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On Some Permanental Conjectures

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Let Ω_n denote the set of all $n \times n$ doubly-stochastic matrices and let $\sigma_k(A)$ be the sum of all subpermanents of order k of matrix A. We prove the Holens–Dokovic conjecture for $k = 4$ and $n \geq 5$. Namely,

$$\sigma_4(A) - \frac{(n-3)^2}{4n}\sigma_3(A) > \sigma_4(J_n) - \frac{(n-3)^2}{4n}\sigma_3(J_n) = 0,$$

$$A \in \Omega_n, \quad n \geq 5, \quad A \neq J_n := \left[\frac{1}{n}\right]_{i,j=1}^{n}.$$

Let Ω_n^0 be the set of all matrices from Ω_n with zero main diagonal and let J_n^0 be the matrix from Ω_n^0 with $1/(n-1)$ in its off-diagonal positions.

We prove the following for $2 \leq k \leq 4$ and $n \geq k+1$:

$$\sigma_k(A) - \frac{(n-k+1)^2}{nk}\sigma_{k-1}(A) > \sigma_k(J_n^0) - \frac{(n-k+1)^2}{nk}\sigma_{k-1}(J_n^0),$$

for any $A \in \Omega_n^0$, $A \neq J_n^0$.

A consequence of the last inequality and also of a result of D. London and H. Minc [8] is the inequality $\sigma_k(A) > \sigma_k(J_n^0)$ for any $A \in \Omega_n^0$, $A \neq J_n^0$, $2 \leq k \leq 4$ and $n \geq k$.

1. INTRODUCTION

We begin by recalling some standard notations. The set of all $n \times n$ doubly stochastic matrices is denoted by Ω_n, the $n \times n$ identity matrix by I_n, the $n \times n$ matrix each of whose entries equals $1/n$ by J_n, and the $n \times n$ matrix with 1/2's in positions $(1,1), (1,2), (2,2), (2,3), \ldots, (n-1,n), (n,n), (n,1)$ by J_n^*. If A is an $n \times n$ matrix and t is an integer with $1 \leq t \leq n$ then the sum of all of the subpermanents of A of order t is denoted by $\sigma_t(A)$. In particular, $\sigma_n(A) = \operatorname{per} A$.

In 1981 D. I. Falikman [3] and G. P. Egorycev [2] proved the van der Waerden conjecture on the minimum of the permanent of the matrix from Ω_n:

$$\text{if} \quad A \in \Omega_n, \quad \text{then} \quad \operatorname{per} A \geq \operatorname{per} J_n = n!/n^n. \tag{1}$$

Egorycev also showed that in (1) the equality holds if and only if $A = J_n$.

In 1982 S. Friedland [4] proved the H. Tverberg's conjecture [15] which generalizes from (1):

$$\text{if} \quad A \in \Omega_n \quad \text{and} \quad A \neq J_n, \quad \text{then} \quad \sigma_t(A) > \sigma_t(J_n), \quad 2 \leq t \leq n. \tag{2}$$

In 1967 D. Z. Dokovic [1] posed and showed the validity of the conjecture for $k \leq 3$ which in its turn generalizes from (2) (see also F. Holens [5]):

CONJECTURE OF F. HOLENS AND D. DOKOVIC

If $A \in \Omega_n$, $n \geq 2$ *and* $2 \leq k \leq n$, *then*

$$\sigma_k(A) \geq \frac{(n-k+1)^2}{nk}\sigma_{k-1}(A). \tag{3}$$

Equality is attained if and only if $2 \leq k \leq n-1$ *and* $A = J_n$, *or* $k = n$ *and* $A = J_n$ *or* A *is a permutation of* J_n^*.

Following [6] we shall call the matrix $A \in \Psi$ an *f-minimizing* matrix on Ψ if $f(A) \leq f(X)$ for all $X \in \Psi$. Here Ψ is a nonempty set of real matrices and f is a real valued function defined on Ψ.

Let

$$F_k(A) := \sigma_k(A) - \frac{(n-k+1)^2}{nk}\sigma_{k-1}(A).$$

D. London [7] showed that if $\operatorname{per} S \geq \sigma_{t-1}(S)/t^2$ for all $S \in \Omega_t$ of rank 2, with equality if and only if $S = J_t$, $t = 2, \dots, n-1$, then $F_k(A) \geq 0$, $t = 2, \dots, n-1$ for all $A \in \Omega_n$ of rank ≤ 2, with equality if and only if $A = J_n$.

S. G. Hwang [6] proved that if A is a F_k*-minimizing* matrix on Ω_n with positive entries, then $A = J_n$ $(k = 2, \dots, n)$.

It is also worth mentioning that the Holens–Dokovic conjecture is equivalent to the assertion that the function $\sigma_k(\theta A + (1-\theta)J_n)$ is increasing in the interval $[0,1]$. This assertion is known as monotonicity conjecture and was partially resolved for some special classes of matrices. More detailed results obtained in connection with this conjecture are discussed in [6, 11, 12].

In the present paper we prove (3) for $k = 4$, $n \geq 5$ and generalize cited Hwang's result to some degree (see Theorem 3).

The other area of research related to the van der Waerden conjecture is the determination of lower bounds of the permanent of matrices on different subsets of Ω_n and the investigation of the forms of per-*minimizing* matrices on these subsets. A clear example of this approach is the London–Minc conjecture [8] (see also Conjecture 44 [12]).

Let Ω_n^0 be the set of all matrices from Ω_n with zero main diagonal, and let J_n^0 be the matrix from Ω_n^0 with $1/(n-1)$ in its off-diagonal positions.

CONJECTURE OF D. LONDON AND H. MINC ([8] AND [12])

If $A \in \Omega_n^0$ *and* $A \neq J_n^0$, *then*

$$\operatorname{per} A > \operatorname{per} J_n^0 = \frac{n!}{(n-1)^n}\left(1 - \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{(-1)^n}{n!}\right). \tag{4}$$

In [8] conjecture (4) was proved for $n \leq 4$.

Now the natural problem arises: Find the minimum of the sum of all subpermanents of order k of the matrix from Ω_n^0. We believe that the following is true.

CONJECTURE 1 *If* $A \in \Omega_n^0$, $A \neq J_n^0$ *and* $2 \leq k \leq n$, *then*

$$\sigma_k(A) > \sigma_k(J_n^0). \tag{5}$$

For $k = n$, clearly, this is the London–Minc conjecture.

It is possible to ask even more: What is the minimum of the function $F_k(A)$ on Ω_n^0? If the Holens–Dokovic conjecture is true, then for $k = n$ we have

$$\min_{A \in \Omega_n^0} F_n(A) = \min_{A \in \Omega_n} F_n(A) = F_n(J_n^*),$$

and thus the minimum of $F_n(A)$ on Ω_n^0 is equal to the minimum on Ω_n. For the other k we conjecture the following.

CONJECTURE 2 *If* $A \in \Omega_n^0$, $A \neq J_n^0$ *and* $2 \leq k \leq n-1$, *then*

$$F_k(A) := \sigma_k(A) - \frac{(n-k+1)^2}{nk}\sigma_{k-1}(A) > F_k(J_n^0). \tag{6}$$

The validity of the London–Minc conjecture and the validity of Conjecture 2 imply that of Conjecture 1. Indeed, for $k = 2$ or $k = n$ the estimate (5) coincides with (6) or (4), respectively (for $k = 2$ functions

$$\sigma_2(A) - \frac{(n-1)^2}{2n}\sigma_1(A) \qquad \text{and} \qquad \sigma_2(A),$$

clearly, achieve their minima at the same points since $\sigma_1(A) = \sigma_1(J_n^0) = n$).

For $2 \leq k \leq n-1$ using (6) we have

$$\begin{aligned}\sigma_k(A) &= \sigma_k(A) - \frac{(n-k+1)^2}{nk}\sigma_{k-1}(A) + \frac{(n-k+1)^2}{nk}\sigma_{k-1}(A) \\ &\geq \sigma_k(J_n^0) - \frac{(n-k+1)^2}{nk}\sigma_{k-1}(J_n^0) + \frac{(n-k+1)^2}{nk}\sigma_{k-1}(J_n^0) = \sigma_k(J_n^0),\end{aligned}$$

and thus (5) follows by induction on k.

In this paper we shall prove (6) for $k \leq 4$. Together with the result of D. London and H. Minc [8] it will prove (5) for $k \leq 4$.

2. ON THE MINIMIZING FUNCTIONS IN THE INTERIOR OF Ω_n

LEMMA A (see [11, pp. 242–243], for example) *The function* F_k, $2 \leq k \leq n$ *has a strict local minimum at* J_n.

THEOREM B ([6]) *If all entries of* F_k*-minimizing matrix* A *on* Ω_n *are positive, then* $A = J_n$.

Theorem B is a consequence of the following theorem.

THEOREM 3 *Let* $\bar{F}(A) = \sum_{i=1}^{n} C_i\sigma_i(A)$, *where* C_i *are any real constants, be such that matrix* J_n *is a point of strict local extremum of* $\bar{F}$. *If all entries of* $\bar{F}$*-minimizing matrix* A *on* Ω_n *are positive, then* $A = J_n$.

Note that the hypothesis of Theorem 3 is satisfied if in particular

$$\sum_{i=2}^{n} \frac{(i-2)!}{n^{i-2}} \binom{n-2}{i-2}^2 C_i \neq 0$$

as

$$\sigma_k(At + J_n(1-t)) = \sum_{\nu=1}^{k} \sigma_\nu(A - J_n) \frac{(k-\nu)!}{n^{k-\nu}} \binom{n-\nu}{k-\nu}^2 t^\nu + \frac{k!}{n^k} \binom{n}{k}^2$$

and therefore the coefficient of t^2 in the decomposition of $\bar{F}(At + J_n(1-t))$ is not zero when $A \neq J_n$ (the coefficient of t in this decomposition is always zero).

The proof of Theorem 3 relies on two results which are based on the ideas of D. W. Sasser and M. L. Slater [13] and S. G. Hwang [6] (see also [9] by M. Marcus and M. Newman).

Let

$$A\begin{bmatrix} i & j \\ x & y \end{bmatrix}$$

denote the $n \times n$ matrix obtained from A by replacement of the columns i and j by the n-vectors x and y, respectively.

LEMMA C *Let* A *be* $\bar{F}$*-minimizing matrix on* Ω_n *with positive entries. Then matrix*

$$\overline{A} = A\begin{bmatrix} i & j \\ \dfrac{a_i + a_j}{2} & \dfrac{a_i + a_j}{2} \end{bmatrix}$$

is $\bar{F}$*-minimizing matrix on* Ω_n *also.*

Proof (see [6, 10], for example). Let

$$B := A\begin{bmatrix} i & j \\ a_j & a_i \end{bmatrix}, \qquad B \neq A.$$

For some $\epsilon > 0$ matrix $A(1-t) + Bt \in \Omega_n$ while $t \in [-\epsilon, 1+\epsilon]$, and $\bar{f}(t) := \bar{F}(A(1-t) + Bt)$ is the polynomial in t of degree not exceeding two such that $\bar{f}'(0) = 0$ and $\bar{f}(0) = \bar{f}(1)$.

This yields the identity $\bar{f}(t) \equiv \bar{f}(0) = \bar{F}(A)$ for $t \in [-\epsilon, 1+\epsilon]$. Thus for every $t \in [-\epsilon, 1+\epsilon]$ (in particular, for $t = \frac{1}{2}$) $A(1-t) + Bt$ is $\bar{F}$*-minimizing* matrix. ■

The following proposition will also be used.

PROPOSITION 4 *Let numbers* $a_i \geq 0$, $1 \leq i \leq n$ *be such that* $\sum_{i=1}^{n} a_i = 1$. *For one step one replaces each of two elements* $a_{i_0} := \max\{a_i,\ i = 1, \ldots, n\}$ *and* $a_{j_0} :=$

$\min\{a_i,\ i=1,\dots,n\}$ *by* $(a_{i_0}+a_{j_0})/2$. *Then for any* $\epsilon>0$ *after a certain finite number of steps all the elements* a_i *are replaced by* $\bar{a}_i$ *such that* $|\bar{a}_i-1/n|<\epsilon$, $i=1,\dots,n$.

Proof of Theorem 3 Let $A\neq J_n$ be an $\bar{F}$-*minimizing* matrix with positive entries. It follows from Proposition 4 that there exists a sequence

$$A=A_1,A_2,\dots,A_k,A_{k+1},\dots$$

such that A_{k+1} is obtained from A_k by "*averaging*"

$$\left(A_{k+1}=A_k\begin{bmatrix} i & j \\ (a_i+a_j)/2 & (a_i+a_j)/2\end{bmatrix}\right) \qquad \text{and} \qquad \lim_{k\to\infty} A_k = J_n.$$

Since $\bar{F}$ is a continuous function it follows that $\lim_{k\to\infty}\bar{F}(A_k)=\bar{F}(J_n)$. But by Lemma C, for every $k\geq 1$ matrix A_k is $\bar{F}$-*minimizing* on Ω_n and $\bar{F}(A_k)=\bar{F}(A)$.

Therefore $\bar{F}(A_k)=\bar{F}(A)=\bar{F}(J_n)$ for all $k\geq 1$.

Since J_n is assumed to be a point of strict local extremum of $\bar{F}$, we immediately conclude that for some $k_0>1$ $A_k=J_n$ for all $k\geq k_0$.

We assume that k_0 is the smallest index, i.e. that $A_k\neq J_n$ while $1\leq k<k_0$ and $A_{k_0}=J_n$. Taking into account the proof of Lemma C one can infer that for every $t\in[0,1]$ matrix

$$(1-t)A_{k_0-1}+tA_{k_0-1}\begin{bmatrix} i & j \\ a_j & a_i\end{bmatrix}$$

is $\bar{F}$-*minimizing* (vectors a_i and a_j are the only columns of A_{k_0-1} which are not equal to the n-vector $(1/n,\dots,1/n)$) and

$$\bar{F}\left((1-t)A_{k_0-1}+tA_{k_0-1}\begin{bmatrix} i & j \\ a_j & a_i\end{bmatrix}\right)=\bar{F}\left(A_{k_0-1}\begin{bmatrix} i & j \\ (a_i+a_j)/2 & (a_i+a_j)/2\end{bmatrix}\right)$$

$$=\bar{F}(A_{k_0})=\bar{F}(J_n) \qquad \text{for every} \quad t\in[0,1].$$

Thus J_n is not a point of *strict* local extremum. This is a contradiction.

The proof of Theorem 3 is now complete. ■

3. AUXILIARY IDENTITIES AND INEQUALITIES

Throughout Sections 3 and 4 we let $A=[a_{ij}]$ be a matrix in Ω_n, and a summation without specified limits is taken over all i and j with $1\leq i,j\leq n$.

Using the formulae for σ_2, σ_3 and σ_4 (see [1, 14])

$$\sigma_2(A)=\frac{1}{2}\sum a_{ij}^2+\frac{n(n-2)}{2},$$

$$\sigma_3(A)=\frac{2}{3}\sum a_{ij}^3+\frac{n-4}{2}\sum a_{ij}^2+\frac{n(n^2-6n+10)}{6}$$

and

$$\sigma_4(A) = \frac{3}{2}\sum a_{ij}^4 + \frac{2}{3}(n-6)\sum a_{ij}^3 + \frac{n^2-10n+28}{4}\sum a_{ij}^2 + \frac{1}{8}\left(\sum a_{ij}^2\right)^2$$
$$+\frac{1}{4}\sum_{1\le i_1<i_2\le n}\left(\sum_{j=1}^n a_{i_1j}a_{i_2j}\right)^2 + \frac{1}{4}\sum_{1\le j_1<j_2\le n}\left(\sum_{i=1}^n a_{ij_1}a_{ij_2}\right)^2$$
$$-\frac{5}{8}\sum_{i=1}^n\left(\sum_{j=1}^n a_{ij}^2\right)^2 - \frac{5}{8}\sum_{j=1}^n\left(\sum_{i=1}^n a_{ij}^2\right)^2 + \frac{n}{24}(n^3-12n^2+52n-84)$$

we get the following presentation of F_2, F_3 and F_4 functions:

$$F_2(A) = \frac{1}{2}\sum a_{ij}^2 - \frac{1}{2}, \tag{7}$$

$$F_3(A) = \frac{2}{3}\sum a_{ij}^3 + \frac{n^2-4n-2}{3n}\sum a_{ij}^2 - \frac{n-4}{3} \tag{8}$$

and

$$F_4(A) = \frac{3}{2}\sum a_{ij}^4 + \frac{n^2-6n-3}{2n}\sum a_{ij}^3 + \frac{n^3-10n^2+23n+36}{8n}\sum a_{ij}^2$$
$$+\frac{1}{8}\left(\sum a_{ij}^2\right)^2 + \frac{1}{4}\sum_{1\le i_1<i_2\le n}\left(\sum_{j=1}^n a_{i_1j}a_{i_2j}\right)^2 + \frac{1}{4}\sum_{1\le j_1<j_2\le n}\left(\sum_{i=1}^n a_{ij_1}a_{ij_2}\right)^2$$
$$-\frac{5}{8}\sum_{i=1}^n\left(\sum_{j=1}^n a_{ij}^2\right)^2 - \frac{5}{8}\sum_{j=1}^n\left(\sum_{i=1}^n a_{ij}^2\right)^2 - \frac{n^2-10n+30}{8}. \tag{9}$$

The proof of the Holens–Dokovic conjecture for $k = 4$, $n \ge 5$ consists of finding bounds on each of the terms in (9). These bounds are obtained using the Jensen and Cauchy inequalities. For convenience we recall these inequalities.

PROPOSITION D ([8, Lemma 1]) *Let $x_1, x_2, \ldots, x_m$ be nonnegative numbers, and let $\sum_{i=1}^m x_i = p$. If $s > 1$, then*

$$\sum_{i=1}^m x_i^s \ge \frac{p^s}{m^{s-1}}. \tag{10}$$

Equality holds if and only if $x_i = p/m$, $i = 1, \ldots, m$.

PROPOSITION E (Cauchy inequality) *For any real numbers a_k and b_k the following inequality holds*

$$\left(\sum_{k=1}^n a_k b_k\right)^2 \le \sum_{k=1}^n a_k^2 \sum_{k=1}^n b_k^2 \tag{11}$$

with equality if and only if $\lambda a_k = \mu b_k$ for all $1 \le k \le n$ and some numbers λ and μ.

Using (11) we obtain some auxiliary inequalities.

First of all, for every $\nu = 1,\ldots,n$ we have

$$(1-a_{\nu 1}^2-a_{\nu 2}^2-\cdots-a_{\nu n}^2)^2=\left(\sum_{i=\overline{1,n},i\neq\nu}\sum_{j=1}^{n}a_{\nu j}a_{ij}\right)^2$$

$$\leq (n-1)\sum_{i=\overline{1,n},i\neq\nu}\left(\sum_{j=1}^{n}a_{\nu j}a_{ij}\right)^2$$

and after the summation in ν, $\nu = 1,\ldots,n$ one gets

$$2\sum_{1\leq i_1<i_2\leq n}\left(\sum_{j=1}^{n}a_{i_1 j}a_{i_2 j}\right)^2\geq\frac{n}{n-1}-\frac{2}{n-1}\sum a_{ij}^2+\frac{1}{n-1}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}a_{ij}^2\right)^2. \tag{12}$$

Now

$$\left(\sum_{j=1}^{n}a_{ij}^2\right)^2=\left(\sum_{j=1}^{n}\sqrt{a_{ij}}(\sqrt{a_{ij}})^3\right)^2\leq\sum_{j=1}^{n}a_{ij}^3\qquad\text{for all}\quad i=1,\ldots,n. \tag{13}$$

Also, for any real number r the following is true for every $i = 1,\ldots,n$:

$$\left(\sum_j a_{ij}^2-r\right)^2=\left(\sum_j a_{ij}(a_{ij}-r)\right)^2\leq n\sum_j a_{ij}^2(a_{ij}-r)^2,$$

and after the summation on i the following emerges

$$\sum a_{ij}^4-2r\sum a_{ij}^3+\left(r^2+\frac{2r}{n}\right)\sum a_{ij}^2\geq\frac{1}{n}\sum_i\left(\sum_j a_{ij}^2\right)^2+r^2,\qquad r\in R. \tag{14}$$

Finally,

$$\left(\sum a_{ij}^2-1\right)^2=\left(\sum_i\left(\sum_j a_{ij}^2-\frac{1}{n}\right)\right)^2\geq\sum_i\left(\sum_j a_{ij}^2-\frac{1}{n}\right)^2,$$

or, equivalently,

$$\left(\sum a_{ij}^2\right)^2-\left(2-\frac{2}{n}\right)\sum a_{ij}^2+1-\frac{1}{n}\geq\sum_i\left(\sum_j a_{ij}^2\right)^2. \tag{15}$$

Note also that the inequalities (12)–(15) are true if we interchange i and j, that corresponds to the transposing of matrix A. Thus the following inequalities hold:

$$\sum_{1\le i_1<i_2\le n}\left(\sum_{j=1}^{n}a_{i_1j}a_{i_2j}\right)^2+\sum_{1\le j_1<j_2\le n}\left(\sum_{i=1}^{n}a_{ij_1}a_{ij_2}\right)^2$$
$$\ge\frac{n}{n-1}-\frac{2}{n-1}\sum a_{ij}^2+\frac{1}{2(n-1)}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}a_{ij}^2\right)^2$$
$$+\frac{1}{2(n-1)}\sum_{j=1}^{n}\left(\sum_{i=1}^{n}a_{ij}^2\right)^2, \tag{16}$$

$$\sum_i\left(\sum_j a_{ij}^2\right)^2+\sum_j\left(\sum_i a_{ij}^2\right)^2\le 2\sum a_{ij}^3, \tag{17}$$

$$\sum a_{ij}^4-2r\sum a_{ij}^3+\left(r^2+\frac{2r}{n}\right)\sum a_{ij}^2$$
$$\ge\frac{1}{2n}\sum_i\left(\sum_j a_{ij}^2\right)^2+\frac{1}{2n}\sum_j\left(\sum_i a_{ij}^2\right)^2+r^2,\qquad r\in R \tag{18}$$

and

$$\left(\sum a_{ij}^2\right)^2-\left(2-\frac{2}{n}\right)\sum a_{ij}^2+1-\frac{1}{n}\ge\frac{1}{2}\sum_i\left(\sum_j a_{ij}^2\right)^2+\frac{1}{2}\sum_j\left(\sum_i a_{ij}^2\right)^2. \tag{19}$$

4. PROOF OF THE HOLDENS–DOKOVIC CONJECTURE FOR $k = 4$, $n > 4$

Now assume that $n \ge 5$. Using (16) one has from (9):

$$F_4(A)\ge\frac{3}{2}\sum a_{ij}^4+\frac{n^2-6n-3}{2n}\sum a_{ij}^3+\left(\frac{n^3-10n^2+23n+36}{8n}-\frac{1}{2(n-1)}\right)\sum a_{ij}^2$$
$$+\frac{1}{8}\left(\sum a_{ij}^2\right)^2-\left(\frac{5}{8}-\frac{1}{8(n-1)}\right)\sum_i\left(\sum_j a_{ij}^2\right)^2$$
$$-\left(\frac{5}{8}-\frac{1}{8(n-1)}\right)\sum_j\left(\sum_i a_{ij}^2\right)^2-\frac{n^2-10n+30}{8}+\frac{n}{4(n-1)}. \tag{20}$$

Now applying the inequalities (18) and (19) we get

$$F_4(A) \geq \left(\frac{n^2-6n-3}{2n}+3r\right)\sum a_{ij}^3$$
$$+\left(\frac{n^3-10n^2+25n+34}{8n}-\frac{1}{2(n-1)}-\frac{3}{2}\left(r^2+\frac{2r}{n}\right)\right)\sum a_{ij}^2$$
$$-\left(\frac{9}{16}-\frac{1}{8(n-1)}-\frac{3}{4n}\right)\sum_i\left(\sum_j a_{ij}^2\right)^2$$
$$-\left(\frac{9}{16}-\frac{1}{8(n-1)}-\frac{3}{4n}\right)\sum_j\left(\sum_i a_{ij}^2\right)^2$$
$$-\frac{n^2-10n+31}{8}+\frac{n}{4(n-1)}+\frac{1}{8n}+\frac{3}{2}r^2. \tag{21}$$

The fact that coefficients of $\sum_i(\sum_j a_{ij}^2)^2$ and $\sum_j(\sum_i a_{ij}^2)^2$ are less than zero for $n > 4$ enables us to use the inequality (17). For every $r \in R$ we have:

$$F_4(A) \geq \left(\frac{4n-33}{8}+3r+\frac{1}{4(n-1)}\right)\sum a_{ij}^3$$
$$+\left(\frac{n^3-10n^2+25n+34}{8n}-\frac{1}{2(n-1)}-\frac{3}{2}\left(r^2+\frac{2r}{n}\right)\right)\sum a_{ij}^2+\frac{3}{2}r^2$$
$$+\frac{n}{4(n-1)}-\frac{n^3-10n^2+31n-1}{8n}.$$

Let $r=\frac{1}{2}$ if $n \geq 6$. Then coefficients of $\sum a_{ij}^3$ and $\sum a_{ij}^2$ are equal to

$$\left(\frac{4n-21}{8}+\frac{1}{4(n-1)}\right) \quad \text{and} \quad \left(\frac{n^2-10n+22}{8}+\frac{43}{20n}+\frac{n-6}{10n(n-1)}\right),$$

respectively. As they are positive for $n \geq 6$ we can use the inequality (10) for $s = 2$ and $s = 3$, and thus

$$F_4(A) \geq \left(\frac{4n-21}{8}+\frac{1}{4(n-1)}\right)\frac{1}{n}+\left(\frac{n^3-10n^2+22n+22}{8n}-\frac{1}{2(n-1)}\right)$$
$$+\frac{1}{4(n-1)}-\frac{n^3-10n^2+26n-1}{8n}=0=F_4(J_n).$$

If $n = 5$, then for $r = 25/48$ the inequality is the following

$$F_4(A) \geq \frac{43}{7680}\sum a_{ij}^2-\frac{43}{7680}\geq 0.$$

It follows from Proposition D that for any $n \geq 5$ and $A \in \Omega_n$ the equality $F_4(A) = 0$ holds if and only if $A = J_n$. The proof is complete.

The case $n = k = 4$ is still open.

5. AUXILIARY INEQUALITIES AND PROOF OF CONJECTURE 2 FOR $k \leq 4$

Throughout all this section we shall assume that a_{ij} are the entries of the matrix $A \in \Omega_n^0$ and thus $a_{ii} = 0$ for $1 \leq i \leq n$. Taking this into account one gets the following corollary of Proposition D.

PROPOSITION 5 *For every $1 \leq i \leq n$ and $s > 1$ the inequality*

$$\sum_{j=1}^{n} a_{ij}^s \geq \frac{1}{(n-1)^{s-1}}$$

is valid. Equality holds only if $a_{ij} = 1/(n-1)$ for all $1 \leq j \leq n$, $j \neq i$. Thus

$$\sum_{i,j=1}^{n} a_{ij}^s \geq \frac{n}{(n-1)^{s-1}}$$

with equality if and only if $A = J_n^0$.

Proof of Conjecture 2 in the cases $k = 2$ and $k = 3$ If $k = 2$, then (6) is a consequence of the formula (7) and Proposition 5 with $s = 2$.

If $k = 3$, then using the inequality

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\left(a_{ij} - \frac{1}{n-1}\right)^2 \geq 0,$$

or, equivalently,

$$\sum_{j=1}^{n} a_{ij}^3 - \frac{2}{n-1}\sum_{j=1}^{n} a_{ij}^2 + \frac{n}{(n-1)^2} \geq 0,$$

one has the following from (8).

$$F_3(A) \geq \left(\frac{n^2-4n-2}{3n} + \frac{4}{3(n-1)}\right)\sum a_{ij}^2 - \frac{n-4}{3} - \frac{2n}{3(n-1)^2}$$

$$= \frac{n^3-5n^2+6n+2}{3n(n-1)}\sum a_{ij}^2 - \frac{n-4}{3} - \frac{2n}{3(n-1)^2}.$$

As $n^3 - 5n^2 + 6n + 2 \geq 0$ for $n \geq 4$ using Proposition 5 we have

$$F_3(A) \geq \frac{n^3-5n^2+6n+2}{3n(n-1)} \cdot \frac{n}{n-1} - \frac{n-4}{3} - \frac{2n}{3(n-1)^2}$$

$$= \frac{n^2-5n+6}{3(n-1)^2} = F_3(J_n^0),$$

where equality holds if and only if $A = J_n^0$.

Thus Conjecture 2 is proved in the cases $k = 2$ and $k = 3$. ■

Proof of Conjecture 2 *for* $k=4$ Using the same method as for the proof of the inequalities (18) and (19) and taking into account that $a_{ii}=0$, $1\le i\le n$, one can verify the following correlations for the entries of the matrix from Ω_n^0:

$$\sum a_{ij}^4 - 2r\sum a_{ij}^3 + \left(r^2+\frac{2r}{n-1}\right)\sum a_{ij}^2$$

$$\ge \frac{1}{2(n-1)}\sum_i\left(\sum_j a_{ij}^2\right)^2 + \frac{1}{2(n-1)}\sum_j\left(\sum_i a_{ij}^2\right)^2 + r^2\frac{n}{n-1}, \qquad r\in R \tag{22}$$

and

$$\left(\sum a_{ij}^2\right)^2 - 2\sum a_{ij}^2 + \frac{n}{n-1} \ge \frac{1}{2}\sum_i\left(\sum_j a_{ij}^2\right)^2 + \frac{1}{2}\sum_j\left(\sum_i a_{ij}^2\right)^2. \tag{23}$$

Now using (22) with $r=\frac{1}{2}$ and (23) we get from (20) for any $n\ge 5$:

$$F_4(A) \ge \left(\frac{n^2-3n-3}{2n}\right)\sum a_{ij}^3 + \left(\frac{n^3-10n^2+22n+36}{8n} - \frac{2}{n-1}\right)\sum a_{ij}^2$$

$$-\left(\frac{9}{16}-\frac{1}{8(n-1)}-\frac{3}{4(n-1)}\right)\sum_i\left(\sum_j a_{ij}^2\right)^2$$

$$-\left(\frac{9}{16}-\frac{1}{8(n-1)}-\frac{3}{4(n-1)}\right)\sum_j\left(\sum_i a_{ij}^2\right)^2$$

$$-\frac{n^2-10n+30}{8}+\frac{n}{4(n-1)}-\frac{n}{8(n-1)}+\frac{3n}{8(n-1)}. \tag{24}$$

Applying (17) one has

$$F_4(A) \ge \left(\frac{4n^2-21n-12}{8n}+\frac{7}{4(n-1)}\right)\sum a_{ij}^3$$

$$+\left(\frac{n^3-10n^2+22n+36}{8n}-\frac{2}{n-1}\right)\sum a_{ij}^2$$

$$-\frac{n^2-10n+30}{8}+\frac{n}{2(n-1)}.$$

It is not difficult to check that the coefficients of $\sum a_{ij}^3$ and $\sum a_{ij}^2$ are positive for $n \geq 5$. Now using Proposition 5 for $s = 2$ and $s = 3$ we have

$$F_4(A) \geq \left(\frac{4n^2 - 21n - 12}{8n} + \frac{7}{4(n-1)}\right)\frac{n}{(n-1)^2} + \left(\frac{n^3 - 10n^2 + 22n + 36}{8n} - \frac{2}{n-1}\right)\frac{n}{n-1} - \frac{n^2 - 10n + 30}{8} + \frac{n}{2(n-1)}$$
$$= \frac{n^4 - 12n^3 + 54n^2 - 107n + 78}{8(n-1)^3} = F_4(J_n^0),$$

with equality if and only if $A = J_n^0$.

Thus Conjecture 2 is proved for $k = 4$. ■

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