# The Finite-time and Infinite-time Ruin Probabilities of a Bivariate Lévy-Driven Risk Process with Heavy Tails

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# 1. Introduction

The surplus process of an insurance company is modeled by

$$Y_t = x - P_t + \int_0^t Y_{s-} \mathrm{d}R_s. \tag{1}$$

- $x > 0$ is the initial surplus level;
- $P$ is the process representing the loss process in a world without economic factors;
- $R$ is an independent process that describes return on investments.

In this study, we assume that both $P$ and $R$ are Lévy processes and call $Y$ a **bivariate Lévy driven risk process**.

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A Lévy process begins from 0, with càdlàg sample paths and stationary and independent increments.

- The classical Cramér-Lundberg model:

$$Y_t = x + ct - S_t$$

  where c is the constant premium rate and $S$ is a compound Poisson process representing the total claim amount process.

- The Cramér-Lundberg risk process with investment in a Black-Scholes type of market:

$$Y_t = x + ct - S_t + \int_0^t Y_{s-}[(1-\pi)r\mathrm{d}s + \pi(\alpha\mathrm{d}s + \sigma\mathrm{d}W_s)]$$

  where $W$ is a standard Brownian motion and $0 < \pi < 1$, $r \geq 0$, $\alpha > 0$, and $\sigma > 0$.

See Paulsen (2008; *Probab. Surv.*), Klüppelberg and Kostadinova (2008; *Insurance Math. Econom.*), and references therein for more examples.

Let's come back to our bivariate Lévy driven risk process (1). Its solution is given by

$$Y_t = \mathrm{e}^{\tilde{R}_t}\left(x - \int_0^t \mathrm{e}^{-\tilde{R}_s}\mathrm{d}P_s\right) := \mathrm{e}^{\tilde{R}_t}(x - Z_t). \tag{2}$$

Here, $\tilde{R}$:

- also a Lévy process;
- $\mathrm{e}^{\tilde{R}_t}$ is the Doléans-Dade exponential of $R$, i.e.,

$$\mathrm{d}\mathrm{e}^{\tilde{R}_t} = \mathrm{e}^{\tilde{R}_{t-}}\mathrm{d}R_t \quad \text{with } \tilde{R}_0 = 0.$$

See, for example, Paulsen (1998; *Stoch. Process. Appl.*) and Paulsen (2002; *Ann. Appl. Probab.*).

As usual, the finite-time ruin probability by $T \geq 0$ is defined as

$$\psi(x,T) = \Pr\left( \inf_{0\leq t\leq T} Y_t < 0 \middle| Y_0 = x \right),$$

and the infinite-time ruin probability as

$$\psi(x,\infty) = \lim_{T\to\infty} \psi(x,T) = \Pr\left( \inf_{0\leq t<\infty} Y_t < 0 \middle| Y_0 = x \right).$$

Paulsen (2002; *Ann. Appl. Probab.*; p.1256) conjectured an asymptotic formula for $\psi(x,\infty)$ as $x \to \infty$ under some certain conditions.

We prove and extend Paulsen's conjecture by showing that his formula holds to $\psi(x,T)$ where $0 \leq T \leq \infty$.

# 2. Preliminaries

## 2.1. A Lévy process $L$

- Its characteristic exponent $\Phi_L(\cdot)$:

$$\mathrm{E}e^{iuL_t} = e^{-t\Phi_L(u)}.$$

- Its Lévy triplet $(p_L, \sigma_L, \nu_L)$ determines $\Phi_L(\cdot)$.
  - $p_L \in (-\infty, \infty)$
  - $\sigma_L \geq 0$
  - Lévy measure $\nu_L$
- When $\overline{\nu}_L(1) = \nu_L((1,\infty)) > 0$, denote $\Pi_L(\cdot) = \frac{\nu_L(\cdot)\mathbb{1}_{(1,\infty)}}{\overline{\nu}_L(1)}$.
- Its Laplace exponent

$$\varphi_L(u) = -\Phi_L(iu) = \log \mathrm{E}e^{-uL_1}.$$

## 2.2. Distributions of extended regular variation

- $F \in \mathrm{ERV}(-\alpha, -\beta), 0 \le \alpha \le \beta < \infty$, if

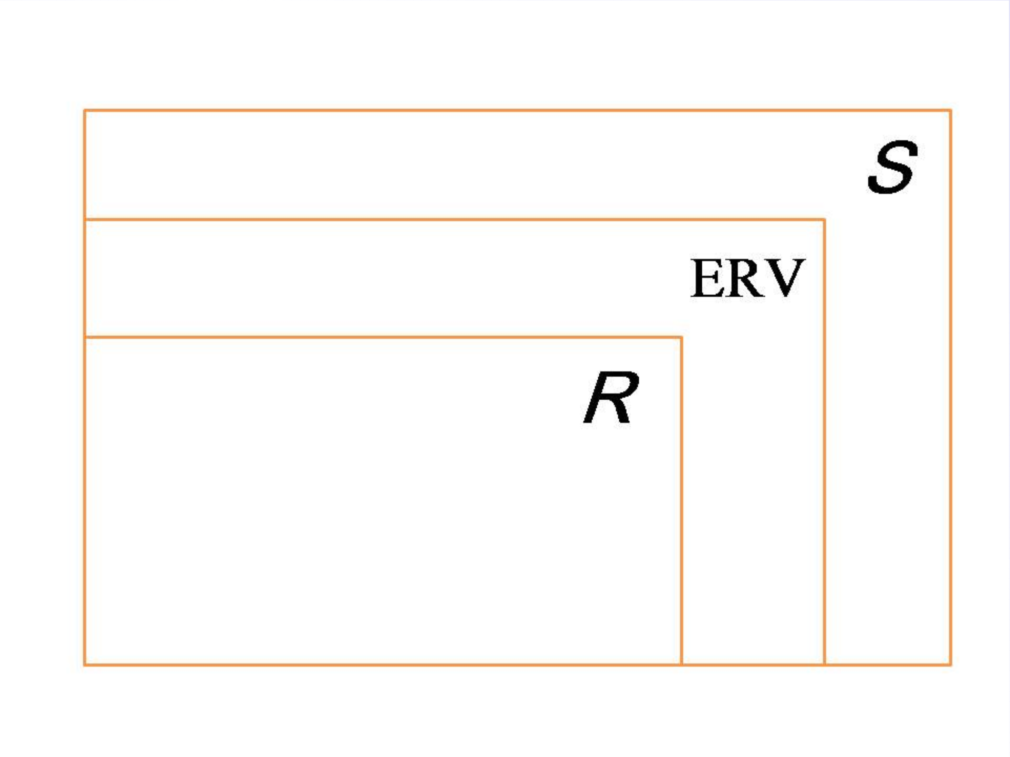
  (*) $\overline{F}(x) = 1 - F(x) > 0$, for all $x$, and

  (*)

$$v^{-\beta} \le \liminf_{x\to\infty} \frac{\overline{F}(vx)}{\overline{F}(x)} \le \limsup_{x\to\infty} \frac{\overline{F}(vx)}{\overline{F}(x)} \le v^{-\alpha}, \quad \text{all } v \ge 1. \qquad (3)$$

- $F \in \mathcal{R}_{-\alpha}$ if $\alpha = \beta$ in relations (3).
- The class ERV is a subset of $\mathcal{S}$, the class of subexponential distributions.

## Relations of classes of distributions:



# 3. Paulsen's Conjecture

Paulsen (2002; *Ann. Appl. Probab.*; p.1256) studied the infinite-time ruin probability of the bivariate Lévy driven risk model and proposed the following conjecture:

**Paulsen's conjecture:** *Consider the surplus process given by (2). If*

- $\Pi_P \in \mathcal{R}_{-\alpha}$ *for some* $\alpha > 0$, *and*
- $\varphi_{\tilde{R}}(\alpha + \varepsilon) < 0$ *for some* $\varepsilon > 0$,

*then*

$$\psi(x, \infty) \sim \frac{\overline{\nu}_P(x)}{|\varphi_{\tilde{R}}(\alpha)|}. \tag{4}$$

**Remark :** Since $\varphi_{\tilde{R}}(\alpha + \varepsilon) < 0$ for some $\varepsilon > 0$, the insurance risk dominates the investment risk. In this case, the ultimate ruin probability is controlled by the probability of large insurance claims. The investment process enters only into the constant $\varphi_{\tilde{R}}(\alpha)$.

# 4. Our Results

**Theorem 1:** *Assume*

- $\Pi_P \in \mathrm{ERV}(-\alpha, -\beta)$ *for some* $0 < \alpha \le \beta < \infty$,
- $\nu_P((-\infty, -x)) = o(\overline{\nu}_P(x))$, *and*
- $\varphi_{\tilde{R}}(\beta + \varepsilon) < 0$ *for some* $\varepsilon > 0$.

*Let* $\lambda = \overline{\nu}_P(1)$ *and* $X$*, distributed by* $\Pi_P$*, be independent of* $P$ *and* $\tilde{R}$*. Then it holds for every* $T \in (0, \infty)$ *that*

$$\psi(x, T) \sim \lambda \int_0^T \Pr\left(X \mathrm{e}^{-\tilde{R}_t} > x\right) \mathrm{d}t. \tag{5}$$

*Furthermore, if*

- $\mathrm{E}\left(\mathrm{e}^{-(\alpha-\varepsilon)\tilde{R}_1} \vee \mathrm{e}^{-(\beta+\varepsilon)\tilde{R}_1}\right) < 1$ *for some* $\varepsilon > 0$,

*then relation (5) also holds with* $T = \infty$.

**Remark :** When $\alpha = \beta$, the condition $\mathrm{E}\left(\mathrm{e}^{-(\alpha-\varepsilon)\tilde{R}_1} \vee \mathrm{e}^{-(\beta+\varepsilon)\tilde{R}_1}\right) < 1$ for some $\varepsilon > 0$ is equivalent to $\varphi_{\tilde{R}}(\beta + \varepsilon) < 0$ for some $\varepsilon > 0$.

**Corollary 1:** *Let $\alpha = \beta$ in Theorem 1. Then it holds for every $T \in (0, \infty]$ that*

$$\psi(x, T) \sim \frac{1 - \mathrm{e}^{\varphi_{\tilde{R}}(\alpha)T}}{|\varphi_{\tilde{R}}(\alpha)|} \overline{\nu}_P(x). \tag{6}$$

Hence, Paulsen's conjecture is true, indeed, under an additional assumption that $\nu_P((-\infty, -x)) = o(\overline{\nu}_P(x))$. Corollary 1 is also consistent with Theorem 4.6a of Klüppelberg and Kostadinova (2008; *Insurance Math. Econom.*).

# 5. Future Problems

- We still don't know whether the additional assumption $\nu_P((-\infty, -x)) = o(\overline{\nu}_P(x))$ is avoidable.
- It's possible to show that relation (5) holds uniformly for all $T \in (0, \infty]$.
- As to the asymptotic relation (5), what is its second order formula?
- If more than one investment asset are available, we need to use multivariate Lévy process instead of univariate Lévy process.

# References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