.3$, $\lambda = 1.5$.

$$\begin{aligned} \mathrm{CVA}_{\text{fixed R}} &= (1-\mathrm{R})\sum_{k=1}^{K} \mathrm{DF}(t_k) \times \mathrm{EE}(t_k) \times \mathrm{PD}(t_{k-1}, t_k) \\ &= 0.006566 \end{aligned}$$

$$\begin{aligned} \mathrm{CVA}_{\text{random R}} &= \sum_{k=1}^{K} \mathrm{DF}(t_k) \times \mathrm{EE}(t_k) \times \mathbb{E}\left[(1-\mathrm{R}_\tau)1_{\{t_{k-1}<\tau\le t_k\}}\right] \\ &= 0.007532 \qquad \longleftarrow \text{15\% higher!} \end{aligned}$$

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Thank you!