On the Impact of Periodic Tax Payments on Asymptotic Ruin Probabilities of the Lévy Insurance Model

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1. Motivations

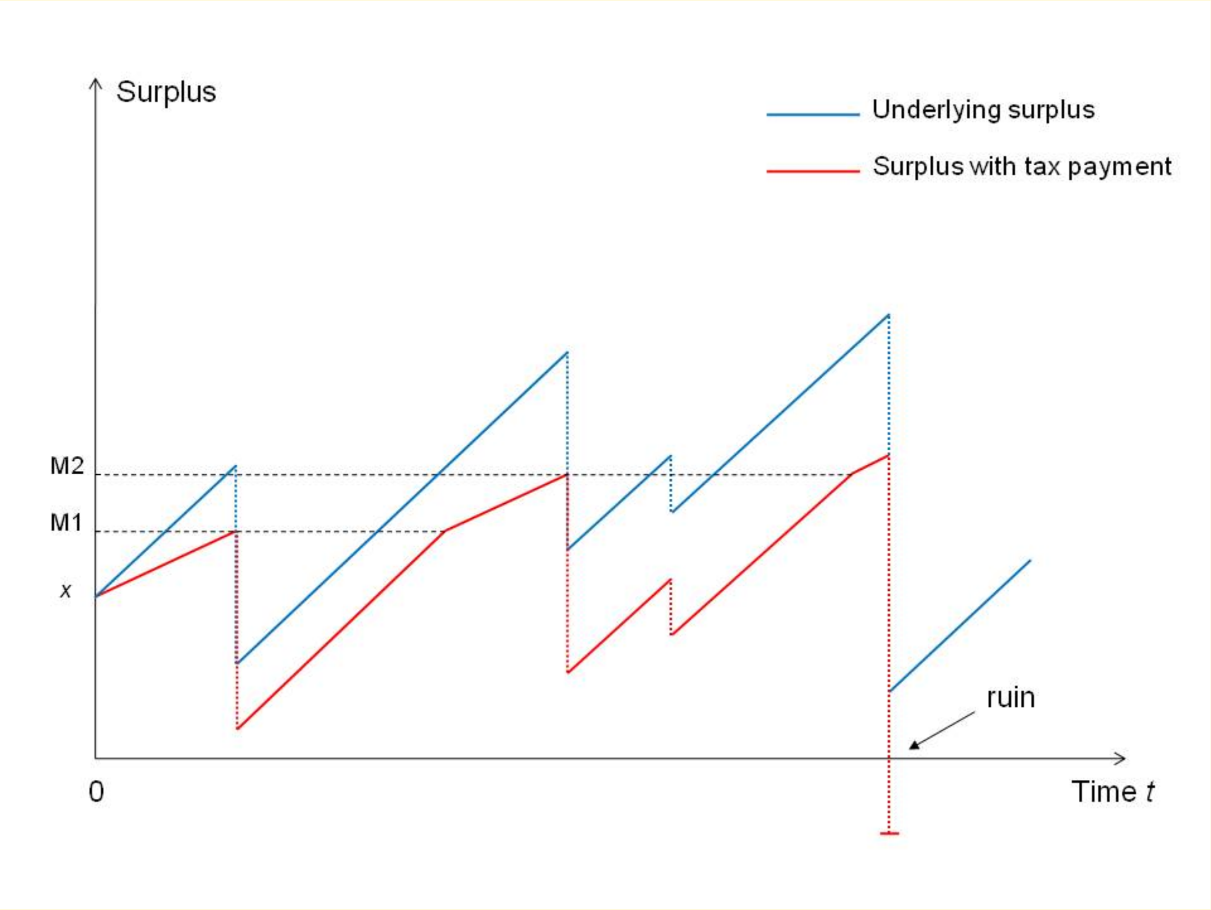
The ruin probability of an insurance company is the probability that its surplus process falls below 0 at some time.

Recently, the influence of tax payment on the ruin probability has become an interesting problem in actuarial science. (See references.)

Their common assumptions:

- $S = (S_t)_{t \geq 0}$ is a stochastic process, with $S_0 = x > 0$, representing the underlying surplus process in a world without economic factors (tax, reinsurance, investment, dividend, etc.).
- Taxes are paid at a fixed rate $\gamma \in [0, 1)$ whenever the surplus process is at a running maximum (called loss-carry-forward taxation).

Loss-carry-forward taxation:



Some known results:

- Assuming that S is a compound Poisson process with positive drift and that taxes are paid at a fixed rate $\gamma \in [0, 1)$ whenever S is at a running maximum, Albrecher and Hipp (2007; *Bl. DGVFM*) and Albrecher *et al.* (2009; *Insurance Math. Econom.*) proved the following strikingly simple relationship between $\psi_\gamma(x)$ and $\psi_0(x)$, the ruin probabilities with and without tax:

$$\psi_\gamma(x) = 1 - (1 - \psi_0(x))^{1/(1-\gamma)}. \tag{1}$$

- Albrecher *et al.* (2008; *J. Appl. Probab.*) further showed that the tax identity (1) still holds for a spectrally negative Lévy surplus process under the loss-carry-forward taxation.
- Albrecher *et al.* (2008; *Insurance Math. Econom.*) proved a similar tax identity for a dual surplus process with general inter-innovation times and exponential innovation sizes under the same type of taxation.

What we do:

Shortcomings of loss-carry-forward taxation:

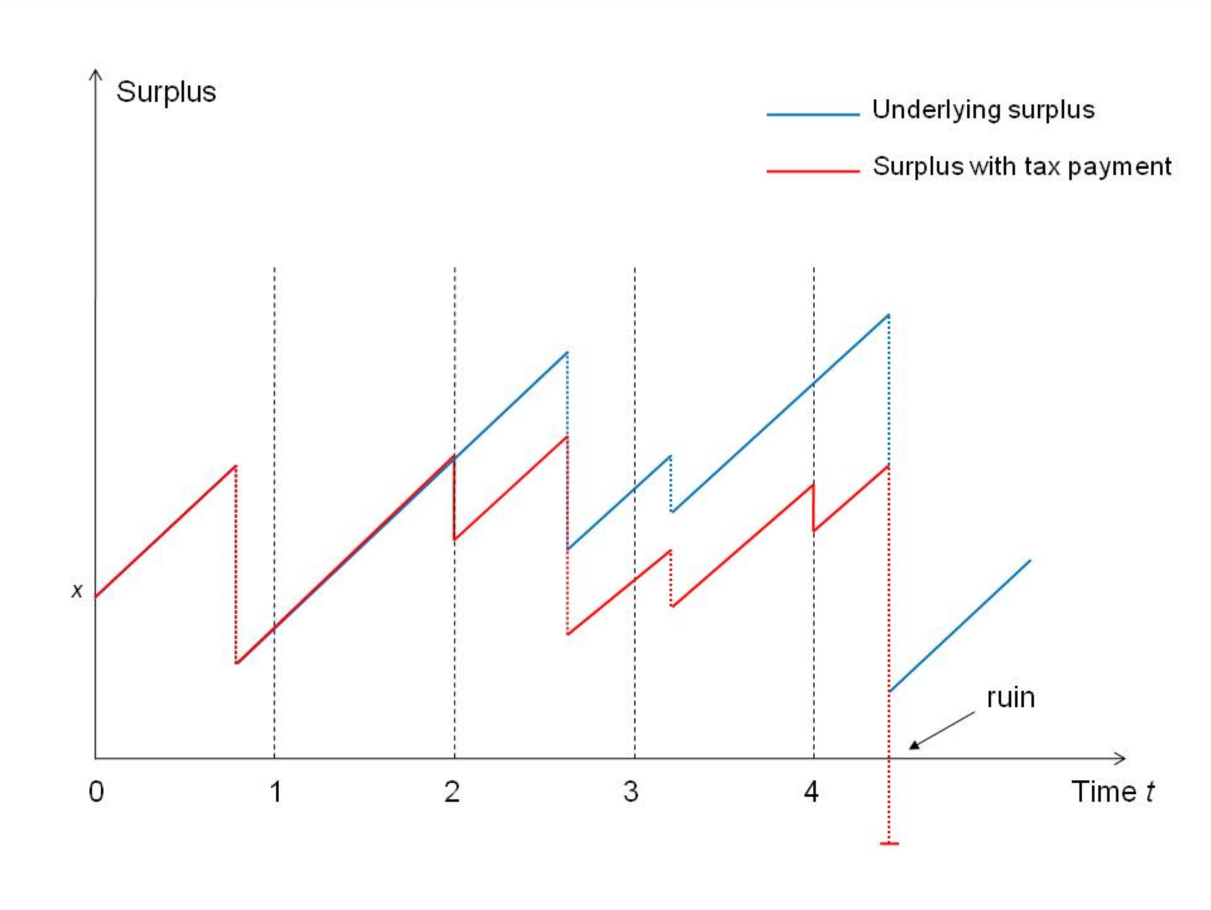
- In reality, taxes are usually paid periodically (e.g. monthly, semi-annually, or annually).
- If S contains a diffusion part, then the moments of running maxima do not form any continuous time interval.

We introduce periodic taxation as well as compensation to the risk model. Given the company survives at time n,

- it pays tax at rate $\gamma \in [0, 1)$ on its **net income** during the period $(n-1, n]$; or,
- it gets compensation at rate $\delta \in [0, 1)$ on its **net loss** during the period $(n-1, n]$.

We investigate the influence of such taxation and compensation rule on the asymptotic behavior of the ruin probability.

Periodic taxation:



2. Model Description

We look at the loss process before tax and reinsurance,

$$L_t = x - S_t, \qquad t \geq 0.$$

For each $n = 1, 2, \ldots$, the maximal net loss and the net loss of the company within the period $(n-1, n]$ are, respectively,

$$Y_n = \sup_{n-1 \leq t \leq n} \left(L_t - L_{n-1} \right), \qquad Z_n = L_n - L_{n-1}.$$

After introducing the periodic taxation at rate $0 \leq \gamma < 1$ and compensation at rate $0 \leq \delta < 1$, the loss of the company within the period $(n-1, n]$ becomes

$$X_n = Z_n + \gamma Z_n^- - \delta Z_n^+ = (1-\delta) Z_n^+ - (1-\gamma) Z_n^-.$$

Then, the ruin probability in this situation is equal to

$$\psi_{\gamma,\delta}(x) = \Pr\left(\sup_{n \geq 1} \left(\sum_{k=1}^{n-1} X_k + Y_n \right) > x \right). \tag{2}$$

Assumptions on the loss process L:

- L is a Lévy process (that is, it starts from 0, is right continuous with left limit, and has stationary and independent increments) with mean $\mathrm{E}L_1 = -\mu < 0$.
- Consequently, the random pairs (X_n, Y_n), $n = 1, 2, \ldots$, appearing in (2) are i.i.d. copies of the random pair

$$(X, Y) =_D \left((1-\delta)L_1^+ - (1-\gamma)L_1^-, \sup_{0 \le t \le 1} L_t \right).$$

- Choose $\gamma \in [0, 1)$ and $\delta \in [0, 1)$ such that

$$\mathrm{E}X = (1-\delta)\mu_+ - (1-\gamma)\mu_- < 0. \tag{3}$$

  So the insurance company still has positive expected profits under such taxation and compensation and that the ruin is not certain.

3. Preliminaries

3.1. Lévy-Khintchine representation

For a Lévy process $L = (L_t)_{t\geq 0}$, its characteristic function can be written in the form

$$\mathrm{E}\mathrm{e}^{\mathrm{i}sL_t} = \mathrm{e}^{-t\Phi(s)},$$

where the characteristic exponent $\Phi(\cdot)$ has the Lévy-Khintchine representation

$$\Phi(s) = \mathrm{i}as + \frac{1}{2}\sigma^2 s^2 + \int_{-\infty}^{\infty} \left(1 - \mathrm{e}^{\mathrm{i}sx} + \mathrm{i}sx\mathbb{1}_{(|x|\leq 1)}\right) \rho(\mathrm{d}x)$$

with $a \in (-\infty, \infty)$, $\sigma \geq 0$, and Lévy measure ρ on $(-\infty, \infty)$ satisfying $\rho(\{0\}) = 0$ and $\int_{-\infty}^{\infty} \left(x^2 \wedge 1\right) \rho(\mathrm{d}x) < \infty$. The triplet (a, σ, ρ) (called Lévy triplet) uniquely determines the distribution of the Lévy process L.

Write $\overline{\rho}(x) = \rho((x, \infty))$. When $\overline{\rho}(1) > 0$, introduce $\Pi(\cdot) = (\overline{\rho}(1))^{-1} \rho(\cdot)\mathbb{1}_{(1,\infty)}$, which is a proper probability measure on $(1, \infty)$.

3.2. Some popular classes of distributions

These classes of distributions have been extensively investigated and applied to various fields by many researchers, such as Embrechts, Klüppelberg, Kyprianou, etc..

Definition: A distribution F on $(-\infty, \infty)$ is said to belong to the class $\mathcal{L}(\alpha)$ for some $\alpha \geq 0$ if $\overline{F}(x) > 0$ for all x and

$$\lim_{x\to\infty} \frac{\overline{F}(x+y)}{\overline{F}(x)} = \mathrm{e}^{-\alpha y}, \qquad y \in (-\infty, \infty)\,.$$

Example: F is a gamma distribution with density

$$f(x;\alpha,\beta) = \frac{\alpha^\beta}{\Gamma(\beta)} x^{\beta-1}\mathrm{e}^{-\alpha x}, \qquad x, \alpha, \beta > 0,$$

$\Longrightarrow \; F \in \mathcal{L}(\alpha).$

A distribution F in $\mathcal{L}(\alpha)$ with $\alpha > 0$ is usually said to have an exponential-like tail.

Definition: A distribution F on $[0, \infty)$ is said to belong to the class $\mathcal{S}(\alpha)$ for some $\alpha \geq 0$ if $F \in \mathcal{L}(\alpha)$ and

$$\lim_{x\to\infty} \frac{\overline{F^{2*}}(x)}{\overline{F}(x)} = 2c$$

exists and is finite.

Example: F is an inverse Gaussian distribution with density

$$f(x;\mu,\lambda) = \left(\frac{\lambda}{2\pi x^3}\right)^{1/2} \exp\left\{\frac{-\lambda(x-\mu)^2}{2\mu^2 x}\right\}, \qquad x, \mu, \lambda > 0,$$

$\implies F \in \mathcal{S}(\alpha)$ with $\alpha = \frac{\lambda}{2\mu^2}$.

A distribution F in $\mathcal{S}(\alpha)$ with $\alpha > 0$ is said to have a convolution-equivalent tail.

$\mathcal{S}(0) = \mathcal{S}$ is the well-known class of subexponential distributions. A useful subclass of $\mathcal{S}$ is $\mathcal{S}^*$. (Klüppelberg(1988; *J. Appl. Probab.*))

Definition: A distribution F on $[0, \infty)$ is said to belong to the class $\mathcal{S}^*$ if $\overline{F}(x) > 0$ for all $x \geq 0$, $\mu_F = \int_0^\infty \overline{F}(x)\mathrm{d}x < \infty$, and

$$\lim_{x\to\infty} \int_0^x \frac{\overline{F}(x-y)}{\overline{F}(x)} \overline{F}(y)\mathrm{d}y = 2\mu_F.$$

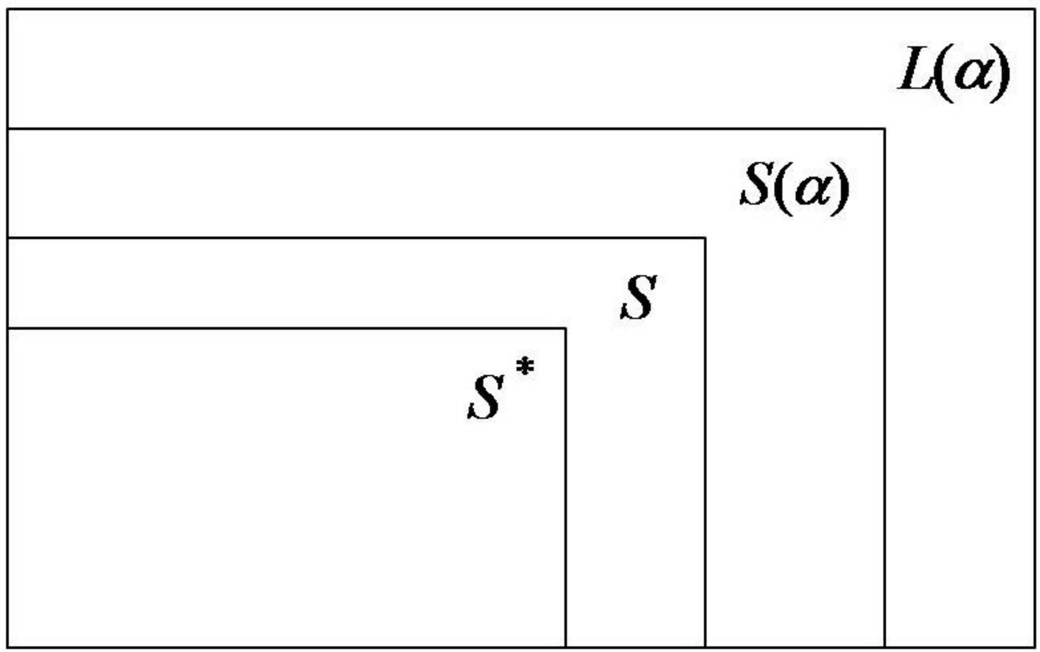
Property: If $F \in \mathcal{S}^*$, then both $F \in \mathcal{S}$ and $F_I \in \mathcal{S}$, where

$$F_I(x) = \frac{1}{\mu_F} \int_0^x \overline{F}(y)\mathrm{d}y, \qquad x \geq 0,$$

denotes the integrated tail distribution of F.

Examples: Pareto (with finite expectation), heavy-tailed Weibull, lognormal distributions.

Relations of these classes of distributions:



4. The Case of Subexponential Tails

Theorem 1 If both Π and Π_I belong to the class $\mathcal{S}$ (which are satisfied when $\Pi \in \mathcal{S}^*$), then for every $0 \leq \gamma < 1$ and $0 \leq \delta < 1$ for which relation (3) holds, we have

$$\psi_{\gamma,\delta}(x) \sim \frac{1}{(1-\gamma)\mu_- - (1-\delta)\mu_+} \int_x^\infty \overline{\rho}(y)\mathrm{d}y. \tag{4}$$

Proof: The proof of Theorem 1 is a direct combination of two results from Rosiński and Samorodnitsky (1993; *Ann. Probab.*) and Palmowski and Zwart (2007; *J. Appl. Probab.*).

Comparison:

The tax identity (1) under loss-carry-forward taxation implies that

$$\psi_\gamma(x) \sim \frac{1}{1-\gamma}\psi_0(x).$$

While under our periodic taxation, substituting $\delta = 0$ to (4) yields that

$$\psi_{\gamma,0}(x) \sim \frac{1}{1-\gamma\frac{\mu_-}{\mu_- - \mu_+}}\psi_{0,0}(x).$$

Note that $\psi_0(x)$ and $\psi_{0,0}(x)$ are identical. The coefficients in the two relations respectively capture the impact of the two taxation rules on the asymptotic behavior of the ruin probability. Now that $\mu_-/(\mu_- - \mu_+) > 1$, we conclude that

$$\psi_\gamma(x) < \psi_{\gamma,0}(x), \qquad \text{for all large } x,$$

i.e., periodic taxation produces more significant impact on the ruin probability than loss-carry-forward taxation does.

5. The Case of Convolution-equivalent Tails

Theorem 2 Assume $\mathrm{E}L_1^2 < \infty$ and $\Pi \in \mathcal{S}(\alpha)$ for some $\alpha > 0$. If $0 \le \gamma < 1$ and $0 < \delta < 1$ are such that

$$\mathrm{E}e^{\alpha'\left((1-\delta)L_1^+ - (1-\gamma)L_1^-\right)} < 1 \tag{5}$$

for some $\alpha' > \alpha$, then,

$$\psi_{\gamma,\delta}(x) \sim \frac{C_\alpha}{1 - \mathrm{E}e^{\alpha\left((1-\delta)L_1^+ - (1-\gamma)L_1^-\right)}}\overline{\rho}(x),$$

where the constant C_α is defined as

$$C_\alpha = \lim_{x\to\infty} \frac{\Pr\left(\sup_{0\le t\le 1} L_t > x\right)}{\overline{\rho}(x)} \in (0, \infty).$$

Proof: The key part of the proof of Theorem 2 is a result from Hao *et al.* (2009; *J. Appl. Probab.*).

Lemma 2.1 (Braverman (1997; *Stochastic Process. Appl.*)) Let L be a Lévy process with Lévy measure ρ such that $\Pi \in \mathcal{S}(\alpha)$ for some $\alpha > 0$. Then for all $t > 0$,

$$\lim_{x\to\infty} \frac{\Pr\left(L_t > x\right)}{\overline{\rho}(x)} = t\mathrm{E}\mathrm{e}^{\alpha L_t} := h(t).$$

There is a unique probability distribution G on $[0,1]$ satisfying $\int_0^1 t^{-1}G(\mathrm{d}t) < \infty$ with moments given by

$$\mu_n(G) = \frac{v_n(n+1)!}{\int_0^1 h(t)\mathrm{d}t}, \qquad n = 1, 2, \ldots,$$

where

$$v_n = \int_{0<t_1\le\cdots\le t_{n+1}\le 1} t_1 \mathrm{E}\mathrm{e}^{\alpha \min_{1\le k\le n+1} L_{t_k}} \mathrm{d}t_1 \cdots \mathrm{d}t_{n+1}.$$

Finally,

$$\lim_{x\to\infty} \frac{\Pr\left(\sup_{0\le t\le 1} L_t > x\right)}{\overline{\rho}(x)} = \int_0^1 t^{-1}G(\mathrm{d}t)\int_0^1 h(t)\mathrm{d}t := C_\alpha. \tag{6}$$

Corollary 2.1 Assume

$$L_t = \sum_{k=1}^{N_t} \xi_k - pt, \qquad t \geq 0,$$

where $p > 0$ represents the constant premium rate, N is a Poisson process with intensity $\lambda > 0$, and $\xi_1, \xi_2, \ldots$ are i.i.d. copies of a random variable ξ independent of N and with distribution F on $(0, \infty)$. Suppose that F has a bounded density $f \in \mathcal{S}d(\alpha)$ for some $\alpha > 0$ and that condition (5) holds. Then,

$$\psi_{\gamma,\delta}(x) \sim \frac{\lambda C_\alpha}{1 - \mathrm{E}e^{\alpha\left((1-\delta)L_1^+ - (1-\gamma)L_1^-\right)}} \overline{F}(x)$$

with the constant C_α given by

$$\begin{aligned} C_\alpha = {} & \mathrm{e}^{\lambda\left(\mathrm{E}e^{\alpha\xi} - 1\right) - \alpha p} \\ & + \alpha \int_0^1 \left(\frac{1}{t} \int_0^t \Pr\left(\sum_{k=1}^{N_t} \xi_k \leq ps \right) \mathrm{d}s \right) (1 - t)\, \mathrm{e}^{\lambda(1-t)\left(\mathrm{E}e^{\alpha\xi} - 1\right) - \alpha p(1-t)} \mathrm{d}t. \end{aligned}$$

6. The Case of Exponential-like Tails

Theorem 3 Assume $\mathrm{E}L_1^2 < \infty$, $\Pi \in \mathcal{L}(\alpha)$ for some $\alpha > 0$, and $\overline{\Pi}(x) = o\left(\overline{\Pi^{2*}}(x)\right)$. If $0 \le \gamma < 1$ and $0 < \delta < 1$ are such that condition (5) holds, then,

$$\psi_{\gamma,\delta}(x) \sim \frac{1}{1 - \mathrm{E}e^{\alpha\left((1-\delta)L_1^+ - (1-\gamma)L_1^-\right)}} \Pr(L_1 > x).$$

Proof: The proof of Theorem 3 is a combination of two results from Hao *et al.* (2009; *J. Appl. Probab.*) and Albin and Sundén (2009; *Stochastic Process. Appl.*).

Two special and important cases of Theorem 3:

(i) A gamma process $U = (U_t)_{t\geq 0}$ starts from 0, with stationary and independent increments, and U_1 having the gamma(α,β) distribution with density

$$f(x) = \frac{\alpha^\beta}{\Gamma(\beta)} x^{\beta-1} \mathrm{e}^{-\alpha x}, \qquad \alpha, \beta, x > 0.$$

Its Lévy triplet is given by $a = \beta\left(\mathrm{e}^{-\alpha} - 1\right)/\alpha$, $\sigma = 0$, and $\rho(\mathrm{d}x) = \beta x^{-1} \mathrm{e}^{-\alpha x} \mathrm{d}x$. It is easy to verify that $\overline{\Pi}(x) = o\left(\overline{\Pi^{2*}}(x)\right)$.

Corollary 3.1 Assume

$$L_t = U_t - pt, \qquad t \geq 0,$$

where $p > 0$ and U is a gamma process as introduced above with parameters $\alpha, \beta > 0$. If $0 \leq \gamma < 1$ and $0 < \delta < 1$ are such that condition (5) holds, then,

$$\psi_{\gamma,\delta}(x) \sim \frac{\alpha^{\beta-1} (x+p)^{\beta-1} \mathrm{e}^{-\alpha(x+p)}}{\left(1 - \mathrm{E}\mathrm{e}^{\alpha\left((1-\delta)L_1^+ - (1-\gamma)L_1^-\right)}\right) \Gamma(\beta)}.$$

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(ii) Consider a compound Poisson process with negative drift.

Corollary 3.2 Assume

$$L_t = \sum_{k=1}^{N_t} \xi_k - pt, \qquad t \geq 0,$$

where $p > 0$ represents the constant premium rate, N is a Poisson process with intensity $\lambda > 0$, and $\xi_1, \xi_2, \ldots$ are i.i.d. copies of a random variable ξ independent of N and follows exponential distribution with mean $1/\alpha$. If $0 \leq \gamma < 1$ and $0 < \delta < 1$ are such that condition (5) holds, then,

$$\psi_{\gamma,\delta}(x) \sim \frac{2\sqrt{\lambda/\pi}}{1 - \mathrm{E}e^{\alpha\left((1-\delta)L_1^+ - (1-\gamma)L_1^-\right)}} \int_0^{\pi/2} \Phi\left(\sqrt{2\lambda}\cos\theta - \sqrt{2\alpha(x+p)}\right) \mathrm{d}\theta,$$

where $\Phi(\cdot)$ is the standard normal distribution.

7. Conclusion

We are the first who have considered the periodic taxation in ruin theory. Our study covers the three cases:

1. the Lévy measure ρ has a subexponential tail;
2. the Lévy measure ρ has a convolution-equivalent tail;
3. the Lévy measure ρ has an exponential-like tail.

For each case, we have derived an asymptotic formula which captures the exact impact of the tax and compensation on the asymptotic behavior of the ruin probability. We have devoted ourselves to deriving explicit formulas.

8. Open Problems

- In Theorems 2 and 3, in order to use the result from Hao *et al.* (2009; *J. Appl. Probab.*) we have to assume that $\delta > 0$. Are the two theorems still true when $\delta = 0$?

- So far we always assume a constant tax rate γ. Can our results be generalized to the case with surplus-dependent tax rate $\gamma(S_t)$, or, even more generally, surplus-dependent and time-inhomogeneous tax rate $\gamma(S_t, t)$?

- The existence of C_α in Theorem 2 was proved by Braverman and Samorodnitsky (1995; *Stochastic Process. Appl.*). Is there any more explicit expression for C_α than that given in (6)?

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