UNCONSTRAINED AND CONVEX POLYNOMIAL APPROXIMATION IN $C[-1,1]$

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Abstract

Some estimates for unconstrained and convex polynomial approximation in the uniform metric are obtained. These results are given in terms of the Ditzian-Totik moduli of smoothness $\omega_\varphi^{r\lambda}(f, n^{-1}(n^{-1}+\varphi(x))^{1-\lambda})$, $0\leqslant\lambda\leqslant1$ *with* $\varphi(x)=\sqrt{1-x^2}$. *The construction of the approximating polynomials does not depend on* λ.

1. Introduction

Let Π_n denote the set of all polynomials of degree not exceeding n and $\varphi(x) := \sqrt{1-x^2}$. Recently, Z. Ditzian and D. Jiang[3] proved that for every function $f(x)\in C[-1,1]$ and $0\leqslant\lambda\leqslant 1$ there exists a sequence of polynomials $p_n\in\Pi_n$ satisfying

$$|f(x)-p_n(x)|\leqslant C(r,\lambda)\omega_\varphi^{r\lambda}(f,n^{-1}(n^{-1}+\varphi(x))^{1-\lambda}), \tag{1}$$

where $\omega_\varphi^{r\lambda}(f,\delta) := \omega_\varphi^{r\lambda}(f,\delta,[-1,1])$ is the Ditzian-Totik modulus of smoothness in the uniform metric, which is given by (see [2]):

$$\omega_\varphi^{r\lambda}(f,\delta,[a,b]) := \sup_{0<h\leqslant\delta}\|\Delta_{h\varphi^\lambda}^r f\|_{C[a,b]}$$

and

$$\Delta_\eta^r f(x) := \begin{cases}\sum_{i=0}^{r}(-1)^i\binom{r}{i}f(x+(\frac{r}{2}-i)\eta), & \text{if } x\pm\frac{r}{2}\eta\in[a,b],\\ 0, & \text{otherwise}\end{cases}$$

is the symmetric rth difference.

This result bridged the gap between pointwise estimates in terms of the usual modulus of smoothness $\omega^r(f,n^{-1}\sqrt{1-x^2}+n^{-2})$ (the case $\lambda=0$), which were studied by A. F. Timan, V. K. Dzjadyk, G. Freud, Yu. A. Brudnyi (see [6], for example), and uniform estimates in terms of $\omega_\varphi^r(f,n^{-1})(\lambda=1)$, which were obtained by Z. Ditzian and V. Totik [2. Chap. 7].

It was shown by Z. Ditzian, D. Jiang and D. Leviatan[4] that for $r=2$ the quantity $n^{-1}+\varphi(x)$ in (1) can be replaced by $\varphi(x)$, and the result can be extended to shape preserving polynomial approximation in $C[-1,1]$.

Theorem A [4, Theorem 1.1]). *For $f\in C[-1,1]$ and $0\leqslant\lambda\leqslant 1$, there exists $p_n\in\Pi_n$ with $p_n(\pm 1)=f(\pm 1)$ such that*

$$|f(x)-p_n(x)|\leqslant C(\lambda)\omega_{\varphi^\lambda}^2(f,n^{-1}\varphi(x)^{1-\lambda}),\ -1\leqslant x\leqslant 1, \tag{2}$$

where $C(\lambda)$ independent of n and f. Moreover, if f is monotone or convex, or both, P_n can be chosen to be monotone or convex or both, respectively. The choice of p_n may be made independent of λ.

For $\lambda=0$ Theorem A was proved by R. A. De Vore and X. M. Yu[1] in the monotone case and by D. Leviatan[11] in the convex one. For $\lambda=1$ it was proved by D. Leviatan[12]. An immediate consequence of the work of A. S. Shvedov[16] is the fact that estimate (2) is not correct with the moduli $\omega_{\varphi^\lambda}^3$ and $\omega_{\varphi^\lambda}^4$ for monotone and convex approximation, respectively.

Thus Theorem A leaves open the case concerning the estimate of the rate of convex polynomial approximation in terms of the $\omega_{\varphi^\lambda}^3$ modulus. the main goal of the paper is to close this gap (see Theorem 1).

It would also be natural to assume that constants in the estimates (1), (2) change " continuously" while λ changes from 0 to 1. This would imply that they can be replaced by those independent of λ. However, it does not follow directly from the proofs in [3,4] as one of the central moments there was the employment of the following equivalence of the Ditzian-Totik modulus and K-functional ([2, Theorem 2.1.1]):

there exists such a constant M which depends on λ that

$$M^{-1}\omega_{\varphi^\lambda}^r(f,t)\leqslant K_{r,\varphi^\lambda}(f,t^r)\leqslant M\omega_{\varphi^\lambda}^r(f,t),$$

where $K_{r,\varphi^\lambda}(f,t^r)=\inf\limits_{g\in C^r}(\|f-g\|+t^r\|\varphi^{\lambda r}g^{(r)}\|)$.

In this paper we strengthen estimate (1) proving, in particular, that constant in (1) can be replaced by $C=C(r)$ (see Theorem 4).

The following theorem on convex approximation is the main result of the paper.

Theorem 1. *For a convex function $f\in C[-1,1]$ and every $n\geqslant 2$ there exists a convex polynomial $p_n\in\Pi_n$ such that for any $\lambda\in[0,1]$ and $x\in[-1,1]$*

$$|f(x)-p_n(x)|\leqslant C\omega_{\varphi^\lambda}^3(f,n^{-1}(n^{-1}+\varphi(x))^{1-\lambda}). \tag{3}$$

If $f\in C^1[-1,1]$, then the following estimate also holds:

$$|f'(x) - p_n'(x)| \leqslant C\omega_{\varphi^\lambda}^2(f', n^{-1}(n^{-1} + \varphi(x))^{1-\lambda}). \qquad (4)$$

Moreover, for $f \in C^2[-1,1]$ *there is also the following estimate*:

$$|f''(x) - p_n''(x)| \leqslant C\omega_{\varphi^\lambda}(f'', n^{-1}(n^{-1} + \varphi(x))^{1-\lambda}), \qquad (5)$$

where C are absolute constants.

It is known that $n^{-1} + \varphi(x)$ in (3) can not be replaced by $\varphi(x)$.

For $\lambda = 0$ and $\lambda = 1$ Theorem 1 was proved in [9]. Also, in the case $\lambda = 0$ a weaker version of the inequality (3) was obtained by Y. Hu, D. Leviatan and X. M. Yu[7] who showed that for a convex function f there exists a convex polynomial $p_n \in \Pi_n$ such that $\| f - p_n \|_\infty \leqslant C\omega^3(f, n^{-1})$.

The following corollaries immediately follow from Theorem 1.

Corollary 2. *For a nondecreasing function* $f \in C[-1,1]$ *there exists a nondecreasing polynomial* $p_n \in \Pi_n$ *such that for any* $\lambda \in [0,1]$ *and* $x \in [-1,1]$

$$|f(x) - p_n(x)| \leqslant C\omega_{\varphi^\lambda}^2(f, n^{-1}(n^{-1} + \varphi(x))^{1-\lambda}).$$

If $f \in C^1[-1,1]$, *then the following estimate also holds*

$$|f'(x) - p_n'(x)| \leqslant C\omega_{\varphi^\lambda}(f', n^{-1}(n^{-1} + \varphi(x))^{1-\lambda}).$$

Corollary 3. *For a nonnegative function* $f \in C[-1,1]$ *and every* $n \geqslant 2$ *there exists a nonnegative polynomial* $p_n \in \Pi_n$ *such that for any* $0 \leqslant \lambda \leqslant 1$

$$|f(x) - p_n(x)| \leqslant C\omega_{\varphi^\lambda}(f, n^{-1}(n^{-1} + \varphi(x))^{1-\lambda}),\ x \in [-1,1].$$

Note that the proof of Theorem 1 is not heavily based on the fact that f is a convex function (of course, we need this condition to construct a convex approximating polynomial). Namely, the same proof works if we drop the condition for f and p_n to be convex. Moreover, in that case we are able to replace, respectively, $\omega_{\varphi^\lambda}^3$, $\omega_{\varphi^\lambda}^2$, and ω_{φ^λ}, by $\omega_{\varphi^\lambda}^r$, $\omega_{\varphi^\lambda}^{r-1}$ and $\omega_{\varphi^\lambda}^{r-2}$ with arbitrary $r \geqslant 3$.

Theorem 4. *For an integer* $r, r \geqslant 3$ *and arbitrary function* $f \in C[-1,1]$ *there exists a sequence of polynomials* $p_n \in \Pi_n$, $n \geqslant r-1$ *satisfying for any* $\lambda \in [0,1]$ *and* $x \in [-1,1]$

$$|f(x) - p_n(x)| \leqslant C(r)\omega_{\varphi^\lambda}^r(f, n^{-1}(n^{-1} + \varphi(x))^{1-\lambda}). \qquad (6)$$

If $f \in C^1[-1,1]$, *then the following estimate also holds*:

$$|f'(x) - p_n'(x)| \leqslant C(r)\omega_{\varphi^\lambda}^{r-1}(f', n^{-1}(n^{-1} + \varphi(x))^{1-\lambda}). \qquad (7)$$

Moreover, for $f \in C^2[-1,1]$ *there is also the following estimate*:

$$|f''(x) - p_n''(x)| \leqslant C(r)\omega_{\varphi^\lambda}^{r-2}(f'', n^{-1}(n^{-1} + \varphi(x))^{1-\lambda}). \qquad (8)$$

It might be of interest to compare the last result with the following theorem on simultaneous approximation of a function f and its derivatives, which was recently proved by Z. Ditzian, D. Jiang and D. Leviatan[5].

Theorem B. *Suppose s and r are integers*, $s\geqslant 0$, $r\geqslant 1$, $0\leqslant\lambda\leqslant 1$ *and* $f^{(s)}(x)\in C[-1,1]$. *Then, for* $n\geqslant r+s-1$ *there exists a polynomial* $p_n\in\Pi_n$ *satisfying*

$$|f^{(j)}(x)-p_n^{(j)}(x)|\leqslant C(r,s,\lambda)(n^{-1}\varphi(x))^{s-j}\omega_{\varphi^\lambda}^r(f^{(s)},n^{-1}(n^{-1}+\varphi(x))^{1-\lambda}),$$
$$0\leqslant j\leqslant s,\ x\in[-1,1]. \tag{9}$$

Theorem 4 is stronger than Theorem B in one sense and weaker in another. The main its disadvantage is that we have simultaneous approximation only up to the second derivative. Secondly, even for $s=2$ estimates (9) for $j=0$ and 1 in some sense are better than (6) and (7) near the endpoints of $[-1,1]$, i.e., when $x\in[-1,-1+n^{-2}]\cup[1-n^{-2},1]$. On the other hand, for $x\notin[-1,-1+n^{-2}]\cup[1-n^{-2},1]$ estimates (6) — (8) are better (in a sense) than (9) for $0\leqslant j\leqslant 2$ and any $s>2$. Also, as a minor advantage of Theorem 4 we mention that constants in (6) — (8) do not depend on λ.

Remark. Estimates (3) and (6) can be improved near the endpoints (see e. g., [15] and [9, ineq. (8)]). However, since it is known that $\Delta_n(x)$ in the estimate by $\omega^r(f,\Delta_n(x))$, generally, can not be replaced by $\varphi(x)n^{-1}$ for $r\geqslant 3$ (it can be replaced by $\frac{\sqrt{1-x^2}}{n}+\sqrt[r]{\frac{1-x^2}{n}}n^{-2+3/r}$ which is less interesting quantity than $\varphi(x)n^{-1}$), we omit the discussions of this subject.

As already was mentioned, the methods of the proofs of Theorems 1 and 4 are the same (in fact, they are not different from the one used in [9]). A complete proof of Theorem 4 is given in Section 5. In Section 6 we discuss what changes are necessary in order to make approximating polynomial p_n convex in the case for convex f and $r=3$.

2. Definitions and notations

Throughout this paper we use the following notations (cf. [8,9,13—15]):

$$I:=[-1,1];\quad \Delta_n(x):=n^{-1}\sqrt{1-x^2}+n^{-2},\quad x\in I;$$

$$x_j:=\cos\frac{j\pi}{n},\quad j=\overline{0,n};\quad \bar{x}_j:=\cos\left(\frac{j\pi}{n}-\frac{\pi}{2n}\right),\quad j=\overline{1,n};$$

$$x_j^0:=\cos\left(\frac{j\pi}{n}-\frac{\pi}{4n}\right),\ \text{if } j<\frac{n}{2},\quad x_j^0:=\cos\left(\frac{j\pi}{n}-\frac{3\pi}{4n}\right),\ \text{if } j\geqslant\frac{n}{2};$$

$$I_j:=[x_j,x_{j-1}],\quad h_j:=x_{j-1}-x_j,\quad j=\overline{1,n};$$

$$t_j:=(x-x_j^0)^{-2}\cos^2 2n\arccos x+(x-\bar{x}_j)^{-2}\sin^2 2n\arccos x,$$

t_j is the algebraic polynomial of degree $4n-2$ (see [6,13]).

Let

$$\Pi_j(\xi,\zeta,\mu) := \int_{-1}^{1} (y-x_j)^{\xi}(x_{j-1}-y)^{\zeta} t_j^{\mu}(y)dy,$$

then

$$Q_j(\xi,\zeta,\mu)(x) := \frac{\int_{-1}^{x} (y-x_j)^{\xi}(x_{j-1}-y)^{\zeta} t_j^{\mu}(y)dy}{\Pi_j(\xi,\zeta,\mu)}, \quad j=\overline{1,n}$$

is the algebraic polynomial of degree $2\mu(2n-1)+\xi+\zeta+1$, which is well defined because $\Pi_j(\xi,\zeta,\mu)$ is never zero (see Proposition E).

We also denote $\Psi_j := \frac{h_j}{|x-x_j|+h_j}$ and $\chi[a,b](x) := \begin{cases}1, & \text{if } x\in[a,b],\\ 0, & \text{otherwise.}\end{cases}$

Obviously, all the above definitions depend on n. To emphasize this in the last section we will use the notations $x_{j(n)}$, $h_{j(n)}$ etc.

$L(t,f;t_1,t_2,\cdots,t_{v+1})$ denotes the Lagrange polynomial, of degree not exceeding v, which interpolates the function $f(x)$ at the points $t_1,t_2,\cdots,t_{v+1}$.

C are positive absolute constants which are not necessarily the same even when they occur on the same line.

A_i, $i\in\mathbb{N}$ denote constants which are larger than 1 and remain fixed throughout the paper.

In order to emphasize that the constant C depends only on the parameters $v_1,\cdots,v_k$ we will use the notation $C(v_1,\cdots,v_k)$.

Without further mentioning the inequalities $h_{j\pm1}<3h_j$ and $\Delta_n(x)<h_j<5\Delta_n(x)$ for $x\in I_j$ are used.

3. Some properties of $\omega_\varphi^{r,\lambda}$, $0\leqslant\lambda\leqslant1$ moduli of smoothness

Lemma 5. *Let $[a,b]\subset[-1,1]$ be such that $b-a\leqslant A_1\Delta_n(a)$ where $A_1\geqslant1$ is an absolute constant. Then for any integer r there exists a constant $C(r)$ such that for any $\lambda\in[0,1]$ and $x\in[a,b]$*

$$C(r)^{-1}\omega^r(f,\Delta_n(x),[a,b]) \leqslant \omega_\varphi^{r,\lambda}(f,n^{-\lambda}\Delta_n(x)^{1-\lambda},[a,b]) \leqslant C(r)\omega^r(f,\Delta_n(x),[a,b]). \tag{10}$$

Lemma 5 shows that moduli $\omega_\varphi^{r,\lambda}(f,n^{-\lambda}\Delta_n(x)^{1-\lambda},[a,b])$ are equivalent for all $0\leqslant\lambda\leqslant1$ if $|b-a|\sim\Delta_n(x)$, $x\in[a,b]$. Of course, neither of the inequalities in (10) is true if we drop the last condition. For examaple,

$$\omega_\varphi(f,n^{-1}) \not\leqslant C\omega(f,\Delta_n(x)),$$

since $\omega_\varphi(f,n^{-1}) \sim n^{-1}$ and $\omega_\varphi(x,\Delta_n(x)) \sim \Delta_n(x)$, and

$$\omega_\varphi(f,n^{-1}) \not\geqslant C\omega(f,\Delta_n(x)),$$

since $\omega_\varphi(\sqrt{x+1},n^{-1}) \sim n^{-1}$ and $\omega(\sqrt{x+1},\Delta_n(x)) \sim \sqrt{\Delta_n(x)}$.

Proof of Lemma 5. First of all, for every $x \in [a,b]$ the quantities $\Delta_n(x)$ and $\Delta_n(a)$ are equivalent, i. e.

$$A_2^{-1}\Delta_n(a) \leqslant \Delta_n(x) \leqslant A_2\Delta_n(a) \tag{11}$$

with some absolute constant A_2 which depends only on A_1.

This is a consequence, for example, of the following inequalities:

$$\Delta_n(x) = \sqrt{1-a^2+a^2-x^2n^{-1}+n^{-2}} \leqslant \sqrt{1-a^2+2A_1\Delta_n(a)n^{-1}+n^{-2}} \leqslant$$

$$\leqslant \Delta_n(a) + \sqrt{2A_1\Delta_n(a)n^{-1}} \leqslant 3\sqrt{A_1}\Delta_n(a)$$

and, using $\sqrt{a+b} \geqslant \sqrt{a} - \sqrt{\max\{-b,0\}}$ for $a \geqslant 0$ and $a+b \geqslant 0$, and recalling that $A_1 \geqslant 1$,

$$\Delta_n(x) = \sqrt{1-a^2+a^2-x^2n^{-1}+n^{-2}} \geqslant \Delta_n(a) - \sqrt{\max\{x^2-a^2,0\}n^{-1}} \geqslant$$

$$\geqslant \Delta_n(a) - \min\{\sqrt{2A_1\Delta_n(a)}, \sqrt{1-a^2}\}n^{-1} \geqslant$$

$$\geqslant \min\left\{1-\sqrt{\frac{2A_1}{2A_1+1}}, \frac{1}{4A_1+1}\right\}\Delta_n(a).$$

Now we are ready to prove (10). For convenience the following two cases are considered separately:

(i) $[a,b] \cap ([-1,-1+n^{-2}] \cup [1-n^{-2},1]) = \varnothing$

and

(ii) $[a,b] \cap ([-1,-1+n^{-2}] \cup [1-n^{-2},1]) \neq \varnothing$.

Case (i). In this case $[a,b]$ is separated from the endpoints of I and, thus, $\varphi(x)n^{-1} < \Delta_n(x) < 2\varphi(x)n^{-1}$ for any $x \in [a,b]$.

Taking this into account one has together with (11)

$$\frac{1}{2}A_2^{-1} \leqslant \frac{\Delta_n(x)}{2\Delta_n(a)} \leqslant \frac{\varphi(x)}{n\Delta_n(a)} \leqslant \frac{\Delta_n(x)}{\Delta_n(a)} \leqslant A_2.$$

Hence the following estimates are valid for $x \in [a,b]$

$$\omega_{\varphi^\lambda}^r(f,n^{-\lambda}\Delta_n(x)^{1-\lambda},[a,b]) = \sup_{0<h\leqslant n^{-\lambda}\Delta_n(x)^{1-\lambda}} \|\Delta_{h\varphi^\lambda}^r f\|_{C[a,b]} \leqslant$$

$$\leqslant \sup_{\bar{h}:=h(n\Delta_n(a))^\lambda \leqslant A_2\Delta_n(a)} \|\Delta_{h\varphi^\lambda}^r f\|_{C[a,b]} =$$

$$= \sup_{\bar{h} \leqslant A_2 \Delta_n(a)} \| \Delta^r_{n^{-\lambda}\varphi^\lambda \Delta_n(a)^{-\lambda}\bar{h}} f \|_{C[a,b]} \leqslant \sup_{\bar{h} \leqslant A_2 \Delta_n(a)} \| \Delta^r_{A_2 \bar{h}} f \|_{C[a,b]} =$$

$$= \omega^r(f, A_2^2 \Delta_n(a), [a,b]) \leqslant C(r)\omega^r(f, \Delta_n(x), [a,b]).$$

Similarly, the inequalities in the other direction can be written as follows:

$$\omega^r_{\varphi^\lambda}(f, n^{-\lambda}\Delta_n(x)^{1-\lambda}, [a,b]) \geqslant \sup_{\bar{h} \leqslant A_2^{-1}\Delta_n(a)} \| \Delta^r_{n^{-\lambda}\varphi^\lambda \Delta_n(a)^{-\lambda}\bar{h}} f \|_{C[a,b]} \geqslant$$

$$\geqslant \sup_{\bar{h} \leqslant A_2^{-1}\Delta_n(a)} \| \Delta^r_{(2A_2)^{-1}\bar{h}} f \|_{C[a,b]} =$$

$$= \omega^r\left(f, \frac{1}{2} A_2^{-2} \Delta_n(a), [a,b] \right) \geqslant C(r)\omega^r(f, \Delta_n(x), [a,b]).$$

This completes the proof of (10) in the case (i).

Case (ii). We assume that $[a,b] \cap [-1, -1+n^{-2}] \neq \emptyset$ and note that for the other case the considerations are analogous.

Inequality (11) implies that $n^{-2} < \Delta_n(x) < 5A_2 n^{-2}$ for every $x \in [a,b]$ and therefore,

$$\omega^r_{\varphi^\lambda}(f, n^{-\lambda}\Delta_n(x)^{1-\lambda}, [a,b]) \leqslant \sup_{0 < h \leqslant 5A_2 n^{\lambda-2}} \| \Delta^r_{h\varphi^\lambda} f \|_{C[a,b]} \leqslant$$

$$\leqslant \sup_{0 < h \leqslant 5A_2 n^{\lambda-2}} \| \Delta^r_{5A_2 n^{-\lambda} h} f \|_{C[a,b]} = \sup_{\bar{h}_1 = 5A_2 n^{-\lambda} h \leqslant 25 A_2^2 n^{-2}} \| \Delta^r_{\bar{h}} f \|_{C[a,b]} =$$

$$= \omega^r(f, 25A_2^2 n^{-2}, [a,b]) \leqslant C(r)\omega^r(f, \Delta_n(x), [a,b]).$$

Thus it remains only to verify (10) in the other direction. At this point we would like to remind that by the definition the difference $\Delta^r_\eta f(x) = 0$ if $x \pm \frac{r}{2}\eta \notin [a,b]$, $\eta = \eta(x)$. This implies the inequality $\| \Delta^r_\eta f \|_{C[a,b]} \geqslant \sup_{x \in S} |\Delta^r_\eta f(x)|$ for any set S. Hence we have

$$\omega^r_{\varphi^\lambda}(f, n^{-\lambda}\Delta_n(x)^{1-\lambda}, [a,b]) \quad \sup_{0 < h \leqslant n^{\lambda-2}} \| \Delta^r_{h\varphi^\lambda} f \|_{C[a,b]} \geqslant$$

$$\geqslant \sup_{h \leqslant n^{\lambda-2}} \sup_{x \in S_1 = \{x \mid x - \frac{r}{4}h^{2/(2-\lambda)} \geqslant -1\}} |\Delta^r_{h\varphi(x)^\lambda} f(x)| \geqslant$$

$$\geqslant \sup_{\bar{h}_1 = \frac{1}{2}h^{2/(2-\lambda)} \leqslant \frac{1}{2}n^{-2}} \sup_{x \in S = \{x \mid x - \frac{r}{2}\bar{h} \geqslant -1\}} |\Delta^r_{\bar{h}} f(x)|.$$

The last inequality is valid because of the following implications:

$$x \in S \Rightarrow 1 - x^2 \geqslant \frac{r}{4} h^{2/(2-\lambda)} \Leftrightarrow$$

$$\Leftrightarrow \varphi(x)^\lambda \geqslant \left(\frac{\sqrt{r}}{2} \right)^\lambda h^{\lambda/(2-\lambda)} \Rightarrow \varphi(x)^\lambda \geqslant \frac{1}{2} h^{\lambda/(2-\lambda)}$$

and therefore for every $x \in S$ the inequality $h\varphi(x)^\lambda \geqslant \frac{1}{2}h^{2/(2-\lambda)} = \bar{h}$ holds. Now using again the definition of the symmetric difference we conclude that if

$$x \notin S \cap \left\{x \,|\, x \pm \frac{r}{2}h \in [a,b]\right\} \text{(in fact, } \left\{x \,|\, x \pm \frac{r}{2}h \in [a,b]\right\} \subset S),$$

then $\Delta_h^r f(x) = 0$ and thus $\| \Delta_h^r f \|_{C(S)} = \| \Delta_h^r f \|_{C[a,b]}$. Finally we have

$$\omega_\varphi^r{}_\lambda(f, n^{-\lambda}\Delta_n(x)^{1-\lambda}, [a,b]) \geqslant \sup_{h\leqslant \frac{1}{2}n^{-2}} \| \Delta_h^r f \|_{C(S)} = \sup_{h\leqslant \frac{1}{2}n^{-2}} \| \Delta_h^r f \|_{C[a,b]}$$

$$= \omega^r\left(f, \frac{1}{2}n^{-2}, [a,b]\right) \geqslant C(r)\omega^r(f, \Delta_n(x), [a,b]).$$

Thus the proof of (10) is complete.

It is well known (see [2, Theorem 4.1.2], for example) that the Ditzian-Totik moduli of smoothness have the following basic property

$$\omega_{\varphi^\lambda}^r(f, \mu t) \leqslant C(r,\lambda)(\mu+1)^r \omega_{\varphi^\lambda}^r(f,t). \tag{12}$$

In our proof we need some inequality analogous to (12) but with a constant independent of λ. Such an inequality is given in the following lemma.

Lemma 6. *For any integer r there exists a constant $C = C(r)$ such that for every $t > 0$, $0 \leqslant \lambda \leqslant 1$ and $\mu \geqslant 1$ the following inequality holds*

$$\omega_{\varphi^\lambda}^r(f, \mu t) \leqslant C(r)\mu^{2r} \omega_{\varphi^\lambda}^r(f,t). \tag{13}$$

Proof. We give a simple combinatorial proof of (13). Obviously, it is sufficient to prove it in the case when μ is an integer. To emphasize this, instead of μ we write n. The following identity which can be easily proved by induction is used:

$$\Delta_{nh}^r f(x) = \sum_{i_1=0}^{n-1} \cdots \sum_{i_r=0}^{n-1} \Delta_h^r f\left(x + \left(i_1 + \cdots + i_r - \frac{(n-1)r}{2}\right) h\right). \tag{14}$$

We write

$$\omega_{\varphi^\lambda}^r(f, nt) = \sup_{0\leqslant h\leqslant nt} \| \Delta_{h\varphi^\lambda}^r f \|_{C(I)} = \sup_{0\leqslant h\leqslant t} \| \Delta_{hn\varphi^\lambda}^r f \|_{C(I)}.$$

It follows from the definition of the symmetric difference that if the inequality $1 - |x| \geqslant \frac{1}{2}hnr\varphi(x)^r$ does not hold, then $\Delta_{hn\varphi^\lambda}^r f(x) = 0$. Therefore, if $\frac{1}{2}hnr > \max_x\{(1-|x|)(1-x^2)^{-\lambda/2}\} = 1$, then $\| \Delta_{hn\varphi^\lambda}^r f \|_{C(I)} = 0$. thus

$$\omega_{\varphi^\lambda}^r(f, nt) = \sup_{0\leqslant h\leqslant \min\{t, \frac{2}{nr}\}} \; \sup_{x,1-|x|\geqslant \frac{1}{2}hnr\varphi(x)^\lambda} |\Delta_{hn\varphi^\lambda}^r f(x)|.$$

Now fixing h and x such that $0 \leqslant h \leqslant \min\{t, \frac{2}{nr}\}$ and $1 - |x| \geqslant \frac{1}{2}hnr\varphi(x)^\lambda$ and using (14) we have

$$|\Delta_{hn\varphi^\lambda}^r f(x)| \leqslant \sum_{i_1=0}^{n-1} \cdots \sum_{i_r=0}^{n-1} \left| \Delta_{h\varphi^\lambda}^r f\left(x + \left(i_1 + \cdots + i_r - \frac{(n-1)r}{2}\right) h\varphi(x)^\lambda\right) \right|. \tag{15}$$

For brevity we denote $\theta(x) := x + \left(i_1 + \cdots + i_r - \frac{(n-1)r}{2}\right) h\varphi(x)^\lambda$. Suppose we showed that $\varphi(x)^\lambda \leqslant A(\sqrt{1-\theta(x)^2})^\lambda$ with A independent of x and h (The best situation is when A in this inequality is an absolute constant. We show its validity with $A = 3\sqrt{n}$, which will do in our case).

Thus we would have

$$|\Delta^r_{h\varphi^\lambda} f(\theta(x))| \leqslant \sup_{0\leqslant h\leqslant \min\{t,\frac{2}{nr}\}} \sup_{x:1-|x|\geqslant\frac{1}{2}hnr\varphi(x)^\lambda} |\Delta_{Ah(1-\theta(x)^2)^{\lambda/2}} f(\theta(x))| \leqslant$$

$$\leqslant \sup_{0\leqslant h\leqslant t} \sup_{y:=\theta(x)\in[-1,1]} |\Delta^r_{Ah(1-y^2)^{\lambda/2}} f(y)| = \sup_{0\leqslant h\leqslant t} \| \Delta^r_{Ah(1-y^2)^{\lambda/2}} f(y) \|_{C(I)} \leqslant \omega^r_{\varphi^\lambda}(f, At).$$

Together with (15) it would yield the estimate $|\Delta^r_{hn\varphi^\lambda} f(x)| \leqslant n^r \omega^r_{\varphi^\lambda}(f, At)$ and, therefore, $\omega^r_{\varphi^\lambda}(f, nt) \leqslant n^r \omega^r_{\varphi^\lambda}(f, At)$.

If $A = 3\sqrt{n}$ then we have the following inequality for all $t \geqslant 0$:

$$\omega^r_{\varphi^\lambda}(f, nt) \leqslant n^r \omega^r_{\varphi^\lambda}(f, 3\sqrt{n}\, t).$$

Now denoting $\tilde{t} := 3\sqrt{n}\, t$ and choosing n so that $\tilde{n} := \frac{\sqrt{n}}{3}$ is an integer one has $\omega^r_{\varphi^\lambda}(f, \tilde{n}\tilde{t}) \leqslant (9\tilde{n}^2)^r \omega^r_{\varphi^\lambda}(f, \tilde{t})$, which is the inequality (13) with $C = 9^r$. Thus to complete the proof of the lemma it is sufficient to show that $\varphi(x)^\lambda \leqslant 3\sqrt{n}(\sqrt{1-\theta(x)^2})^\lambda$ for h and x such that the following holds:

$$0 \leqslant hnr \leqslant 2 \tag{16}$$

and

$$1 - |x| \geqslant \frac{1}{2} hnr\varphi(x)^\lambda. \tag{17}$$

We write

$$1 - \theta(x)^2 = 1 - \left(x + \left(i_1 + \cdots + i_r - \frac{(n-1)r}{2}\right) h\varphi(x)^\lambda\right)^2 \geqslant$$

$$\geqslant 1 - \left(|x| + \frac{1}{2}h(n-1)r\varphi(x)^\lambda\right)^2.$$

Note that if $x \neq \pm 1$, then it follows from (17) that $\varphi(x)^{2-\lambda} \geqslant \frac{1}{2}hnr$. Now let us consider the following two cases:

(i) $\frac{1}{2}hnr \leqslant \varphi(x)^{2-\lambda} \leqslant 3hnr$

and

(ii) $\varphi(x)^{2-\lambda} \leqslant 3hnr$.

Case (i). Inequality (17) yields

$$|x| + \frac{1}{2}h(n-1)r\varphi(x)^\lambda = |x| + \frac{1}{2}hnr\varphi(x)^\lambda - \frac{1}{2}hr\varphi(x)^\lambda \leqslant 1 - \frac{1}{2}hr\varphi(x)^\lambda.$$

It follows from the inequality (16) that

$$1-\theta(x)^2 \geqslant 1-\left(1-\frac{1}{2}hr\varphi(x)^\lambda\right)^2 =$$

$$=\frac{1}{2}hr\varphi(x)^\lambda\left(2-\frac{1}{2}hr\varphi(x)^\lambda\right) \geqslant \frac{1}{2}hr\varphi(x)^\lambda =$$

$$=\frac{1}{2}hr\varphi(x)^2\varphi(x)^{\lambda-2} \geqslant \frac{1}{2}hr\varphi(x)^2(3hrn)^{-1} = \frac{\varphi(x)^2}{6n}.$$

Hence

$$\varphi(x)^\lambda \leqslant (6n(1-\theta(x)^2))^{\lambda/2} \leqslant 3\sqrt{n}\,(\sqrt{1-\theta(x)^2})^\lambda.$$

Case (ii). Inequality $\varphi(x)^{2-\lambda} > 3hnr$ is equaivalent to $\varphi(x)^\lambda < \frac{1-x^2}{3hnr}$. Therefore

$$1-\theta(x)^2 \geqslant 1-\left(|x|+\frac{1}{2}h(n-1)r\varphi(x)^\lambda\right)^2 \geqslant 1-\left(|x|+\frac{1}{2}hnr\frac{1-x^2}{3hnr}\right)^2 =$$

$$=1-\left(|x|+\frac{1-x^2}{6}\right)^2 \geqslant 1-\left(|x|+\frac{1-x^2}{6}\right) \geqslant \frac{1-x^2}{3} > \frac{1-x^2}{6n}.$$

The proof is now complete.

4. Auxiliary statements and results

Let us recall that a spline $s(t)$ of degree m has the defect $k_i\,(1\leqslant k_i\leqslant m+1)$ at a knot t_i if functions $s(t)$, $s'(t),\cdots,s^{(m-k_i)}(t)$ are continuous at t_i and the derivative $s^{(m-k_i+1)}(t)$ is discontinuous at this point.

Lemma C ([10, Proposition 2.3.1]). *The spline $s(t)$ of degree m which has defect k_i $(1\leqslant k_i\leqslant m+1)$ at the knots $t_i\,(i=1,2,\cdots,N-1)$ $(t_0<t_1<\cdots<t_N)$ is uniquely presented in the form*

$$s(t)=\sum_{v=0}^{m}c_v(t-t_0)^v+\sum_{i=1}^{N-1}\sum_{j=0}^{k_i-1}a_{ij}(t-t_i)_+^{m-j},\quad t_0\leqslant t\leqslant t_N$$

where

$$c_v=\frac{s^{(v)}(t_0)}{v!},\qquad v=0,1,\cdots,m;$$

$$a_{ij}=(s^{(m-j)}(t_i+0)-s^{(m-j)}(t_i-0))/(m-j)!,$$

$$i=2,\cdots,N-1,\ j=0,1,\cdots,k_i-1.$$

Using Lemma C and sufficiently good polynomial approximation of the truncated power functions $(t-t_i)_+^{m-j}$ we are thus able to construct the polynomial which approximates the spline $s(t)$.

Proposition D (see [13,14], for example). *The following inequalities hold*:

$$\min\{(x-x_j^0)^{-2},(x-\bar{x}_j)^{-2}\}\leqslant t_j(x)\leqslant\max\{(x-x_j^0)^{-2},(x-\bar{x}_j)^{-2}\},\ x\in I,$$

$$t_j(x)\leqslant 10^3h_j^{-2},\ x\in I,$$

$$x_j^0-x_j>\frac{\bar{x}_j-x_j}{2}>\frac{1}{4}h_j,\ x_{j-1}-\bar{x}_j>\frac{1}{4}h_j,\ \bar{x}_j-x_j^0\leqslant\frac{3}{8}h_j \text{ while } j\leqslant\frac{n}{2},$$

$$x_{j-1}-x_j^0>\frac{x_{j-1}-\bar{x}_j}{2}>\frac{1}{4}h_j,\ \bar{x}_j-x_j>\frac{1}{4}h_j,\ x_j^0-\bar{x}_j\leqslant\frac{3}{8}h_j \text{ while } j>\frac{n}{2},$$

$$\max\{(x-x_j^0)^{-2},(x-\bar{x}_j)^{-2}\}\leqslant 64(|x-x_j|+h_j)^{-2},\quad x\notin I_j$$

and

$$(|x-x_j|+h_j)^{-2}\leqslant t_j(x)\leqslant 4\cdot 10^3(|x-x_j|+h_j)^{-2},\quad x\in I.$$

Proposition E[9]. *The following inequalities are valid*:

$$C(\mu)^{-1}h_j^{-2\mu+\xi+\zeta+1}\leqslant \Pi_j(\xi,\zeta,\mu):=$$

$$:=\int_{-1}^{1}(y-x_j)^{\xi}(x_{j-1}-y)^{\zeta}t_j^{\mu}(y)dy\leqslant C(\mu)h_j^{-2\mu+\xi+\zeta+1},$$

where ξ, ζ *and* μ *are integers satisfying* $\xi\geqslant 0$, $\zeta\geqslant 0$ *and* $\mu\geqslant 9(\xi+\zeta+1)$.

The remaining statements in this section are generalizations of Lemmas 1 and 2 in [9].

Lemma 7. *The following inequalities hold*:

$$1-x_{j-1}<\int_{-1}^{1}Q_j(\xi,\zeta,\mu)(y)dy<1-x_j, \tag{18}$$

$$|Q_j'(\xi,\zeta,\mu)(x)|\leqslant C(\mu)\Psi_j^{2\mu-\xi-\zeta}h_j^{-1}, \tag{19}$$

$$|\chi_j(x)-Q_j(\xi,\zeta,\mu)(x)|\leqslant C(\mu)\Psi_j^{2\mu-\xi-\zeta-1}, \tag{20}$$

where $\mu\geqslant 9(\xi+\zeta+2)$, $x\in I$ *and* $\chi_j(x):=\chi[x_j,1](x)$.

Proof. The lemma was proved in [9] for $Q_j(1,0,10)$, $Q_j(0,1,10)$ and $Q_j(0,0,9)$ (see [9, Lemma 1]). Since the method of the proof for $Q_j(\xi,\zeta,\mu)$ is exactly the same we omit the details in this paper.

It follows from the inequality (18) that for $j=\overline{1,n-1}$ one can choose $\alpha_j\in[0,1]$ so that for polynomail

$$R_{j,0}(x):=R_{j,0}(\xi,\zeta,\mu)(x):=$$

$$=\int_{-1}^{x}(\alpha_jQ_j(\xi,\zeta,\mu)(y)+(1-\alpha_j)Q_{j+1}(\xi,\zeta,\mu)(y))dy,$$

the following equality occurs:

$$R_{j,0}(1)=1-x_j.$$

Let $R_{j,m}(x):=R_{j,m}(\xi,\zeta,\mu)(x):=(x-x_j)^mR_{j,0}(\xi,\zeta,\mu)(x)$, $m\geqslant 0$.

Lemma 8. *The following inequalities hold*:

$$|(x-x_j)_+^m - R_{j,m-1}(x)| \leqslant C(\mu)\Psi_j^{2\mu-\xi-\zeta-m-1} h_j^m, \tag{21}$$

$$|m(x-x_j)_+^{m-1} - R'_{j,m-1}(x)| \leqslant C(\mu)\Psi_j^{2\mu-\xi-\zeta-m} h_j^{m-1}, \tag{22}$$

$$|m(m-1)(x-x_j)_+^{m-2} - R''_{j,m-1}(x)| \leqslant C(\mu)\Psi_j^{2\mu-\xi-\zeta-m+1} h_j^{m-2}, \tag{23}$$

where $x \in I$, $m \geqslant 1$, $\mu \geqslant 9(\xi+\zeta+m+2)$ *and*

$$(x-x_j)_+^v := \begin{cases} (x-x_j)^v \chi[x_j,1](x), & \text{if } v \geqslant 0, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. The case $m = 1$ for $(\xi,\zeta,\mu) = (1,0,10)$, $(0,1,10)$ and $(0,0,9)$ was considered in [9, Lemma 2]. In the general case the considerations are similar and, therefore, we omit the details.

Now the inequalities (21), (22) and (23) for $m > 1$ follow from the case $m = 1$ and the estimates

$$|(x-x_j)_+^m - R_{j,m-1}(x)| = |x-x_j|^{m-1}|(x-x_j)_+ - R_{j,0}(x)| \leqslant$$
$$\leqslant C(\mu)\Psi_j^{2\mu-\xi-\zeta-m-1} h_j^m,$$
$$|R'_{j,m-1}(x) - m(x-x_j)_+^{m-1}| =$$
$$= |(x-x_j)^{m-1}R'_{j,0}(x) + (m-1)(x-x_j)^{m-2}R_{j,0}(x) - m(x-x_j)_+^{m-1}| \leqslant$$
$$\leqslant (m-1)|x-x_j|^{m-2}|R_{j,0}(x) - (x-x_j)_+| + |x-x_j|^{m-1}|R'_{j,0}(x) - \chi_j(x)| \leqslant$$
$$\leqslant C(\mu)\Psi_j^{2\mu-\xi-\zeta-m} h_j^{m-1},$$
$$|R''_{j,m-1}(x) - m(m-1)(x-x_j)_+^{m-2}| =$$
$$= |(x-x_j)^{m-1}R''_{j,0}(x) + 2(m-1)(x-x_j)^{m-2}R'_{j,0}(x) +$$
$$+ (m-1)(m-2)(x-x_j)^{m-3}R_{j,0}(x) - m(m-1)(x-x_j)x_+^{m-2}| \leqslant$$
$$\leqslant |x-x_j|^{m-1}|R''_{j,0}(x)| + 2(m-1)|x-x_j|^{m-2}|R'_{j,0}(x) - \chi_j(x)| +$$
$$+ (m-1)(m-2)|x-x_j|^{m-3}|R_{j,0}(x) - (x-x_j)_+| \leqslant$$
$$\leqslant C(\mu)\Psi_j^{2\mu-\xi-\zeta-m+1} h_j^{m-2},$$

respectively.

5. Unconstrained polynomial approximation

Following the ideas of [1] we construct a spline $S(x)$ of degree $\leqslant r-1$ which sufficiently approximates the function $f = f(x)$, $f \in C(I)$, that is

$$|f(x) - S(x)| \leqslant C(r)\omega_\varphi^{\lambda}(f, n^{-\lambda}\Delta_n(x)^{1-\lambda}), \ x \in I. \tag{24}$$

Then we approximate $S(x)$ by an algebraic polynomial so that

$$|S(x) - p_n(x)| \leqslant C(r)\omega_\varphi^{\lambda}(f, n^{-\lambda}\Delta_n(x)^{1-\lambda}), \ x \in I. \tag{25}$$

This proves the estimate (6).

Construction of the spline

Let $S(x):=L(x,f;x_j,x_{j-1},\cdots,x_{j-r+1})$, $x\in I_j$, $j=\overline{r-1,n}$ and $S(x):=L(x,f;x_{r-1},x_{r-2},\cdots,x_0)$ for the other x. Then $S(x)$ is a spline of degree $r-1$ which satisfies (24). To show this we need the well known Whitney's inequality:

$$|g(x)-L(x,g;t_0,t_0+h,\cdots,t_0+kh)|\leqslant C(k)\omega^{k+1}(g,h,[t_0,t_0+kh]),$$

where $g\in C[t_0,t_0+kh]$ and $x\in[t_0,t_0+kh]$. Using Lemma 5 and the same considerations as in [9, ineq. (57)] one has the estimate

$$|f(x)-S(x)|\leqslant C(r)\omega^r(f,\Delta_n(x),[x_j,x_{j-r+1}])\leqslant$$
$$\leqslant C(r)\omega^r_{\varphi^\lambda}(f,n^{-1}(n^{-1}+\varphi(x))^{1-\lambda}),\qquad x\in I_j,$$

which is the inequality (24).

Remark. In fact, without loss in the rate of approximation instead of the spline $S(x)$ for $x\in I_j$ we could choose $\tilde{S}(x):=L(x,f;x_{\tilde{j}},x_{\tilde{j}-1},\cdots,x_{\tilde{j}-r+1})$ where $\tilde{j}$ is an index such that $|j-\tilde{j}|\leqslant C(r)$ (i.e. $\tilde{j}$ is "not far from" j). This observation is used in Section 6 for the proof of Theorem1.

Taking into account that the spline $S(x)$ is of degree $r-1$ one gets the following analytic representation (see Lemma C) which is used for construction of the approximating polynomial.

$$S(x)=\sum_{v=0}^{r-1}\frac{1}{v!}S^{(v)}(-1)(x+1)^v+$$
$$+\sum_{i=1}^{n-1}\sum_{j=0}^{k_i-1}\frac{1}{(r-1-j)!}(S^{(r-1-j)}(x_i+0)-S^{(r-1-j)}(x_i-0))\,(x-x_i)_+^{r-1-j},\ x\in I.$$

Construction of the polynomial

Let

$$P_n(x):=\sum_{v=0}^{r-1}\frac{1}{v!}S^{(v)}(-1)(x+1)^v+$$
$$+\sum_{i=1}^{n-1}\sum_{j=0}^{k_i-1}\frac{1}{(r-1-j)!}(S^{(r-1-j)}(x_i+0)-S^{(r-1-j)}(x_i-0))\,R_{i,r-j-2}(x),\ x\in I.$$

This polynomial is well defined since $r-j-2\geqslant r-k_i-1\geqslant 0$.

Let us estimate $|p_n(x)-S(x)|$, $x\in I$. First of all, using Markov's and then Whitney's inequalities we have

$$|S^{(r-1-j)}(x_i+0)-S^{(r-1-j)}(x_i-0)|=$$
$$=|L^{(r-1-j)}(x_i,f;x_i,x_{i-1},\cdots,x_{i-r+1})-L^{(r-1-j)}(x_i,f;x_{i+1},x_i,\cdots,x_{i-r+2}|\leqslant$$
$$\leqslant h_i^{-r+1+j}(r-1)^{2(r-1-j)}\times$$
$$\times\|L(x,f;x_i,x_{i-1},\cdots,x_{i-r+1})-L(x,f;x_{i+1},x_i,\cdots,x_{i-r+2})\|_{C[x_i,x_{i-1}]}\leqslant$$
$$\leqslant C(r)h_i^{-r+1+j}\omega(f,\Delta_n(x_i),[x_{i+1},x_{i-r+1}]). \tag{26}$$

Now we choose μ, ξ and ζ so that the conditions of Lemma 8 are valid. For example, let $\mu=36r$ and $0\leqslant\xi,\zeta\leqslant 1$.

The inequalities (26), (21) and (10) imply the estimate

$$|p_n(x)-S(x)|\leqslant\sum_{i=1}^{n-1}\sum_{j=0}^{k_i-1}\frac{1}{(r-1-j)!}|s^{(r-1-j)}(x_i+0)-S^{(r-1-j)}(x_j-0)|\cdot$$
$$\cdot|R_{i,r-j-2}(x)-(x-x_i)_+^{r-j-1}|\leqslant$$
$$\leqslant\sum_{i=1}^{n-1}\sum_{j=0}^{k-1}C(r)h_i^{-r+j+1}\omega(f,\Delta_n(x_i),[x_{i+1},x_{i-r+1}])\Psi_i^{60r}h_i^{r-j-1}\leqslant$$
$$\leqslant C(r)\sum_{i=1}^{n-1}\omega(f,\Delta_n(x_i),[x_{i+1},x_{i-r+1}])\Psi_i^{60r}\leqslant$$
$$\leqslant C(r)\sum_{i=1}^{n-1}\omega_{\varphi^\lambda}^r(f,n^{-\lambda}\Delta_n(x_i)^{1-\lambda})\Psi_i^{60r}.$$

Taking into account Lemma 6 and also the inequalities (see [13,14])

$$\Delta_n^2(y)<4\Delta_n(x)(|x-y|+\Delta_n(x))$$

and

$$2(|x-y|+\Delta_n(x))>|x-y|+\Delta_n(y)>\frac{1}{2}(|x-y|+\Delta_n(x)),\ x\in I,\ y\in I$$

we have

$$\omega_{\varphi^\lambda}^r(f,n^{-\lambda}\Delta_n(x_i)^{1-\lambda}\leqslant$$
$$\leqslant\omega_{\varphi^\lambda}^r(f,n^{-\lambda}(4\Delta_n(x)(|x-x_i|+\Delta_n(x)))^{(1-\lambda)/2})=$$
$$=\omega_{\varphi^\lambda}^r\left(f,n^{-\lambda}\Delta_n(x)^{1-\lambda}\left(4\cdot\frac{|x-x_i|+\Delta_n(x)}{\Delta_n(x)}\right)^{(1-\lambda)/2}\right)\leqslant$$
$$\leqslant C(r)\left(4\cdot\frac{|x-x_i|+\Delta_n(x)}{\Delta_n(x)}\right)^{r(1-\lambda)}\omega_{\varphi^\lambda}^r(f,n^{-\lambda}\Delta_n(x)^{1-\lambda})\leqslant$$
$$\leqslant C(r)\left(\frac{|x-x_i|+\Delta_n(x_i)}{\Delta_n(x_i)}\right)^{2r(1-\lambda)}\omega_{\varphi^\lambda}^r(f,n^{-\lambda}\Delta_n(x)^{1-\lambda})\leqslant$$
$$\leqslant C(r)\Psi_i^{-2r}\omega_{\varphi^\lambda}^r(f,n^{-\lambda}\Delta_n(x)^{1-\lambda}),\quad x\in I. \tag{27}$$

Now using (27) one has the following inequalities

$$|p_n(x) - S(x)| \leqslant C(r)\omega_\varphi^{\lambda}(f, n^{-\lambda}\Delta_n(x)^{1-\lambda}) \sum_{i=1}^{n-1} \Psi_i^{48r} \leqslant$$
$$\leqslant C(r)\omega_\varphi^{\lambda}(f, n^{-\lambda}\Delta_n(x)^{1-\lambda}),\ x \in I.$$

This completes the proof of (25) and, therefore, of (6).

For the proofs of the inequalities (7) and (8) we need the following proposition.

Proposition F (see [15, Lemma 1.4.2]). *For the function* $f \in C^k[x_j, x_{j-r+1}]$, $k \in \mathbb{N}$, $j \geqslant r-1 \geqslant k$ *the following inequality holds*

$$|f^{(k)}(x) - L^{(k)}(x, f; x_j, x_{j-1}, \cdots, x_{j-r+1})| \leqslant$$
$$\leqslant C(r)\omega^{r-k}(f^{(k)}, h_j, [x_j, x_{j-r+1}]),\ x \in [x_j, x_{j-r+1}]. \tag{28}$$

Recalling the construction of the spline $S(x)$, that is,

$$S(x) := L(x, f; x_j, x_{j-1}, \cdots, x_{j-r+1}),\ x \in I_j,\ j = \overline{r-1, n},$$
$$S(x) := L(x, f; x_{r-1}, x_{r-2}, \cdots, x_0), \quad \text{for the other } x,$$

and using Proposition F we conclude that for $x \in I_j$

$$|f^{(v)}(x) - S^{(v)}(x)| \leqslant C(r)\omega^{r-v}(f^{(v)}, \Delta_n(x), [x_j, x_{j-r+1}]),$$

where $v = 1$ or 2 when $f \in C^1(I)$ or $f \in C^2(I)$, respectively, and

$$[x_j, x_{j-r+1}] := [x_{r-1}, x_0] \quad \text{if } j < r-1.$$

Hence it follows from Lemma 5 that

$$|f^{(v)}(x) - S^{(v)}(x)| \leqslant C(r)\omega_\varphi^{\lambda-v}(f^{(v)}, n^{-\lambda}\Delta_n(x)^{1-\lambda}),\ x \in I.$$

It remains to estimate $|p^{(v)}(x) - S^{(v)}(x)|$.

It follows from (26) that for $f \in C^v(I)$ the estimate

$$|S^{(r-1-j)}(x_i + 0) - S^{(r-1-j)}(x_i - 0)| \leqslant$$
$$\leqslant C(r)h_i^{-r+1+j+v}\omega^{r-v}(f^{(v)}, \Delta_n(x_i), I_{i+1} \cup \cdots \cup I_{i-r+2})$$

is valid, and therefore by (10)

$$|S^{(r-1-j)}(x_i + 0) - S^{(r-1-j)}(x_i - 0)| \leqslant$$
$$\leqslant C(r)h_i^{-r+1+j+v}\omega_\varphi^{\lambda-v}(f^{(v)}, n^{-\lambda}\Delta_n(x_i)^{1-\lambda}), \quad v = 1 \text{ or } 2.$$

Now using Lemma 8 and the same arguments as for (27) we obtain the following estimate for any $x \in I$

$$|P_n^{(v)}(x) - S^{(v)}(x)| \leqslant$$
$$\leqslant \sum_{i=1}^{n-1}\sum_{j=0}^{k_i-1} \frac{1}{(r-1-j)!}|S^{(r-1-j)}(x_i + 0) - S^{(r-1-j)}(x_i - 0)| \times$$
$$\times \left| R_{i,r-j-2}^{(v)}(x) - \frac{\partial^v}{\partial x^v}(x - x_i)_+^{r-j-1} \right| \leqslant$$

$$\leqslant \sum_{i=1}^{n-1}\sum_{j=0}^{k_i-1} C(r)h_i^{-r+j+1+v}\omega_\varphi^{\frac{r}{\lambda}-v}(f^{(v)},n^{-\lambda}\Delta_n(x_i)^{1-\lambda})\Psi_i^{60r}h_i^{r-j-1-v} \leqslant$$

$$\leqslant C(r)\omega_\varphi^{\frac{r}{\lambda}-v}(f^{(v)},n^{-\lambda}\Delta_n(x)^{1-\lambda})\sum_{i=1}^{n-1}\Psi_i^{48r} \leqslant$$

$$\leqslant C(r)\omega_\varphi^{\frac{r}{\lambda}-v}(f^{(v)},n^{-\lambda}\Delta_n(x)^{1-\lambda}).$$

Therefore for any $x \in I$ and $v = 1$ or 2

$$|f^{(v)}(x) - p_n^{(v)}(x)| \leqslant C(r)\omega_\varphi^{\frac{r}{\lambda}-v}(f^{(v)},n^{-\lambda}\Delta_n(x)^{1-\lambda}).$$

The proof of Theorem 4 is now complete.

6. Convex polynomial approximation

Theorem 1 would be proved as long as we chose a convex spline $S(x)$ to be constructed so that for $x \in I_j$, $S(x) = L(x,f;x_{\bar{j}},x_{\bar{j}-1},x_{\bar{j}-2})$ with $|\bar{j} - j| \leqslant$ const and convex polynomial $p_n(x)$ in the same form as the polynomial in the proof of Theorem 4.

Such spline and polynomial are presented in [9].

Namely,

$$S(x) = \max\{L(x,f;x_j,x_{j-1},x_{j-2}),\ L(x,f;x_{j+1},x_j,x_{j-1})\},$$

$$x \in I, \qquad j = \overline{2,n-1},$$

$$S(x) = L(x,f;x_2,x_1,x_0), \qquad x \in I_1,$$

and

$$S(x) = L(x,f;x_n,x_{n-1},x_{n-2}), \qquad x \in I_n,$$

or in its analytic representation

$$S(x) = f(-1) + S'(-1)(x+1) + \frac{1}{2}S''(-1)(x+1)^2 +$$

$$+ \sum_{\substack{i=\overline{2,n-1}\\ x_i \in \mathrm{I}\cup\mathrm{III}}} A_i\{h_i(x-x_i)_+ - (x-x_i)_+^2\} +$$

$$+ \sum_{\substack{i=\overline{1,n-2}\\ x_i \in \mathrm{II}\cup\mathrm{III}}} B_i\{h_{i+1}(x-x_i)_+ + (x-x_i)_+^2\},$$

where $A_i = [x_{i+1},x_i,x_{i-1};f] - [x_i,x_{i-1},x_{i-2};f]$, $i = \overline{2,n-1}$ and $B_i = -A_{i+1}$ for $i = \overline{1,n-2}$.

There exists such an absolute constant M that if $n_1 := Mn$, and i_1 is chosen so that $x_{i_1(n_1)} = x_i$, then the polynomial

$$p_n(x) = f(-1) + S'(-1)(x+1) + \frac{1}{2}S''(-1)(x+1)^2 +$$

$$+\sum_{\substack{i=\overline{2,n-1}\\ x_i\in \mathrm{I}\cup\mathrm{III}}} A_i\{h_i R_{i_{(n_1)},0}(0,0,9)(x) - R_{i_{(n_1)},1}(1,0,10)(x)\} +$$

$$+\sum_{\substack{i=\overline{1,n-2}\\ x_i\in \mathrm{II}\cup\mathrm{III}}} B_i\{h_{i+1} R_{i_{(n_1)},0}(0,0,9)(x) - R_{i_{(n_1)},1}(0,1,10)(x)\}$$

is convex on I.

All the necessary proofs and definitions (for example, $x_i \in \mathrm{I}\cup\mathrm{III}$ and $x_i\in \mathrm{II}\cup\mathrm{III}$) can be found in [9].

Now the same arguments as in section 5 can be used to verify the inequalities (3)−(5).

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