Evaluation of Credit Value Adjustment in K-forwards

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K-forwards

CBD two-factor mortality model:

$$\ln\left(\frac{q_{x,t}}{1-q_{x,t}}\right) = \kappa_t^{(1)} + \kappa_t^{(2)}(x-\bar{x}) + \varepsilon_{x,t}$$

- $q_{x,t}$: the probability that an individual aged x at time t will die by $t+1$
- $\bar{x}$: the average of the ages used in the dataset
- $\kappa_t^{(1)}$: the level of the logit-transformed mortality curve
- $\kappa_t^{(2)}$: the slope of the logit-transformed mortality curve
- $\varepsilon_{x,t}$: the error term

Cairns *et al.* (2006, JRI)

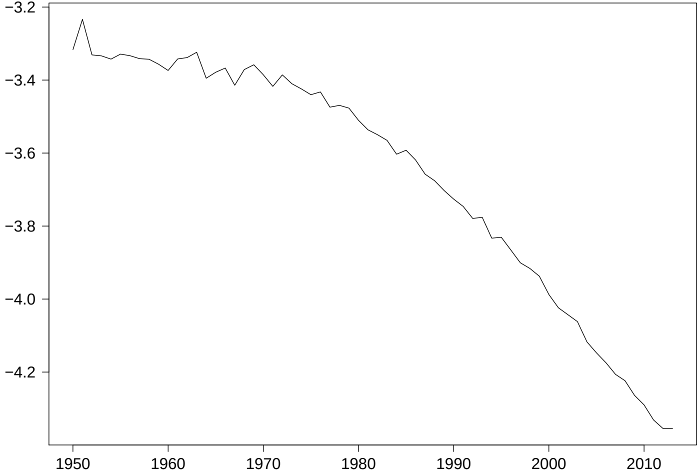


Figure: Estimates of $k_t^{(1)}$ for England and Wales males, aged from 40 to 90, 1950–2013.

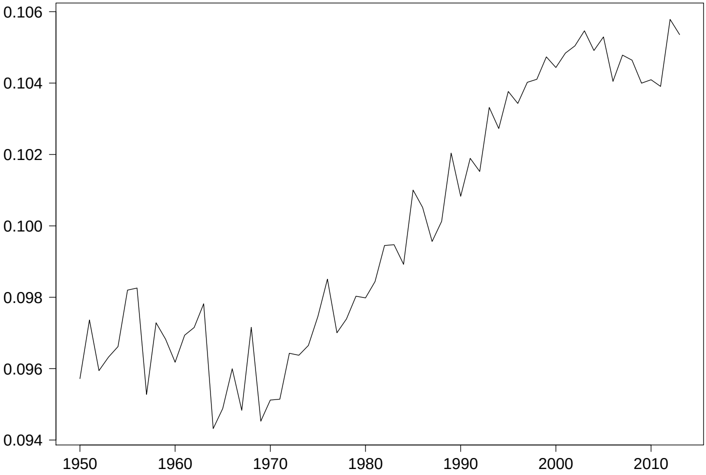
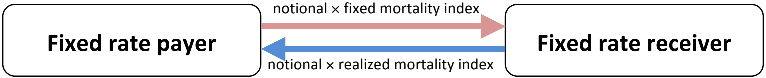


Figure: Estimates of $k_t^{(2)}$ for England and Wales males, aged from 40 to 90, 1950–2013.

A *K-forward* is a zero-coupon swap, which exchanges a fixed forward mortality index with a realized reference mortality index at maturity.



The payoff to the fixed rate receiver can be written as

$$\Upsilon \times \left(\tilde{\kappa}_T^{(i)} - \kappa_T^{(i)} \right), \qquad i = 1, 2.$$

- T: the reference year
- $\tilde{\kappa}_T^{(i)}$: the fixed forward mortality index, determined at time 0
- $\kappa_T^{(i)}$: the reference mortality index, unknown at time 0
- Υ: the notional amount

VARIMA model for $k_t^{(1)}$ and $k_t^{(2)}$

Suggested by Chan *et al.* (2014, NAAJ), we model the process $\boldsymbol{k}_t = (k_t^{(1)}, k_t^{(2)})^\mathsf{T}$ by a VARIMA model

$$\Delta^d \boldsymbol{k}_t - \sum_{i=1}^{p} \Phi_i \Delta^d \boldsymbol{k}_{t-i} = \Phi_0 + \boldsymbol{\epsilon}_t - \sum_{j=1}^{q} \Theta_j \boldsymbol{\epsilon}_{t-j}$$

with

- $\Delta^d \boldsymbol{k}_t = \Delta^{d-1} \boldsymbol{k}_t - \Delta^{d-1} \boldsymbol{k}_{t-1}$
- $\boldsymbol{\epsilon}_t \overset{i.i.d.}{\sim} N_2(\mathbf{0}, \Sigma)$

It turns out that for the England and Wales males, aged 40–90, from 1950 to 2013, a VARIMA($p = 3, d = 1, q = 1$) model fits well.

Table: SCCM and $M(l)$ of SPAM for $\Delta \boldsymbol{k}_t$

lag(l)	1	2	3	4	5	6	7	8
Sample cross-correlation matrices (SCCM)	$\begin{bmatrix} - & . \\ - & - \end{bmatrix}$	$\begin{bmatrix} . & . \\ . & . \end{bmatrix}$	$\begin{bmatrix} . & . \\ . & . \end{bmatrix}$	$\begin{bmatrix} . & . \\ . & . \end{bmatrix}$	$\begin{bmatrix} . & . \\ . & . \end{bmatrix}$	$\begin{bmatrix} . & . \\ . & . \end{bmatrix}$	$\begin{bmatrix} + & . \\ . & . \end{bmatrix}$	$\begin{bmatrix} . & . \\ . & . \end{bmatrix}$
Test statistics $M(l)$ of sample partial autocorrelation matrices (SPAM) with a critical value at $\chi^2_{4,0.95} = 9.45$	17.525	3.990	10.279	6.938	8.947	8.229	5.617	6.663

Table: AIC and BIC of fitted VARIMA(p, d, q) models

	(0,1,1)	(3,1,0)	ref. (3,1,0)	(3,1,1)	ref. (3,1,1)
AIC	-21.53	-21.58	-21.71	-21.96	**-22.32**
BIC	-21.33	-21.10	-21.51	-21.35	**-22.01**

Table: Estimated coefficients for the refined VARIMA(3,1,1)

Φ_0	Φ_1	Φ_2
$\begin{pmatrix} 0 \\ .000145 \end{pmatrix}$	$\begin{pmatrix} .613 & -5.078 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
Φ_3	Θ_1	Σ
$\begin{pmatrix} .349 & -4.010 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} -1.115 & .212 \\ -.031 & -.485 \end{pmatrix}$	$\begin{pmatrix} .00027315 & .00000371 \\ .00000371 & .00000061 \end{pmatrix}$

Table: SCCM and $M(l)$ of SPAM for residuals of the refined VARIMA(3,1,1)

lag(l)							
1	2	3	4	5	6	7	8
Sample cross-correlation matrices (SCCM)							
$\begin{bmatrix} . & . \\ . & . \end{bmatrix}$	$\begin{bmatrix} . & . \\ . & . \end{bmatrix}$	$\begin{bmatrix} . & . \\ . & . \end{bmatrix}$	$\begin{bmatrix} . & . \\ . & . \end{bmatrix}$	$\begin{bmatrix} . & . \\ . & . \end{bmatrix}$	$\begin{bmatrix} . & . \\ . & . \end{bmatrix}$	$\begin{bmatrix} . & . \\ . & . \end{bmatrix}$	$\begin{bmatrix} . & . \\ . & . \end{bmatrix}$
Test statistics $M(l)$ of sample partial autocorrelation matrices (SPAM) with a critical value at $\chi^2_{4,0.95} = 9.45$							
2.089	0.533	0.417	2.299	1.363	5.292	8.217	3.741

Table: Normality test results for residuals of the refined VARIMA(3,1,1)

	uniNorm test for residuals in $k^{(1)}$	uniNorm test for residuals in $k^{(2)}$	Royston's test	Henze-Zirkler's test
Statistics	0.9907	0.9707	2.7546	0.8505
p-value	0.9301	0.1586	0.2527	0.0687

Credit value adjustment (CVA)

From the perspective of the fixed rate receiver in a K-forward, CVA is its loss due to the potential default of the fixed rate payer.

$$\mathrm{CVA} = (1 - \mathrm{Rec}) \sum_{t=1}^{T} \mathrm{DF}(t) \times \mathrm{EE}(t) \times \mathrm{PD}(t-1, t)$$

- $1 - \mathrm{Rec}$: the loss given default
- $\mathrm{DF}(t)$: the risk-free discount factor for time t
- $\mathrm{EE}(t)$: the expected risk exposure for the institution at t
- $\mathrm{PD}(t-1, t)$: the default probability between dates $t-1$ and t

Reduced-form model

Suppose there exists a deterministic default intensity $h(t)$, $t \geq 0$. Using Nelson-Siegel yield rate function, we assume

$$h(t) = \beta_0 + \beta_1 e^{-t/\beta_3} + \beta_2 e^{-t/\beta_3} t/\beta_3, \qquad t \geq 0.$$

$$\Downarrow$$

$$H(t) = \frac{1}{t}\int_0^t h(s)\mathrm{d}s = \beta_0 + (\beta_1 + \beta_2)(1 - e^{-t/\beta_3})\beta_3/t - \beta_2 e^{-t/\beta_3}$$

and

$$S(t) = \exp\left(-tH(t)\right), \qquad t \geq 0.$$

- $\beta_0(> 0)$ is the long-term component.
- $\beta_1(\geq -\beta_0)$ is the short-term component.
- β_2 ($\geq \beta^*$, determined by β_0 and β_1) is the medium-term component.
- $\beta_3(> 0)$ is the time-scalar parameter.

For a non-callable bond, we assume the recovery is a fraction of par.

$$\text{“dirty price”} = \sum_{i=1}^{n} \mathrm{DF}(t_i) \cdot c\Delta_i \cdot S(t_i) + V \cdot \mathrm{DF}(t_n) \cdot S(t_n) + V \cdot \mathrm{Rec} \cdot \int_0^{t_n} \mathrm{DF}(t) F(\mathrm{d}t)$$

- t_i: coupon payment time
- c: the coupon rate
- Δ_i: the fraction of year between t_{i-1} and t_i
- V: par value

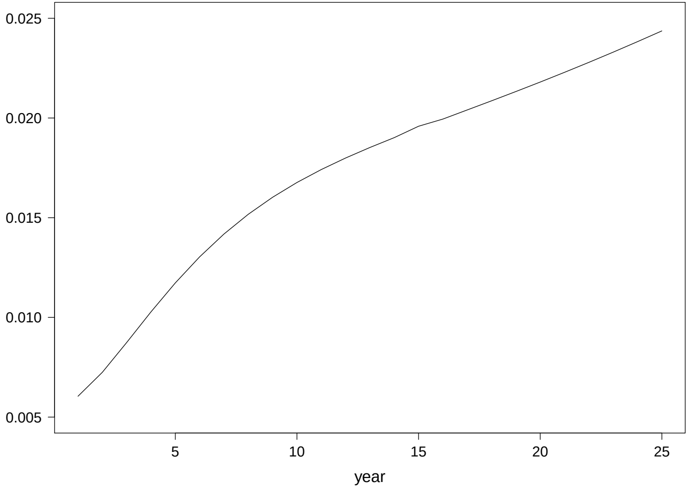


Figure: US treasury zero yield curve on June 16, 2016.

Assuming a fixed Rec, we calibrate the default model parameters for a specific investor by minimizing the root mean square percentage error:

$$\underset{\boldsymbol{\beta}=(\beta_0,\beta_1,\beta_2,\beta_3)^\mathsf{T}}{\arg\min} \sqrt{\frac{1}{N}\sum_{j=1}^{N}\left(\frac{\text{dirty price}_j - \text{market price}_j}{\text{market price}_j}\right)^2}$$

Calibration results

Table: Two potential investors of K-forwards

Investor	Moody's rating	Rating grade
JP Morgan Chase	Baa1	Lower medium
RBS Group	Ba1	Non-investment

Table: Ten bonds of JP Morgan Chase on June 16, 2016

Maturity	Coupon	Frequency	Last price	Model price	Currency
0.668	1.350	semiannually	100.166	100.166	USD
1.584	6.000	semiannually	107.050	108.646	USD
3.351	2.200	semiannually	101.290	100.188	USD
4.101	4.400	semiannually	109.164	109.114	USD
5.167	4.350	semiannually	109.158	109.531	USD
6.614	3.200	semiannually	103.199	103.800	USD
7.912	3.625	semiannually	105.290	105.290	USD
21.926	6.400	semiannually	135.326	135.326	USD
24.348	5.500	semiannually	124.220	123.242	USD
25.575	5.400	semiannually	122.221	122.524	USD

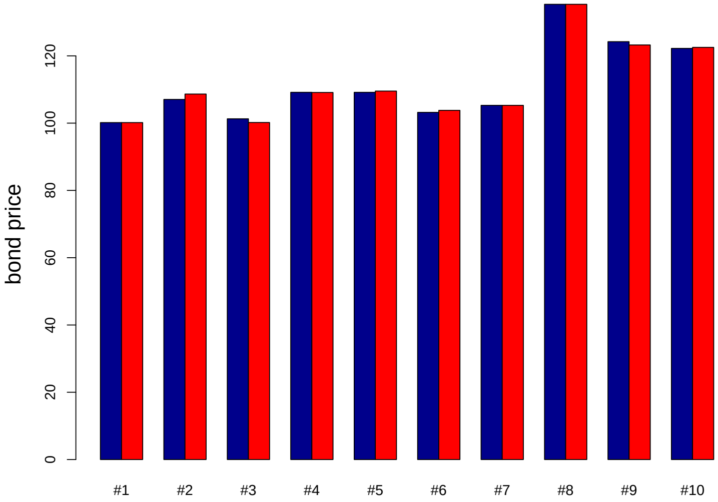


Figure: Calibration of Nelson-Siegel model on bonds of JP Morgan Chase. ($\beta_0 = 0.0120, \beta_1 = 0.0067, \beta_2 = 0.0185, \beta_3 = 3.4164$)

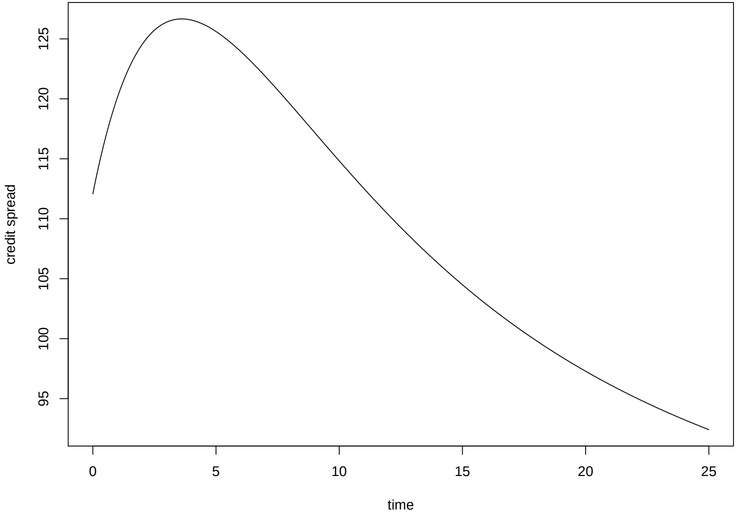


Figure: Credit spread term structure of JP Morgan Chase on June 16, 2016.

Table: Seven bonds of Royal Bank of Scotland Group on June 16, 2016

Maturity	Coupon	Frequency	Last price	Market price	Currency
1.249	2.650	monthly	99.000	99.189	USD
1.499	2.250	monthly	99.513	98.259	USD
1.668	2.150	monthly	97.938	97.938	USD
3.668	2.750	monthly	95.313	95.313	USD
6.501	3.650	monthly	97.563	93.379	USD
6.986	6.100	semiannually	103.000	103.743	USD
7.926	5.125	semiannually	98.076	98.076	USD

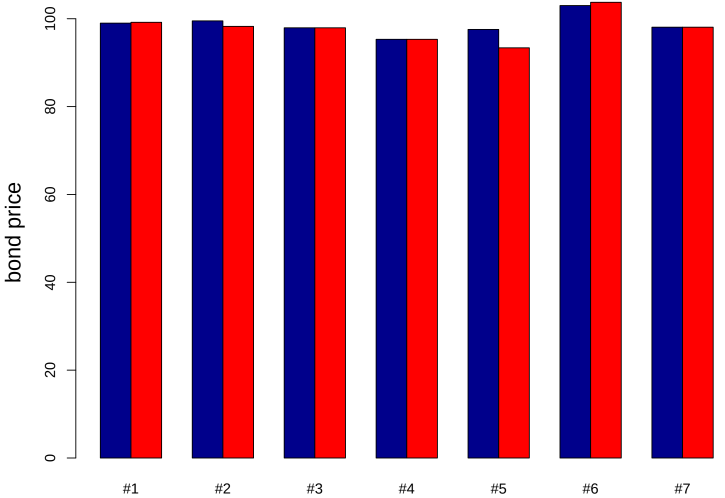


Figure: Calibration of Nelson-Siegel model on bonds of RBS Group. ($\beta_0 = .0516, \beta_1 = -.0288, \beta_2 = .0275, \beta_3 = .9014$)

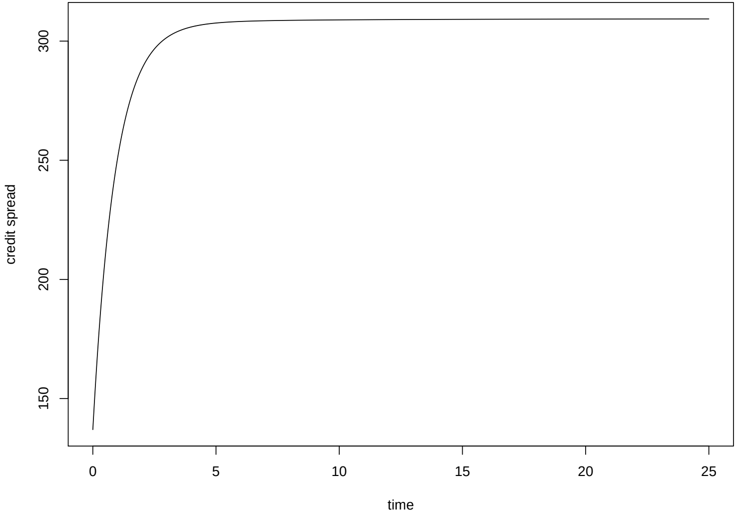


Figure: Credit spread term structure of RBS on June 16, 2016.

Numerical results

Consider K-forwards for England and Wales males aged from 40 to 90.

$$\Upsilon \times \left(\tilde{\kappa}_T^{(i)} - \kappa_T^{(i)} \right), \qquad i = 1, 2,$$

with $\Upsilon = \$1$.

Table: CVA (bps) in K-forwards when Rec $= 0.4$.

	$T = 15$			$T = 20$			$T = 25$		
	$K1$	$K2$	sprd	$K1$	$K2$	sprd	$K1$	$K2$	sprd
JPMC	32	.68	104	50	.86	97	69	1.0	92
RBS	76	1.6	309	115	2.0	309	155	2.2	309

Table: CVA (bps) in K-forwards when Rec $= 0.3$.

	$T = 15$			$T = 20$			$T = 25$		
	$K1$	$K2$	sprd	$K1$	$K2$	sprd	$K1$	$K2$	sprd
JPMC	35	.75	112	55	.95	105	76	1.1	101
RBS	83	1.8	322	128	2.2	323	174	2.5	324

Table: CVA (bps) in K-forwards when Rec $= 0.5$.

	$T = 15$			$T = 20$			$T = 25$		
	$K1$	$K2$	sprd	$K1$	$K2$	sprd	$K1$	$K2$	sprd
JPMC	28	.59	95	43	.73	88	59	.86	83
RBS	54	1.1	243	76	1.3	208	96	1.3	181

References

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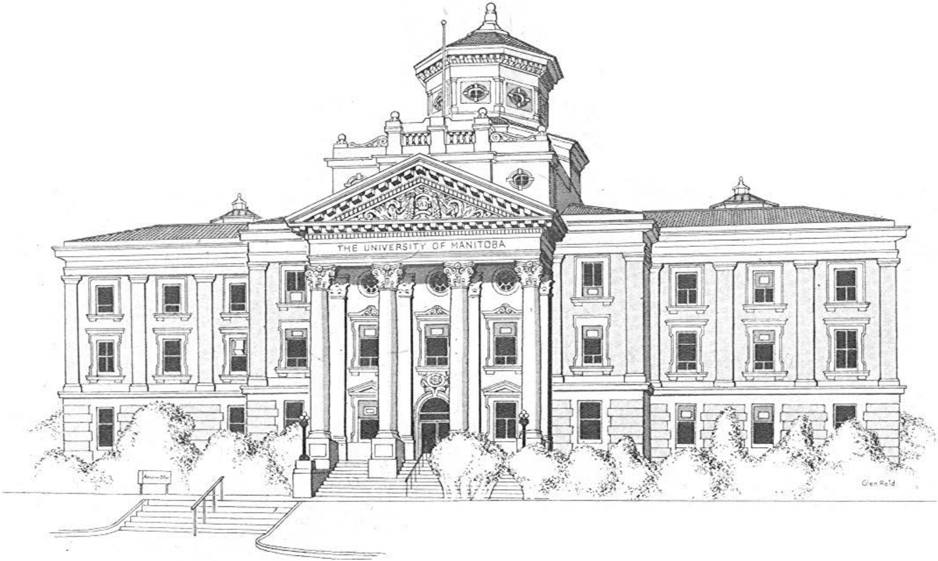
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