Asymptotic Tail Probability of the Maximum Exceedance over a Renewal Threshold and Its Application in Insurance Mathematics

XUEMIAO HAO

(Based on a joint work with Drs. Qihe Tang and Li Wei)

Department of Statistics and Actuarial Science
The University of Iowa

Outline

- *Motivation and Objective*
- *Some Classes of Distributions*
- *Main Result*
- *Lévy Risk Model*
- *The Case of Exponential-like Tails*

1. Motivation and Objective

Many problems in applied fields, including corporate finance, insurance risk, and production systems, can be reduced to the study of the distribution of *the maximum exceedance of a sequence of random variables over a renewal threshold*. In our paper, we are motivated to investigate the tail probability of such a maximum exceedance.

Precisely, we investigate the tail probability of

$$M = \sup_{n\geq 1}\left(Y_n - \sum_{i=1}^{n-1} X_i\right) \tag{1}$$

under the following assumptions:

- $\{(X_n, Y_n), n = 1, 2, \ldots\}$ is a sequence of independent and identically distributed (i.i.d.) random pairs with generic random pair (X, Y);
- $\mathbb{E}X = \mu > 0$, Y follows a distribution F on $(-\infty, \infty)$, and $0 < \nu_F = \int_0^\infty \overline{F}(y)\mathrm{d}y < \infty$.

2. Some Classes of Distributions

Some classes of distributions have been extensively investigated and applied to various fields by many researchers, such as Embrechts, Klüppelberg, Kyprianou, etc..

Definition: A distribution F on $(-\infty, \infty)$ is said to belong to the class $\mathcal{L}(\alpha)$ for some $\alpha \geq 0$ if $\overline{F}(x) > 0$ for all x and

$$\lim_{x\to\infty} \frac{\overline{F}(x+y)}{\overline{F}(x)} = \mathrm{e}^{-\alpha y}, \qquad y \in (-\infty, \infty)\,.$$

Example: A gamma distribution F with density

$$f(x;\alpha,\beta) = \frac{\alpha^\beta}{\Gamma(\beta)} x^{\beta-1}\mathrm{e}^{-\alpha x}, \qquad x,\alpha,\beta > 0.$$

$\Longrightarrow \quad F \in \mathcal{L}(\alpha).$

$\mathcal{L}(0)$ reduces to the well-known class $\mathcal{L}$ of *long-tailed distributions*. A distribution F in $\mathcal{L}(\alpha)$ with $\alpha > 0$ is usually said to have *an exponential-like tail*.

Definition: A distribution F on $[0, \infty)$ is said to belong to the class $\mathcal{S}(\alpha)$ for some $\alpha \geq 0$ if $F \in \mathcal{L}(\alpha)$ and

$$\lim_{x \to \infty} \frac{\overline{F^{2*}}(x)}{\overline{F}(x)} = 2c$$

exists and is finite.

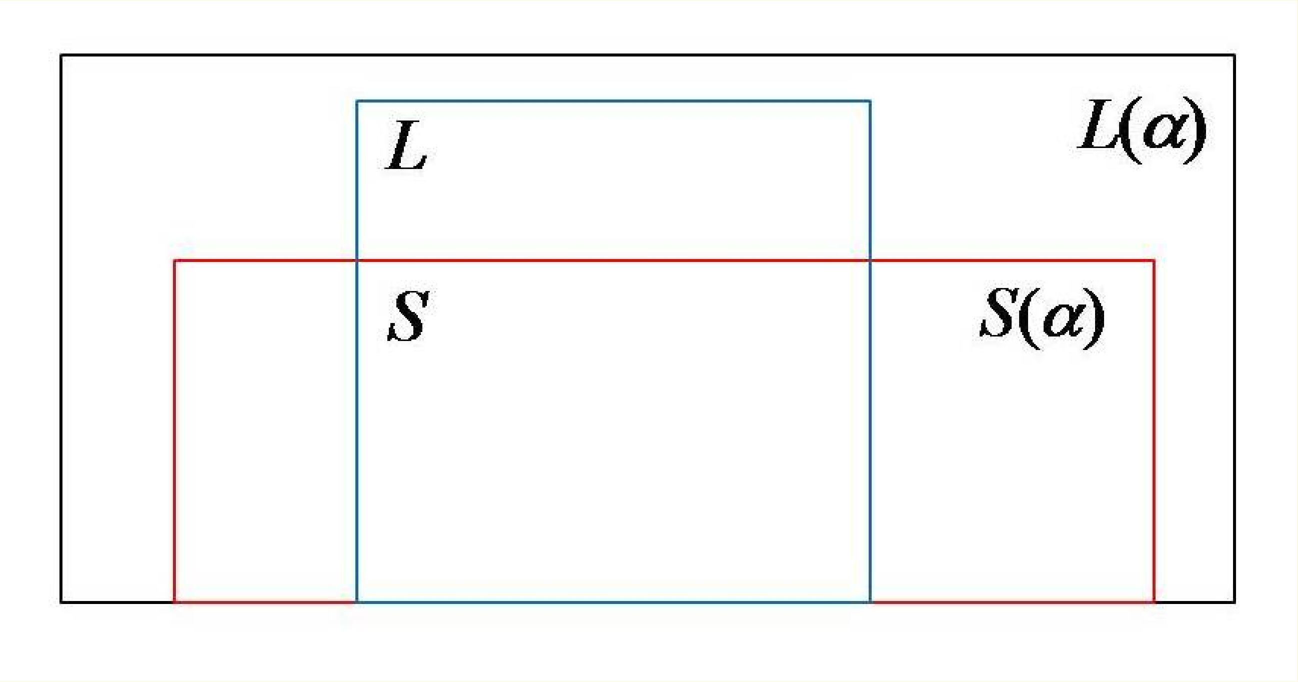
Example: An inverse Gaussian distribution F with density

$$f(x; \mu, \lambda) = \left(\frac{\lambda}{2\pi x^3} \right)^{1/2} \exp\left\{ \frac{-\lambda(x - \mu)^2}{2\mu^2 x} \right\}, \qquad x, \mu, \lambda > 0,$$

$\Longrightarrow \quad F \in \mathcal{S}(\alpha)$ with $\alpha = \frac{\lambda}{2\mu^2}$.

$\mathcal{S}(0) = \mathcal{S}$ is the well-known class of *subexponential distributions*. A distribution F in $\mathcal{S}(\alpha)$ with $\alpha > 0$ is said to have *a convolution-equivalent tail*.

Relations of these classes of distributions:



3. Main Result

We make a convention that

$$\left.\frac{\alpha}{1-\mathbb{E}\mathrm{e}^{-\alpha X}}\right|_{\alpha=0}=\frac{1}{\mu}.$$

Recalling the equilibrium distribution of F, $F_e(x)=\frac{1}{\nu_F}\int_0^x \overline{F}(y)\mathrm{d}y$, $x\geq 0$, we give the following theorem:

Theorem 1 Consider the i.i.d. sequence $\{(X_n, Y_n), n=1,2,\ldots\}$ and the maximum M defined in (1), where $\mathbb{E}X=\mu>0$ and Y is distributed by F. Then, the relation

$$\lim_{x\to\infty}\frac{\mathbb{P}(M>x)}{\int_x^\infty \overline{F}(y)\mathrm{d}y}=\frac{\alpha}{1-\mathbb{E}\mathrm{e}^{-\alpha X}} \tag{2}$$

holds under one of the following groups of conditions:

(i) $F_e\in\mathcal{L}(\alpha)$ for some $\alpha\geq 0$, $\mathbb{E}X^2<\infty$, and $\mathbb{E}\mathrm{e}^{-\beta X}<1$ for some $\beta>\alpha$;

(ii) $F_e\in\mathcal{S}(\alpha)$ for some $\alpha\geq 0$, $\mathbb{P}(-X>x)=o(\overline{F}(x))$, and $\mathbb{E}\mathrm{e}^{-\alpha X}<1$ provided $\alpha>0$.

4. Lévy Risk Model

The underlying surplus process of an insurance company is denoted by

$$S = (S_t)_{t \geq 0}$$

with $S_0 = x > 0$ representing the initial capital. Assuming the loss process

$$L = (L_t)_{t \geq 0} \text{ with } L_t = x - S_t$$

be a Lévy process going to $-\infty$ almost surely, we have the so-called Lévy risk model, which has attracted a lot of interest in insurance mathematics.

A *Lévy process* starts from 0, is right continuous with left limit, and has stationary and independent increments. Its characteristic exponent $\Psi(\cdot)$ has the Lévy-Khintchine representation

$$\Psi(s) = \mathrm{i}as + \frac{1}{2}\sigma^2 s^2 + \int_{-\infty}^{\infty} \left(1 - \mathrm{e}^{\mathrm{i}sx} + \mathrm{i}sx\mathbb{1}_{(|x| \leq 1)}\right) \rho(\mathrm{d}x)$$

The triplet $\left(a, \sigma^2, \rho\right)$ (called Lévy triplet) completely determines the distribution of the Lévy process.

We introduce periodic taxation as well as loss compensation to the risk model. Given the company survives at time n,

- it pays tax at rate $\gamma \in [0,1)$ on its **net income** during the period $(n-1,n]$; or,
- it gets compensation at rate $\delta \in [0,1)$ on its **net loss** during the period $(n-1,n]$.

Then, for each $n = 1, 2, \ldots$, the maximal net loss and the net loss of the company within the period $(n-1,n]$ are, respectively,

$$Y_n = \sup_{n-1 \le t \le n} \left(L_t - L_{n-1}\right), \qquad Z_n = L_n - L_{n-1}.$$

The actual loss of the company within the period $(n-1,n]$ becomes

$$X_n = Z_n + \gamma Z_n^- - \delta Z_n^+ = (1-\delta) Z_n^+ - (1-\gamma) Z_n^-.$$

Hence, the ruin probability in this situation is equal to

$$\psi_{\gamma,\delta}(x) = \mathbb{P}\left(\sup_{n \ge 1} \left(\sum_{k=1}^{n-1} X_k + Y_n\right) > x\right). \tag{3}$$

5. The Case of Exponential-like Tails

Theorem 2 Consider the ruin probability $\psi_{\gamma,\delta}(x)$ in (3). Assume $\mathbb{E}L_1^2 < \infty$, the Lévy measure ρ has an exponential-like tail for some $\alpha > 0$. If $0 \le \gamma < 1$ and $0 < \delta < 1$ are such that

$$\mathbb{E}\mathrm{e}^{\alpha'\left((1-\delta)L_1^+ - (1-\gamma)L_1^-\right)} < 1 \tag{4}$$

for some $\alpha' > \alpha$, then,

$$\psi_{\gamma,\delta}(x) \sim \frac{1}{1 - \mathbb{E}\mathrm{e}^{\alpha\left((1-\delta)L_1^+ - (1-\gamma)L_1^-\right)}} \mathbb{P}(L_1 > x).$$

Two special and important cases of Theorem 2:

(i) A gamma process $U = (U_t)_{t \geq 0}$ starts from 0, with stationary and independent increments, and U_1 having the gamma(α,β) distribution with density

$$f(x) = \frac{\alpha^\beta}{\Gamma(\beta)} x^{\beta-1} \mathrm{e}^{-\alpha x}, \qquad \alpha, \beta, x > 0.$$

Its Lévy triplet is given by $a = \beta \left(\mathrm{e}^{-\alpha} - 1\right) / \alpha$, $\sigma = 0$, and $\rho(\mathrm{d}x) = \beta x^{-1} \mathrm{e}^{-\alpha x} \mathrm{d}x$.

Corollary 2.1 Assume

$$L_t = U_t - pt, \qquad t \geq 0,$$

where $p > 0$ and U is a gamma process as introduced above with parameters $\alpha, \beta > 0$. If $0 \leq \gamma < 1$ and $0 < \delta < 1$ are such that condition (4) holds, then,

$$\psi_{\gamma,\delta}(x) \sim \frac{\alpha^{\beta-1} (x+p)^{\beta-1} \mathrm{e}^{-\alpha(x+p)}}{\left(1 - \mathbb{E}\mathrm{e}^{\alpha\left((1-\delta)L_1^+ - (1-\gamma)L_1^-\right)}\right) \Gamma(\beta)}.$$

(ii) Consider a compound Poisson process with negative drift.

Corollary 2.2 Assume

$$L_t = \sum_{k=1}^{N_t} \xi_k - pt, \qquad t \geq 0,$$

where $p > 0$ represents the constant premium rate, N is a Poisson process with intensity $\lambda > 0$, and $\xi_1, \xi_2, \ldots$ are i.i.d. copies of a random variable ξ independent of N and follows exponential distribution with mean $1/\alpha$. If $0 \leq \gamma < 1$ and $0 < \delta < 1$ are such that condition (4) holds, then,

$$\psi_{\gamma,\delta}(x) \sim \frac{2\sqrt{\lambda/\pi}}{1 - \mathbb{E}e^{\alpha\left((1-\delta)L_1^+ - (1-\gamma)L_1^-\right)}} \int_0^{\pi/2} \Phi\left(\sqrt{2\lambda}\cos\theta - \sqrt{2\alpha(x+p)}\right) \mathrm{d}\theta,$$

where $\Phi(\cdot)$ is the standard normal distribution.

∼ The End ∼