Finite-time survival probability and credit default swaps pricing under geometric Lévy markets

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Credit Default Swaps

Credit risk is an investor's risk of loss arising from a borrower who does not make payments as promised.

The Depository Trust & Clearing Corporation estimates that the size of the global credit derivatives market in 2010 was \$1.66 quadrillion US Dollars. Credit default swaps (CDSs) are the simplest and most popular credit derivatives.

Single-name CDS: A bilateral agreement where the protection buyer transfers the credit risk of a reference entity to the protection seller for a determined time period T.

An example of CDS

Suppose a person owns a defaultable bond of a firm, called *reference entity*. To protect himself from the default risk of the reference entity, the bondholder buys protection from a protection seller through a CDS contract:

- face value $N = 10,000$ euros
- maturity $T = 3$ year
- recovery rate $R = 0.4$
- CDS spread $c = 400$ bps

An example of CDS (cont'd)

Then there are two different scenarios that can happen:

- The bond does not default before maturity. The protection buyer has paid each year $N \times c/10000 = 400$ euros to the protection seller. The protection seller has no outward cash-flow.
- The bond defaults before maturity. The protection buyer has paid 400 euros per year up to default. The protection seller will pay an amount equal to $N(1 - R) = 6,000$ euros. The CDS contract expires after the default.

How to determine the CDS spread?

Use the equivalence principle in insurance:

$$\mathrm{PV}_{\mathrm{fees}}(T) = \mathrm{PV}_{\mathrm{loss}}(T).$$

Suppose spread payments occur at $t_i, i = 1, 2, \ldots, n$ with $t_n = T$.

- $\delta(t)$: the survival probability up to time t, i.e., no default occurs by time t.
- $D(0, t_i)$: the discount factor for time t_i.

$$\mathrm{PV}_{\mathrm{fees}}(T) = cN \sum_{i=1}^{n} D(0, t_i)\delta(t_i)\Delta t_i,$$

$$\mathrm{PV}_{\mathrm{loss}}(T) = (1 - R)N \sum_{i=1}^{n} D(0, t_i)(\delta(t_{i-1}) - \delta(t_i)).$$

How to determine the CDS spread? (cont'd)

Hence,

$$c(T) = \frac{(1-R)\sum_{i=1}^{n} D(0,t_i)(\delta(t_{i-1}) - \delta(t_i))}{\sum_{i=1}^{n} D(0,t_i)\delta(t_i)\Delta t_i}.$$

If CDS spread is continuously paid throughout the year, then

$$c(T) = \frac{(1-R)\left(-\int_0^T D(0,t)\mathrm{d}\delta(t)\right)}{\int_0^T D(0,t)\delta(t)\mathrm{d}t}.$$

Recall: Lévy process

Lévy process: A stochastic process $S = \{S_t, t \geq 0\}$ is a Lévy process if the following conditions hold:

1. $S_0 = 0$.
2. The process has stationary increments, i.e., the distribution of the increment $S_{t+s} - S_t$ over the interval $[t, t+s]$ does not depend on t, but only on the length s of the interval.
3. The process has independent increments, i.e., if $l < s \leq t < u$, $S_u - S_t$ and $S_s - S_l$ are independent random variables.

Recall: Lévy process (cont'd)

A subordinator is a Lévy process whose paths are almost surely non-decreasing.

Examples of subordinator:

- Gamma process: A Lévy process $S = \{S_t, t \geq 0\}$ is called a gamma(a, b) process if S_t follows the gamma(at, b) distribution for every $t > 0$.

$$f(x; a, b) = \frac{b^a}{\Gamma(a)} x^{a-1} \mathrm{e}^{-bx}, \qquad x > 0.$$

- Inverse Gaussian process: A Lévy process $S = \{S_t, t \geq 0\}$ is called an IG(a, b) process if S_t follows the IG(at, b) distribution for every $t > 0$.

$$f(x; a, b) = \frac{a}{\sqrt{2\pi}} \mathrm{e}^{ab} x^{-3/2} \exp\{-(a^2 x^{-1} + b^2 x)/2\}, \qquad x > 0.$$

The Lévy first-passage model

Under the risk-neutral setting:

- A firm's asset process $V = \{V_t, t \geq 0\}$ follows

$$V_t = V_0 \, \mathrm{e}^{\mu t - S_t},$$

  where $S = \{S_t, t \geq 0\}$ is a pure jump subordinator.
- $\mathbb{E}(V_t) = V_0 \, \mathrm{e}^{rt}$, with r the constant interest rate.
- For a predetermined barrier level $L < V_0$, default is defined to occur the first time before maturity when

$$V_t \leq L,$$

  or, equivalently,

$$S_t - \mu t \geq \log(V_0 / L).$$

Remarks:

- This model was proposed by Madan and Schoutens (2008; *J. of Credit Risk*). It reasonably includes jumps and incorporates skewness in the underlying return distribution.
- A firm's asset value is exposed to shocks (represented by negative jumps), which is the main concern in risk management practice.
- According to Carr *et al.* (2002; *J. of Business*), risk-neutral processes for equity prices should be processes of infinite activity and finite variation.

The Lévy first-passage model (cont'd II)

In the Lévy first-passage model,

$$\begin{aligned}\delta(t) &= \mathbb{P}\left(V_s > L, \text{ for all } 0 \le s \le t\right) \\ &= \mathbb{P}\left(\sup_{0 \le s \le t} (S_s - \mu s) < \log(V_0/L)\right).\end{aligned}$$

Our contribution is to give a closed-form formula for the following finite-time survival probability

$$\delta(t, u) = \mathbb{P}\left(\sup_{0 \le s \le t} (S_s - \mu s) < u\right), \qquad t, u > 0,$$

for a pure jump subordinator S.

Finite-time survival probability

We denote the cdf and pdf of S_t by

$$F_t(x) := \mathbb{P}(S_t \leq x), \quad f_t(x) := \frac{\partial}{\partial x} F_t(x), \quad t, x > 0.$$

Theorem 1. Let $\mu > 0$ and $S = \{S_t, t \geq 0\}$ be a pure jump subordinator. Suppose that $F_t \in C^1(0, \infty)$. For $t, u > 0$, we have

$$\begin{aligned} \delta(t,u) \;=\; & F_t(u+\mu t) - \mu \int_0^t F_s(\mu s) f_{t-s}(u+\mu(t-s))\,\mathrm{d}s \\ & + \int_0^t \frac{1}{s}\left(\int_0^{\mu s} x f_s(x)\,\mathrm{d}x\right) f_{t-s}(u+\mu(t-s))\,\mathrm{d}s. \end{aligned}$$

Remark: When S is a gamma process, this result was obtained by Dickson and Waters (1993; *ASTIN Bulletin*).

Selected calibration results for CDX and iTraxx indices

Company	Moody's		1y	2y	3y	4y	5y	7y	10y	a	b
Nestle	Aa1	market	22	27	36	43	50	57	65		
		gamma	19	28	36	43	48	57	65	0.7023	4.8862
		IG	20	29	36	43	48	57	65	0.5958	2.7197
Credit Suisse	Aa2	market	111	125	138	153	163	164	165		
		gamma	111	130	143	151	157	164	167	0.4373	2.5501
		IG	111	129	142	151	157	164	167	0.4652	1.8850
Munich Reinsurance	Aa3	market	53	61	70	79	85	92	97		
		gamma	52	64	73	79	85	92	97	0.4003	3.1307
		IG	53	64	73	79	85	92	98	0.4043	2.1323
Siemens	A1	market	34	50	63	73	81	92	100		
		gamma	34	50	63	73	81	92	100	0.8195	4.5252
		IG	35	50	62	72	80	92	101	0.7284	2.6239
McDonald's	A2	market	8	12	18	22	25	31	39		
		gamma	7	12	17	22	26	32	39	0.7844	6.0141
		IG	8	12	17	21	25	32	39	0.6503	3.0983
Baxter International	A3	market	13	19	25	32	36	45	55		
		gamma	12	19	26	32	37	45	53	0.8030	5.5796
		IG	13	19	26	32	37	45	53	0.6590	2.9377
Black & Decker	Baa1	market	14	21	30	38	46	57	66		
		gamma	12	21	31	39	46	56	66	1.1855	6.3079
		IG	12	21	30	38	45	56	67	0.9254	3.1384
British Telecom	Baa2	market	54	75	91	106	114	124	131		
		gamma	54	76	92	104	112	124	132	0.8117	4.0355
		IG	54	75	91	103	113	125	133	0.7731	2.4842
Cardinal Health	Baa3	market	19	27	37	49	56	65	76		
		gamma	19	30	40	49	56	66	76	0.9365	5.3741
		IG	19	29	39	48	55	66	76	0.8048	2.9138

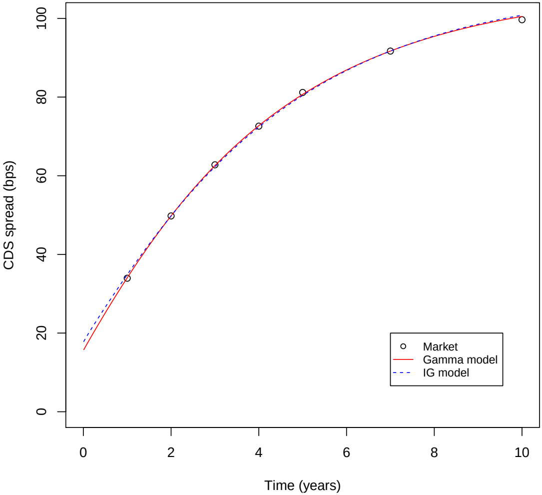

Figure 1: Calibration for Siemens CDS term structure

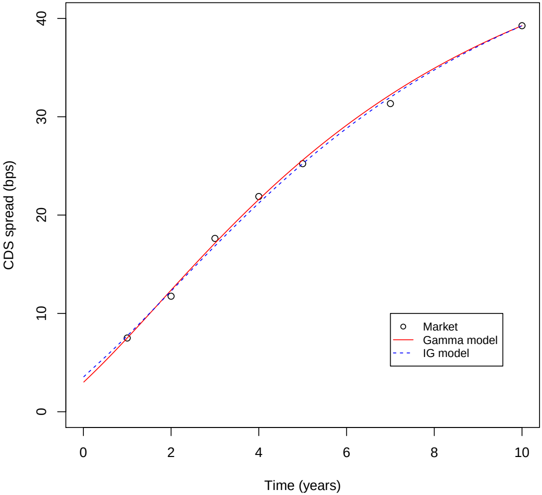

Figure 2: Calibration for McDonald's CDS term structure

Why CDS spread is not negligible for short maturity?

Consider the credit spread $c(T)$ in the Lévy first-passage model.

Corollary 1. If S is a gamma(a, b) process with $a, b > 0$, then

$$\lim_{T \to 0} c(T) = (1 - R) \int_{\log(V_0/L)}^{\infty} a s^{-1} \mathrm{e}^{-bs} \mathrm{d}s.$$

Corollary 2. If S is a IG(a, b) process with $a, b > 0$, then

$$\lim_{T \to 0} c(T) = (1 - R) \int_{\log(V_0/L)}^{\infty} \frac{a \mathrm{e}^{-b^2 s/2}}{\sqrt{2\pi s^3}} \mathrm{d}s.$$

Future research

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Thank you for your attention!