Asymptotic Ruin Probabilities of the Lévy Insurance Model under Periodic Taxation

Xuemiao Hao[a,b,*] and Qihe Tang[a]

[a] Department of Statistics and Actuarial Science
The University of Iowa
241 Schaeffer Hall, Iowa City, IA 52242, USA

[b] Warren Centre for Actuarial Studies and Research
University of Manitoba
181 Freedman Crescent, Winnipeg, MB R3T 5V4, Canada

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Abstract

Recently, Albrecher and his coauthors have published a series of papers on the ruin probability of the Lévy insurance model under the so-called loss-carry-forward taxation, meaning that taxes are paid at a certain fixed rate immediately when the surplus of the company is at a running maximum. In this paper we assume periodic taxation under which the company pays tax at a fixed rate on its net income during each period. We devote ourselves to deriving explicit asymptotic relations for the ruin probability in the most general Lévy insurance model in which the Lévy measure has a subexponential tail, a convolution-equivalent tail, or an exponential-like tail.

Keywords: Asymptotics; convolution-equivalent tail; Lévy process; periodic taxation; ruin probability; subexponentiality.

1 Introduction

The ruin probability of an insurance company is the probability that its surplus process falls below 0 at some time. Recently, the influence of tax payment on the ruin probability has become an interesting problem in actuarial science. Let $S = (S_t)_{t\geq 0}$ be a stochastic process, with $S_0 = x > 0$, representing the underlying surplus process in a world without economic factors (tax, reinsurance, or investment, etc.) of an insurance company. Assuming that S is a compound Poisson process with positive drift and that taxes are paid at a fixed rate $\gamma \in [0, 1)$ whenever S is at a running maximum (called the loss-carry-forward taxation),

*Corresponding author. E-mail: hao@cc.umanitoba.ca; phone: 1-204-474-8710; fax: 1-204-474-7545.

Albrecher and Hipp (2007) and Albrecher *et al.* (2009) proved the following strikingly simple relationship between $\psi_\gamma(x)$ and $\psi_0(x)$, the ruin probabilities with and without tax:

$$\psi_\gamma(x) = 1 - (1 - \psi_0(x))^{1/(1-\gamma)}. \tag{1.1}$$

Albrecher *et al.* (2008b) further showed that the tax identity (1.1) still holds for a spectrally negative Lévy surplus process under the loss-carry-forward taxation. Also, Albrecher *et al.* (2008a) proved a similar tax identity for a dual surplus process with general inter-innovation times and exponential innovation sizes under the same type of taxation.

All these papers cited above assume the loss-carry-forward taxation. In reality, however, taxes are usually paid periodically (e.g. monthly, semi-annually, or annually). Furthermore, if the surplus process contains a diffusion part, then the moments of running maxima do not form any continuous time interval. In this case, the loss-carry-forward type taxation is rather unrealistic, as was also commented by Albrecher and Hipp (2007).

In this paper, we introduce periodic taxation as well as reinsurance to the risk model. Precisely, we assume that at each discrete moment $n = 1, 2, \ldots$, the company, given that it survives, pays tax at rate $\gamma \in [0, 1)$ on its net income during the period $(n-1, n]$ and it gets paid by reinsurance at rate $\delta \in [0, 1)$ on its net loss during the period $(n-1, n]$. We are interested in the influence of such taxation rule and reinsurance policy on the asymptotic behavior of the ruin probability.

An example of such a reinsurance is the so-called quota-share reinsurance in which the reinsurer assumes an agreed percentage of an insurer being reinsured and shares all premiums and losses accordingly with the insurer. In this paper we shall assume that the loss process before tax and reinsurance is a Lévy process, which, after paying reinsurance premiums at a constant rate, is still a Lévy process. Therefore, for simplicity, in our formulation we shall ignore the reinsurance premiums or we understand the loss process as after paying the reinsurance premiums.

Let us intuitively compare these two types of taxation. Under the loss-carry-forward taxation, as long as the surplus does not hit its historical peak, the insurance company can legally evade any tax payment possibly for a long time, even if it makes profits every single period during that time. While under the periodic taxation, the insurance company has to pay tax whenever it survives and its net income is positive in that period. Hence, the latter imposes a more strict taxation rule and produces more significant impact on the ruin probability than the former does. This will be demonstrated in Section 3.

It is convenient for us to look at the loss process before tax and reinsurance,

$$L_t = x - S_t, \qquad t \geq 0.$$

For each $n = 1, 2, \ldots$, the maximal net loss and the net loss of the company within the period $(n-1, n]$ are, respectively,

$$Y_n = \sup_{n-1 \le t \le n} (L_t - L_{n-1}), \qquad Z_n = L_n - L_{n-1}.$$

After introducing the periodic taxation at rate $0 \le \gamma < 1$ and reinsurance at rate $0 \le \delta < 1$, the loss of the company within the period $(n-1, n]$ becomes

$$X_n = Z_n + \gamma Z_n^- - \delta Z_n^+ = (1-\delta) Z_n^+ - (1-\gamma) Z_n^-,$$

where $z^+ = z \vee 0$ and $z^- = -(z \wedge 0)$ for a real number z. Then, it is easy to see that the ruin probability in this situation is equal to

$$\psi_{\gamma,\delta}(x) = \Pr\left(\sup_{n \ge 1} \left(\sum_{k=1}^{n-1} X_k + Y_n\right) > x\right), \tag{1.2}$$

where, as usual, a summation over an empty set of indices produces a value 0. Notice that we have used $\psi_\gamma(x)$ (with only one subscript γ) for the ruin probability under the loss-carry-forward taxation and used $\psi_{\gamma,\delta}(x)$ (with two subscripts γ and δ) for the ruin probability under the periodic taxation and reinsurance. We shall let the notation speak for itself.

In this paper, we shall assume that the loss process L is a Lévy process (that is, it starts with 0, is right continuous with left limit, and has stationary and independent increments) with mean $\mathrm{E}L_1 = -\mu < 0$ (so that ignoring possible ruin it converges to $-\infty$ almost surely). Consequently, the random pairs (X_n, Y_n), $n = 1, 2, \ldots$, appearing in (1.2) are independent and identically distributed (i.i.d.) copies of the random pair

$$(X, Y) \overset{D}{=} \left((1-\delta) L_1^+ - (1-\gamma) L_1^-, \sup_{0 \le t \le 1} L_t\right). \tag{1.3}$$

Write $\mu_+ = \mathrm{E}L_1^+$ and $\mu_- = \mathrm{E}L_1^-$, which are assumed to be finite. Throughout this paper, we always choose $\gamma \in [0, 1)$ and $\delta \in [0, 1)$ such that

$$\mathrm{E}X = (1-\delta)\mu_+ - (1-\gamma)\mu_- < 0, \tag{1.4}$$

so that the insurance company still has positive expected profits under such taxation and reinsurance and that the ruin is not certain.

The rest of this paper consists of four sections. After listing some preliminaries on Lévy processes and several important distribution classes in Section 2, we present our main results in Sections 3-5 for the cases that the Lévy measure of the loss process L has a subexponential tail, a convolution-equivalent tail, and an exponential-like tail, respectively.

2 Preliminaries

For a Lévy process $L = (L_t)_{t\geq 0}$, its characteristic function can be written in the form

$$\mathrm{E}\mathrm{e}^{\mathrm{i}sL_t} = \mathrm{e}^{-t\Phi(s)},$$

where the characteristic exponent $\Phi(\cdot)$ has the Lévy-Khintchine representation

$$\Phi(s) = \mathrm{i}as + \frac{1}{2}\sigma^2 s^2 + \int_{-\infty}^{\infty} \left(1 - \mathrm{e}^{\mathrm{i}sx} + \mathrm{i}sx1_{(|x|\leq 1)}\right) \rho(\mathrm{d}x)$$

with $a \in (-\infty, \infty)$, $\sigma \geq 0$, and Lévy measure ρ on $(-\infty, \infty)$ satisfying $\rho(\{0\}) = 0$ and $\int_{-\infty}^{\infty} (x^2 \wedge 1)\, \rho(\mathrm{d}x) < \infty$. The triplet (a, σ^2, ρ) (called Lévy triplet) uniquely determines the distribution of the Lévy process L.

Throughout this paper, for a Lévy measure ρ and a distribution F on $(-\infty, \infty)$, write $\overline{\rho}(x) = \rho((x, \infty))$ and $\overline{F}(x) = 1 - F(x)$ for $x \geq 0$. When $\overline{\rho}(1) > 0$, introduce $\Pi(\cdot) = (\overline{\rho}(1))^{-1} \rho(\cdot)1_{(1,\infty)}$, which is a proper probability measure on $(1, \infty)$. Hereafter, all limit relationships are according to $x \to \infty$ unless otherwise stated, and for two positive functions $a(\cdot)$ and $b(\cdot)$, write $a(x) \sim b(x)$ if $a(x)/b(x) \to 1$.

A distribution F on $[0, \infty)$ is said to belong to the class $\mathcal{L}(\alpha)$ for some $\alpha \geq 0$ if $\overline{F}(x) > 0$ for all $x \geq 0$ and the relation

$$\lim_{x\to\infty} \frac{\overline{F}(x+y)}{\overline{F}(x)} = \mathrm{e}^{-\alpha y} \tag{2.1}$$

holds for all $y \in (-\infty, \infty)$. Furthermore, a distribution F on $[0, \infty)$ is said to belong to the class $\mathcal{S}(\alpha)$ for some $\alpha \geq 0$ if $F \in \mathcal{L}(\alpha)$ and the limit

$$\lim_{x\to\infty} \frac{\overline{F^{2*}}(x)}{\overline{F}(x)} = 2c \tag{2.2}$$

exists and is finite, where F^{2*} denotes the 2-fold distribution convolution of F, i.e., $F^{2*}(x) = \int_{0-}^{x} F(x-y)F(\mathrm{d}y)$ for all $x \geq 0$. For later use, we write $F^{1*} = F$ and $F^{n*} = F^{(n-1)*} * F$ for $n = 2, 3, \ldots$. It is known that the constant c in (2.2) is equal to $\int_{0-}^{\infty} \mathrm{e}^{\alpha x} F(\mathrm{d}x)$; see Rogozin (2000) and references therein. In the literature, a distribution F in $\mathcal{L}(\alpha)$ with $\alpha > 0$ is usually said to have an exponential-like tail, and F in $\mathcal{S}(\alpha)$ with $\alpha > 0$ is said to have a convolution-equivalent tail.

We shall assume that the Lévy measure ρ of the Lévy process L in our model has a tail $\overline{\rho}$ asymptotically equivalent to a convolution-equivalent tail. This is a natural assumption when studying the tail probability of Lévy processes. In risk theory, this assumption has recently been used by e.g. Klüppelberg *et al.* (2004) and Doney and Kyprianou (2006).

Note that $\mathcal{S}(0) = \mathcal{S}$ is the well-known subexponential class. A useful subclass of $\mathcal{S}$ is $\mathcal{S}^*$. By definition, a distribution F on $[0, \infty)$ is said to belong to the class $\mathcal{S}^*$ if $\overline{F}(x) > 0$ for all

$x \geq 0$, $\mu_F = \int_0^\infty \overline{F}(x)\mathrm{d}x < \infty$, and

$$\lim_{x\to\infty} \int_0^x \frac{\overline{F}(x-y)}{\overline{F}(x)} \overline{F}(y)\mathrm{d}y = 2\mu_F.$$

This class was introduced by Klüppelberg (1988), who pointed out that if $F \in \mathcal{S}^*$, then both $F \in \mathcal{S}$ and $F_I \in \mathcal{S}$, where F_I denotes the integrated tail distribution of F, defined as

$$F_I(x) = \frac{1}{\mu_F} \int_0^x \overline{F}(y)\mathrm{d}y, \qquad x \geq 0.$$

According to Chover *et al.* (1973) and Klüppelberg (1989), a measurable function $f : [0, \infty) \to [0, \infty)$, not necessarily a probability density function on $[0, \infty)$, is said to belong to the class $\mathcal{L}_d(\alpha)$ for some $\alpha \geq 0$ if $f(x) > 0$ for all large x and the relation

$$\lim_{x\to\infty} \frac{f(x+y)}{f(x)} = \mathrm{e}^{-\alpha y} \tag{2.3}$$

holds for all $y \in (-\infty, \infty)$. Furthermore, a measurable function $f : [0, \infty) \to [0, \infty)$ is said to belong to the class $\mathcal{S}_d(\alpha)$ for some $\alpha \geq 0$ if $f \in \mathcal{L}_d(\alpha)$ and the limit

$$\lim_{x\to\infty} \frac{f^{2\star}(x)}{f(x)} = 2c \tag{2.4}$$

exists and is finite, where $f^{2\star}$ denotes the 2-fold density convolution of f, i.e., $f^{2\star}(x) = \int_0^x f(x-y)f(y)\mathrm{d}y$ for all $x \geq 0$. For later use, we write $f^{1\star} = f$ and $f^{n\star} = f^{(n-1)\star} \star f$ for $n = 2, 3, \ldots$. It is known that the constant c in (2.4) is equal to $\int_0^\infty \mathrm{e}^{\alpha x} f(x)\mathrm{d}x$. For a distribution F with a density $f \in \mathcal{L}_d(\alpha)$ for some $\alpha > 0$, it is easy to see that $f(x)/\overline{F}(x) \to \alpha$. Furthermore, for this case $F \in \mathcal{S}(\alpha)$ if and only if $f \in \mathcal{S}_d(\alpha)$. The convergence in both (2.1) and (2.3) is automatically uniform on compact y-intervals. See Klüppelberg (1989) for these assertions.

Lemma 2.1 *(Embrechts and Goldie (1982)) If $F \in \mathcal{S}(\alpha)$ for some $\alpha \geq 0$, then for every $n = 1, 2, \ldots$,*

$$\lim_{x\to\infty} \frac{\overline{F^{n*}}(x)}{\overline{F}(x)} = n \left(\int_{0-}^\infty \mathrm{e}^{\alpha y} F(\mathrm{d}y) \right)^{n-1}.$$

Furthermore, for every $\varepsilon > 0$ there exists some $K_\varepsilon > 0$ such that for all $n = 1, 2, \ldots$ and $x > 0$,

$$\overline{F^{n*}}(x) \leq K_\varepsilon \left(\varepsilon + \int_{0-}^\infty \mathrm{e}^{\alpha y} F(\mathrm{d}y) \right)^n \overline{F}(x).$$

Lemma 2.2 *(Chover* et al. *(1973); Klüppelberg (1989)) If $f \in \mathcal{S}_d(\alpha)$ for some $\alpha \geq 0$, then for every $n = 1, 2, \ldots$,*

$$\lim_{x\to\infty} \frac{f^{n\star}(x)}{f(x)} = n \left(\int_0^\infty \mathrm{e}^{\alpha y} f(y)\mathrm{d}y \right)^{n-1}.$$

Furthermore, if f is bounded, then for every $\varepsilon > 0$ there exists some $K_\varepsilon > 0$ such that for all $n = 1, 2, \dots$ and $x > 0$,

$$f^{n\star}(x) \le K_\varepsilon \left(\varepsilon + \int_0^\infty \mathrm{e}^{\alpha y} f(y) \mathrm{d}y \right)^n f(x).$$

3 The Case of Subexponential Tails

In our first main result below we look at the case that the Lévy measure ρ has a subexponential tail.

Theorem 3.1 *Consider the Lévy insurance model introduced in Section 1. If both Π and Π_I belong to the class $\mathcal{S}$ (which is satisfied when $\Pi \in \mathcal{S}^*$), then for every $0 \le \gamma < 1$ and $0 \le \delta < 1$ for which relation (1.4) holds, we have*

$$\psi_{\gamma,\delta}(x) \sim \frac{1}{(1-\gamma)\mu_- - (1-\delta)\mu_+} \int_x^\infty \overline{\rho}(y) \mathrm{d}y. \tag{3.1}$$

Klüppelberg *et al.* (2004) systematically studied the asymptotic behavior of the ruin probability in the Lévy insurance model without tax or reinsurance. Restricting to the case that L is spectrally positive with Lévy measure ρ such that $\Pi \in \mathcal{S}^*$, we see that Theorem 6.2(i) of Klüppelberg *et al.* (2004) corresponds to our Theorem 3.1 with $\gamma = \delta = 0$.

Clearly, the tax identity (1.1) under the loss-carry-forward taxation implies that

$$\psi_\gamma(x) \sim \frac{1}{1-\gamma} \psi_0(x); \tag{3.2}$$

see also Albrecher and Hipp (2007). While under our periodic taxation, plugging $\delta = 0$ into (3.1) yields that

$$\psi_{\gamma,0}(x) \sim \frac{1}{1 - \gamma \frac{\mu_-}{\mu_- - \mu_+}} \psi_{0,0}(x). \tag{3.3}$$

Note that $\psi_0(x)$ in (3.2) and $\psi_{0,0}(x)$ in (3.3) are identical. The coefficients in relations (3.2) and (3.3) respectively capture the impact of the two taxation rules on the asymptotic behavior of the ruin probability. Now that $\mu_-/(\mu_- - \mu_+) > 1$ in (3.3), comparing (3.2) with (3.3) we conclude that, at least for the current heavy-tailed case, periodic taxation produces more significant impact on the ruin probability than the loss-carry-forward taxation does. As explained in Section 1, this is natural since with loss-carry-forward taxation one does not need to pay tax until a large loss is fully recuperated, whereas with periodic taxation every time unit counts anew.

To prove Theorem 3.1, we need the following two lemmas:

Lemma 3.1 *Let L be a Lévy process with Lévy measure ρ such that $\Pi \in \mathcal{S}$. Then*

$$\Pr\left(\sup_{0\le t\le 1} L_t > x\right) \sim \overline{\rho}(x). \tag{3.4}$$

Lemma 3.2 *Let random pairs (X_n, Y_n), $n = 1, 2, \ldots$, be i.i.d. copies of a random pair (X, Y). Denote $M = X \vee Y$. If $-\infty < \mathrm{E}X < 0$, $\mathrm{E}|M| < \infty$, and $\int_x^\infty \Pr(M > y)\,\mathrm{d}y$ is asymptotically equivalent to a subexponential tail, then*

$$\Pr\left(\sup_{n\ge 1}\left(\sum_{k=1}^{n-1} X_k + Y_n\right) > x\right) \sim \frac{1}{|\mathrm{E}X|}\int_x^\infty \Pr(M > y)\,\mathrm{d}y.$$

Lemma 3.1 is an implication of Theorem 3.1 of Rosiński and Samorodnitsky (1993) (see the example of Lévy motion on their page 1006). Lemma 3.2 is exactly Theorem 1 of Palmowski and Zwart (2007).

Proof of Theorem 3.1. Recall (1.2) in which the random pairs (X_n, Y_n), $n = 1, 2, \ldots$, are i.i.d. copies of the random pair (X, Y) given in (1.3). Use the notation $M = X \vee Y$ in Lemma 3.2. Since $\Pi \in \mathcal{S}$, from Lemma 3.1 we have

$$\Pr(Y > x) \sim \overline{\rho}(x). \tag{3.5}$$

It is clear that $Y \ge L_1^+ \ge X^+$. Hence by (3.5) and $\Pi_I \in \mathcal{S}$,

$$\int_x^\infty \Pr(M > y)\,\mathrm{d}y = \int_x^\infty \Pr(Y > y)\,\mathrm{d}y \sim \int_x^\infty \overline{\rho}(y)\mathrm{d}y,$$

a subexponential tail. Then by Lemma 3.2, we obtain (3.1). ■

4 The Case of Convolution-equivalent Tails

Next, we consider the case that the Lévy measure ρ has a light tail such that $\Pi \in \mathcal{S}(\alpha)$ for some $\alpha > 0$.

Theorem 4.1 *Consider the Lévy insurance model introduced in Section 1. Assume $\mathrm{E}L_1^2 < \infty$ and $\Pi \in \mathcal{S}(\alpha)$ for some $\alpha > 0$. If $0 \le \gamma < 1$ and $0 < \delta < 1$ are such that*

$$\mathrm{E}e^{\alpha'\left((1-\delta)L_1^+ - (1-\gamma)L_1^-\right)} < 1 \tag{4.1}$$

for some $\alpha' > \alpha$, then

$$\psi_{\gamma,\delta}(x) \sim \frac{C_\alpha}{1 - \mathrm{E}e^{\alpha\left((1-\delta)L_1^+ - (1-\gamma)L_1^-\right)}}\overline{\rho}(x), \tag{4.2}$$

where the constant C_α is defined as

$$C_\alpha = \lim_{x\to\infty}\frac{\Pr(Y > x)}{\overline{\rho}(x)} = \lim_{x\to\infty}\frac{\Pr\left(\sup_{0\le t\le 1} L_t > x\right)}{\overline{\rho}(x)} \in (0, \infty). \tag{4.3}$$

The existence of the limit C_α in (4.3) was proved by Braverman and Samorodnitsky (1995). Condition (4.1) is feasible in view of (1.4) and $\mathrm{E}e^{\alpha L_1^+} < \infty$.

Lemma 3.1 of Tang (2007) says that, for $\alpha > 0$, $\Pi \in \mathcal{L}(\alpha)$ if and only if $\overline{\rho}(x) \sim \alpha \int_x^\infty \overline{\rho}(y)\mathrm{d}y$. Therefore, relation (4.2) may be rewritten as

$$\psi_{\gamma,\delta}(x) \sim \frac{\alpha C_\alpha}{1 - \mathrm{E}e^{\alpha\left((1-\delta)L_1^+ - (1-\gamma)L_1^-\right)}} \int_x^\infty \overline{\rho}(y)\mathrm{d}y. \tag{4.4}$$

With the understanding that $C_0 = 1$ by relation (3.4) and that the fraction on the right-hand side of relation (4.4) converges to $\left((1-\gamma)\mu_- - (1-\delta)\mu_+\right)^{-1}$ as $\alpha \to 0$, relation (3.1) in Theorem 3.1 indicates that relation (4.4) still holds when $\alpha = 0$.

To prove Theorem 4.1 we need the following lemma, which corresponds to the light-tailed case of Theorem 1.1(i) of Hao *et al.* (2009):

Lemma 4.1 *Let random pairs (X_n, Y_n), $n = 1, 2, \ldots$, be i.i.d. copies of a random pair (X, Y). If $\mathrm{E}X < 0$, $\mathrm{E}X^2 < \infty$, the distribution of Y belongs to $\mathcal{L}(\alpha)$ for some $\alpha > 0$, and $\mathrm{E}e^{\alpha' X} < 1$ for some $\alpha' > \alpha$, then*

$$\Pr\left(\sup_{n\ge1}\left(\sum_{k=1}^{n-1} X_k + Y_n\right) > x\right) \sim \frac{1}{1 - \mathrm{E}e^{\alpha X}} \Pr\left(Y > x\right).$$

Proof of Theorem 4.1. Use the notation in (1.3). By relation (4.3) and closure of the class $\mathcal{S}(\alpha)$ under tail equivalence, the distribution of Y also belongs to the class $\mathcal{S}(\alpha)$. The moment conditions on X required in Lemma 4.1 are clearly satisfied. Then, using Lemma 4.1 we obtain (4.2). ∎

To apply Theorem 4.1, an immediate problem is how to determine the constant C_α in (4.3). This has been a very difficult problem for a Lévy process L whose Lévy measure ρ has a convolution-equivalent tail. For related discussions see Albin and Sundén (2009) and references therein. The following lemma gives an expression for C_α:

Lemma 4.2 *Let L be a Lévy process with Lévy measure ρ such that $\Pi \in \mathcal{S}(\alpha)$ for some $\alpha > 0$. Then for all $t > 0$,*

$$\lim_{x\to\infty} \frac{\Pr\left(L_t > x\right)}{\overline{\rho}(x)} = t\mathrm{E}e^{\alpha L_t} = h(t).$$

There is a unique probability distribution G on $[0, 1]$ satisfying $\int_0^1 t^{-1}G(\mathrm{d}t) < \infty$ with moments given by

$$\mu_n(G) = \frac{v_n(n+1)!}{\int_0^1 h(t)\mathrm{d}t}, \qquad n = 1, 2, \ldots,$$

where

$$v_n = \int_{0<t_1\leq\cdots\leq t_{n+1}\leq 1} t_1 \mathrm{E}\mathrm{e}^{\alpha \min_{1\leq k\leq n+1} L_{t_k}} \mathrm{d}t_1 \cdots \mathrm{d}t_{n+1}. \tag{4.5}$$

Finally,

$$\lim_{x\to\infty} \frac{\Pr\left(\sup_{0\leq t\leq 1} L_t > x\right)}{\overline{\rho}(x)} = \int_0^1 t^{-1} G(\mathrm{d}t) \int_0^1 h(t)\mathrm{d}t = C_\alpha. \tag{4.6}$$

Lemma 4.2 is a combination of Proposition 1.3 and Theorem 2.1 of Braverman (1997). Here we need to point out that the constants v_n defined by Braverman (1997) are not correct. This is due to a calculation error in his Lemma 3.1. Indeed, under his assumptions and in his notation, instead of his relation (3.1) we should have

$$\Pr\left(\Sigma_k > x, 1 \leq k \leq n\right) \sim \overline{F_1}(x) \mathrm{E}\mathrm{e}^{\alpha\left(\min_{1\leq k\leq n} \Sigma_k - X_1\right)},$$

where $\Sigma_k = \sum_{i=1}^k X_i, 1 \leq k \leq n$. Therefore, to qualify his Theorem 2.1, the constants v_n should be given by our (4.5) above. However, we remark that the expression for C_α given in (4.6) is far from being explicit and can not be evaluated unless L is a subordinator.

To pursue a more explicit expression for C_α, we then restrict the Lévy process L to a compound Poisson process with negative drift:

$$L_t = \sum_{k=1}^{N_t} \xi_k - pt, \qquad t \geq 0, \tag{4.7}$$

where $p > 0$ represents the constant premium rate, N is a Poisson process with intensity $\lambda > 0$, and $\xi_1, \xi_2, \ldots$ are i.i.d. copies of a random variable ξ independent of N and with distribution F on $(0, \infty)$.

Corollary 4.1 *Consider the Lévy insurance model introduced in Section 1 in which the loss process L is given by (4.7). Suppose that F has a bounded density $f \in \mathcal{S}_d(\alpha)$ for some $\alpha > 0$ and that condition (4.1) holds. Then*

$$\psi_{\gamma,\delta}(x) \sim \frac{\lambda C_\alpha}{1 - \mathrm{E}\mathrm{e}^{\alpha\left((1-\delta)L_1^+ - (1-\gamma)L_1^-\right)}} \overline{F}(x)$$

with the constant C_α given by

$$C_\alpha = \mathrm{e}^{\lambda\left(\mathrm{E}\mathrm{e}^{\alpha\xi}-1\right)-\alpha p} + \alpha \int_0^1 \int_0^t \Pr\left(\sum_{k=1}^{N_t} \xi_k \leq ps\right) \mathrm{d}s \frac{1-t}{t} \mathrm{e}^{\lambda(1-t)\left(\mathrm{E}\mathrm{e}^{\alpha\xi}-1\right)-\alpha p(1-t)} \mathrm{d}t. \tag{4.8}$$

For example, if F is an inverse Gaussian distribution with density

$$f(x) = \left(\frac{a}{2\pi x^3}\right)^{1/2} \exp\left(\frac{-a(x-b)^2}{2b^2 x}\right), \qquad a, b, x > 0,$$

which is a typical example of $f \in \mathcal{S}_d(\alpha)$ with $\alpha = a/(2b^2)$, then we can appropriately choose the constants p, γ, and δ such that condition (4.1) is satisfied.

While the expression for C_α defined in (4.8) is still not completely explicit, with the only unknown part $\int_0^t \Pr\left(\sum_{k=1}^{N_t} \xi_k \le ps\right) \mathrm{d}s$ for $0 < t \le 1$, it is simple enough for simulations, especially when ξ follows an inverse Gaussian distribution.

To prove Corollary 4.1, we need a result below. Let $F(\cdot, t)$ be the distribution of aggregate claims,

$$F(x,t) = \Pr\left(\sum_{k=1}^{N_t} \xi_k \le x\right),$$

and let $f(\cdot, t)$ be its density. Write $Y_t = \sup_{0 \le s \le t} L_s$. Then $Y_1 = Y$. The lemma below is a combination of Theorems 2.1 and 2.2 of Asmussen (2000):

Lemma 4.3 *For the compound Poisson model (4.7), we have*

$$\Pr(Y_t \le 0) = \frac{1}{t}\int_0^t F(ps, t)\,\mathrm{d}s, \qquad t > 0,$$

and

$$1 - \Pr(Y_T > x) = F(x + pT, T) - \int_0^T \Pr(Y_{T-t} \le 0)\, f(x + pt, t)\,\mathrm{d}t, \qquad T > 0.$$

Proof of Corollary 4.1. By Theorem 4.1, it suffices to verify (4.8). By Lemma 4.3, we have

$$\Pr(Y > x) = \overline{F}(x + p, 1) + \int_0^1 \Pr(Y_{1-t} \le 0)\, f(x + pt, t)\,\mathrm{d}t = I_1(x) + I_2(x). \tag{4.9}$$

Since $f \in \mathcal{S}_d(\alpha)$ for $\alpha > 0$ implies $F \in \mathcal{S}(\alpha)$, we apply the dominated convergence theorem justified by Lemma 2.1 to obtain

$$\begin{aligned} I_1(x) &= \sum_{n=1}^{\infty} \Pr\left(\sum_{k=1}^{n} \xi_k > x + p\right) \Pr(N_1 = n) \\ &\sim \sum_{n=1}^{\infty} n\left(\mathrm{E}\mathrm{e}^{\alpha\xi}\right)^{n-1} \overline{F}(x+p) \frac{\lambda^n}{n!}\mathrm{e}^{-\lambda} \\ &\sim \lambda \mathrm{e}^{\lambda\left(\mathrm{E}\mathrm{e}^{\alpha\xi}-1\right)-\alpha p}\overline{F}(x). \end{aligned} \tag{4.10}$$

Similarly, by Lemma 2.2, for each fixed $t \in (0, 1]$,

$$f(x + pt, t) \sim \lambda t \mathrm{e}^{\lambda t\left(\mathrm{E}\mathrm{e}^{\alpha\xi}-1\right)-\alpha pt} f(x). \tag{4.11}$$

In order to plug (4.11) into $I_2(x)$ in (4.9), we need to apply the dominated convergence theorem again. We notice that, by Lemma 2.2, there exists some $K > 0$ such that for all

$x \geq 0$ for which $f(x) > 0$ and for all $t \in (0,1]$,

$$\begin{aligned}\frac{\Pr\left(Y_{1-t} \leq 0\right) f\left(x+pt,t\right)}{f\left(x\right)} &\leq \sum_{n=1}^{\infty} \frac{\left(\lambda t\right)^n}{n!} \mathrm{e}^{-\lambda t} \cdot \frac{f^{n\star}\left(x+pt\right)}{f\left(x+pt\right)} \cdot \frac{f\left(x+pt\right)}{f\left(x\right)} \\ &\leq \sum_{n=1}^{\infty} \frac{\left(\lambda t\right)^n}{n!} \mathrm{e}^{-\lambda t} \cdot K\left(\mathrm{E}\mathrm{e}^{\alpha\xi}+1\right)^n \\ &\leq K\mathrm{e}^{\lambda \mathrm{E}\mathrm{e}^{\alpha\xi}},\end{aligned}$$

where in the second step we used the local uniformity of the convergence in (2.3). Then, applying the dominated convergence theorem and using Lemma 4.3 again,

$$I_2(x) \sim f(x) \int_0^1 \left(\frac{1}{1-t} \int_0^{1-t} F\left(ps, 1-t\right) \mathrm{d}s\right) \lambda t \mathrm{e}^{\lambda t\left(\mathrm{E}\mathrm{e}^{\alpha\xi}-1\right)-\alpha pt} \mathrm{d}t. \tag{4.12}$$

Plugging (4.10) and (4.12) into (4.9) and using (4.3) and the facts $\overline{\rho}(\cdot) = \lambda\overline{F}(\cdot)$, $f(x)/\overline{F}(x) \to \alpha$, we obtain (4.8). ■

5 The Case of Exponential-like Tails

Finally, we consider the case that the Lévy measure ρ has a light tail such that $\Pi \in \mathcal{L}(\alpha)\backslash\mathcal{S}(\alpha)$ for some $\alpha > 0$.

Theorem 5.1 *Consider the Lévy insurance model introduced in Section 1. Assume* $\mathrm{E}L_1^2 < \infty$*,* $\Pi \in \mathcal{L}(\alpha)$ *for some* $\alpha > 0$*, and* $\overline{\Pi}(x) = o\left(\overline{\Pi^{2*}}(x)\right)$*. If* $0 \leq \gamma < 1$ *and* $0 < \delta < 1$ *are such that condition (4.1) holds, then*

$$\psi_{\gamma,\delta}(x) \sim \frac{1}{1-\mathrm{E}\mathrm{e}^{\alpha\left((1-\delta)L_1^+-(1-\gamma)L_1^-\right)}} \Pr(L_1 > x). \tag{5.1}$$

We need the following result, which is a combination of Theorem 3.3 and Corollary 6.2 of Albin and Sundén (2009):

Lemma 5.1 *Let* L *be a Lévy process with Lévy measure* ρ *such that* $\Pi \in \mathcal{L}(\alpha)$ *for some* $\alpha > 0$ *and* $\overline{\Pi}(x) = o\left(\overline{\Pi^{2*}}(x)\right)$*. Then for all* $t > 0$*, the distribution of* L_t *belongs to* $\mathcal{L}(\alpha)\backslash\mathcal{S}(\alpha)$ *and*

$$\Pr\left(\sup_{0\leq s\leq t} L_s > x\right) \sim \Pr\left(L_t > x\right).$$

Note that the conditions on the Lévy measure ρ in Lemma 5.1 are for instance fulfilled if $\overline{\rho}$ is asymptotically equivalent to the tail of an exponential distribution, a gamma distribution, or, more generally, an Erlang distribution.

Proof of Theorem 5.1. Use the notation in (1.3). By Lemma 5.1 we know that the distribution of L_1 belongs to $\mathcal{L}(\alpha)\backslash\mathcal{S}(\alpha)$ and

$$\Pr\left(Y > x\right) \sim \Pr\left(L_1 > x\right).$$

Hence, the distribution of Y belongs to $\mathcal{L}(\alpha)\backslash\mathcal{S}(\alpha)$ as well. Finally, using Lemma 4.1 again we obtain relation (5.1). ■

The asymptotic relation (5.1) is in terms of the tail of L_1 instead of the tail of the Lévy measure ρ. In case the tail of L_1 is unknown, relation (5.1) is not completely explicit. We are going to show two special, but important, cases of Theorem 5.1 in which a completely explicit asymptotic relation for the ruin probability is given.

First, we consider a gamma process $U = (U_t)_{t\geq 0}$, which starts with 0, has stationary and independent increments, with U_1 having a gamma(α, β) distribution with density

$$g(x) = \frac{\alpha^\beta}{\Gamma(\beta)} x^{\beta-1} \mathrm{e}^{-\alpha x}, \qquad \alpha, \beta, x > 0.$$

Its Lévy triplet is given by $a = \beta\left(\mathrm{e}^{-\alpha} - 1\right)/\alpha$, $\sigma = 0$, and $\rho(\mathrm{d}x) = \beta x^{-1}\mathrm{e}^{-\alpha x}\mathrm{d}x$; see Section 1.2.4 of Kyprianou (2006) for details. For this case, it is easy to verify that $\overline{\Pi}(x) = o\left(\overline{\Pi^{2*}}(x)\right)$. By Theorem 5.1, we immediately have the following:

Corollary 5.1 *Consider the Lévy insurance model introduced in Section 1. Assume*

$$L_t = U_t - pt, \qquad t \geq 0,$$

where $p > 0$ and U is a gamma process as introduced above with parameters $\alpha, \beta > 0$. If $0 \leq \gamma < 1$ and $0 < \delta < 1$ are such that condition (4.1) holds, then

$$\psi_{\gamma,\delta}(x) \sim \frac{\alpha^{\beta-1}\left(x+p\right)^{\beta-1}\mathrm{e}^{-\alpha(x+p)}}{\left(1 - \mathrm{E}\mathrm{e}^{\alpha\left((1-\delta)L_1^+ - (1-\gamma)L_1^-\right)}\right)\Gamma(\beta)}.$$

Next, we again consider the compound Poisson process with negative drift. The following is another corollary of Theorem 5.1:

Corollary 5.2 *Consider the Lévy insurance model introduced in Section 1 in which the loss process L is given by (4.7). Suppose F is an exponential distribution with mean $1/\alpha$. If $0 \leq \gamma < 1$ and $0 < \delta < 1$ are such that condition (4.1) holds, then*

$$\psi_{\gamma,\delta}(x) \sim \frac{1}{1 - \mathrm{E}\mathrm{e}^{\alpha\left((1-\delta)L_1^+ - (1-\gamma)L_1^-\right)}} \frac{\sqrt{\lambda}}{\pi\sqrt{\alpha x}} \mathrm{e}^{-\alpha(x+p)-\lambda} \int_0^{\pi/2} \mathrm{e}^{2\sqrt{\alpha\lambda}\sqrt{x+p}\cos\theta}\mathrm{d}\theta. \tag{5.2}$$

To prove Corollary 5.2 we need an elementary result below:

Lemma 5.2 *Let $l(\cdot)$ be a bounded function on $[-1,1]$, left continuous at 1 with $l(1)\neq 0$, and let $c>0$ be a constant. Then*

$$\int_0^{\pi} \mathrm{e}^{cx\cos\theta} l(\cos\theta)\mathrm{d}\theta \sim l(1)\int_0^{\pi/2} \mathrm{e}^{cx\cos\theta}\mathrm{d}\theta. \tag{5.3}$$

Proof. Choose some small $0<\varepsilon\leq\pi/2$ such that $|l(x)|\geq 0.5\,|l(1)|>0$ for all $x\in[\cos\varepsilon,1]$. We split the integral on the left-hand side of (5.3) into two parts as $\int_0^{\varepsilon}+\int_{\varepsilon}^{\pi}$. It is easy to see that the second part is asymptotically negligible, as

$$\frac{\int_{\varepsilon}^{\pi} \mathrm{e}^{cx\cos\theta} l(\cos\theta)\mathrm{d}\theta}{\int_0^{\varepsilon} \mathrm{e}^{cx\cos\theta} l(\cos\theta)\mathrm{d}\theta} = O(1)\frac{\mathrm{e}^{cx\cos\varepsilon}}{\int_0^{\varepsilon/2}\mathrm{e}^{cx\cos\theta}\mathrm{d}\theta} \to 0.$$

Hence,

$$\int_0^{\pi} \mathrm{e}^{cx\cos\theta} l(\cos\theta)\mathrm{d}\theta \sim \int_0^{\varepsilon} \mathrm{e}^{cx\cos\theta} l(\cos\theta)\mathrm{d}\theta. \tag{5.4}$$

Note that, since l is left continuous at 1, if $\varepsilon>0$ in (5.4) is chosen to be sufficiently close to 0, then $l(\cos\theta)$ is sufficiently close to $l(1)$. Moreover, $\int_0^{\varepsilon}\mathrm{e}^{cx\cos\theta}\mathrm{d}\theta\sim\int_0^{\pi/2}\mathrm{e}^{cx\cos\theta}\mathrm{d}\theta$. Therefore, by (5.4) and the arbitrariness of $\varepsilon>0$, we obtain (5.3). ■

Under the conditions of Corollary 5.2, the Lévy triplet of L is given by $a=p-\lambda\int_0^1\alpha x\mathrm{e}^{-\alpha x}\mathrm{d}x$, $\sigma=0$, and $\rho(\mathrm{d}x)=\lambda\alpha\mathrm{e}^{-\alpha x}\mathrm{d}x$; see Section 1.2.2 of Kyprianou (2006) for details.

Proof of Corollary 5.2. Clearly, $\mathrm{E}L_1^2<\infty$, $\Pi\in\mathcal{L}(\alpha)$, and $\overline{\Pi}(x)=o\left(\overline{\Pi^{2*}}(x)\right)$. Therefore by Theorem 5.1, we only need to focus on derivation of the tail probability $\Pr(L_1>x)$. Since the n-fold convolution of an exponential distribution with mean $1/\alpha$ is a gamma distribution with parameters (α,n), we have

$$\begin{aligned}\Pr(L_1>x) &= \int_{x+p}^{\infty}\sum_{n=1}^{\infty}\frac{\alpha^n}{(n-1)!}y^{n-1}\mathrm{e}^{-\alpha y}\cdot\frac{\lambda^n}{n!}\mathrm{e}^{-\lambda}\mathrm{d}y\\ &= \sqrt{\alpha\lambda}\mathrm{e}^{-\lambda}\int_{x+p}^{\infty}\sum_{n=0}^{\infty}\frac{\left(\sqrt{\alpha\lambda y}\right)^{2n+1}}{n!(n+1)!}y^{-1/2}\mathrm{e}^{-\alpha y}\mathrm{d}y. \end{aligned}\tag{5.5}$$

The last series in the above is of the structure of the modified Bessel function of order 1; that is,

$$\sum_{n=0}^{\infty}\frac{\left(\sqrt{\alpha\lambda y}\right)^{2n+1}}{n!(n+1)!} = \frac{1}{\pi}\int_0^{\pi}\mathrm{e}^{2\sqrt{\alpha\lambda y}\cos\theta}\cos\theta\mathrm{d}\theta. \tag{5.6}$$

Using Lemma 5.2, as $y\to\infty$,

$$\int_0^{\pi}\mathrm{e}^{2\sqrt{\alpha\lambda y}\cos\theta}\cos\theta\mathrm{d}\theta \sim \int_0^{\pi/2}\mathrm{e}^{2\sqrt{\alpha\lambda y}\cos\theta}\mathrm{d}\theta.$$

Plugging this into (5.6) then plugging (5.6) into (5.5), we obtain

$$\begin{aligned}\Pr(L_1 > x) &\sim \frac{\sqrt{\alpha\lambda}}{\pi}\mathrm{e}^{-\lambda}\int_{x+p}^{\infty}\left(\int_0^{\pi/2}\mathrm{e}^{2\sqrt{\alpha\lambda y}\cos\theta}\mathrm{d}\theta\right)y^{-1/2}\mathrm{e}^{-\alpha y}\mathrm{d}y \\ &= \frac{2\sqrt{\alpha\lambda}}{\pi}\mathrm{e}^{-\lambda}\int_0^{\pi/2}\int_{\sqrt{x+p}}^{\infty}\mathrm{e}^{-\alpha u^2+2u\sqrt{\alpha\lambda}\cos\theta}\mathrm{d}u\mathrm{d}\theta. \end{aligned}\tag{5.7}$$

We are going to simplify this expression. It is easy to see that, for each real number c,

$$\lim_{z\to\infty} 2\alpha z\int_0^{\infty}\mathrm{e}^{-\alpha(u^2+2zu)+cu}\mathrm{d}u = 1. \tag{5.8}$$

It follows that the relation

$$\int_z^{\infty}\mathrm{e}^{-\alpha u^2+2u\sqrt{\alpha\lambda}\cos\theta}\mathrm{d}u \sim \frac{1}{2\alpha z}\mathrm{e}^{-\alpha z^2+2z\sqrt{\alpha\lambda}\cos\theta}, \qquad z\to\infty, \tag{5.9}$$

holds uniformly for all $\theta\in(-\infty,\infty)$; that is,

$$\lim_{z\to\infty}\sup_{-\infty<\theta<\infty}\left|\frac{2\alpha z\int_z^{\infty}\mathrm{e}^{-\alpha u^2+2u\sqrt{\alpha\lambda}\cos\theta}\mathrm{d}u}{\mathrm{e}^{-\alpha z^2+2z\sqrt{\alpha\lambda}\cos\theta}}-1\right| = 0.$$

Actually, this can easily be verified by applying relation (5.8) to the upper and lower bounds shown below:

$$2\alpha z\int_0^{\infty}\mathrm{e}^{-\alpha(u^2+2zu)-2u\sqrt{\alpha\lambda}}\mathrm{d}u \le \frac{2\alpha z\int_z^{\infty}\mathrm{e}^{-\alpha u^2+2u\sqrt{\alpha\lambda}\cos\theta}\mathrm{d}u}{\mathrm{e}^{-\alpha z^2+2z\sqrt{\alpha\lambda}\cos\theta}} \le 2\alpha z\int_0^{\infty}\mathrm{e}^{-\alpha(u^2+2zu)+2u\sqrt{\alpha\lambda}}\mathrm{d}u.$$

Plugging (5.9) into (5.7), we obtain

$$\Pr(L_1 > x) \sim \frac{\sqrt{\lambda}}{\pi\sqrt{\alpha x}}\mathrm{e}^{-\alpha(x+p)-\lambda}\int_0^{\pi/2}\mathrm{e}^{2\sqrt{x+p}\sqrt{\alpha\lambda}\cos\theta}\mathrm{d}\theta.$$

Finally, plugging this into (5.1) yields (5.2). ■

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