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COMONOTONE POLYNOMIAL APPROXIMATION IN $\mathbf{L}_p[-1, 1]$, $0 < p \leqq \infty$

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1. Introduction and main result

Let $\mathbf{P}_n$ denote the set of all algebraic polynomials of degree $\leqq n$, $\mathbf{L}_p[a, b]$, $0 < p \leqq \infty$, be the set of all measurable functions on $[a, b]$ such that the (quasi)norm $\|f\|_{\mathbf{L}_p[a,b]}$ is finite, where as always,

$$\|f\|_{\mathbf{L}_p[a,b]} := \left(\int_a^b |f(x)|^p \, dx\right)^{1/p}, \qquad 0 < p < \infty,$$

and it is the sup-norm for $p = \infty$. Thus throughout the paper, $\mathbf{L}_\infty[a, b]$ is understood to be $\mathbf{C}[a, b]$ with the usual uniform norm. Also for brevity, we denote $\|\cdot\|_p := \|\cdot\|_{\mathbf{L}_p[-1,1]}$.

Let $Y_r := \{y_1, \dots, y_r | y_0 := -1 < y_1 < y_2 < \dots < y_r < 1 =: y_{r+1}\}$, $r \geqq 0$. We denote by $\Delta^1(Y_r)$ the set of all functions f such that f is nondecreasing on $[y_{r-2k}, y_{r-2k+1}]$, and is nonincreasing on $[y_{r-2k-1}, y_{r-2k}]$, i.e., those that have $0 \leqq r < \infty$ monotonicity changes at the points in Y_r and are nondecreasing near 1. Also, let $\Delta^1 := \Delta^1(Y_0)$ denote the set of all nondecreasing functions on $[-1, 1]$. Functions from the class $\Delta^1(Y_r)$ are said to be *comonotone* with one another.

Comonotone polynomial approximation is the approximation of a function $f \in \Delta^1(Y_r)$, by polynomials which are comonotone with it. For $f \in \mathbf{L}_p[-1, 1] \cap \Delta^1(Y_r)$, $r \geqq 0$, let

$$E_n^{(1)}(f, Y_r)_p := \inf_{P_n \in \mathbf{P}_n \cap \Delta^1(Y_r)} \|f - P_n\|_p,$$

be the degree of comonotone polynomial approximation of f. (In particular,

$$E_n^{(1)}(f)_p := E_n^{(1)}(f, Y_0)_p := \inf_{P_n \in \mathbf{P}_n \cap \Delta^1} \|f - P_n\|_p,$$

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is the degree of monotone approximation.)

Recall that the m-th order Ditzian–Totik modulus of smoothness $\omega_m^\varphi(f,\delta)_p$ (see [6]) is given by

$$\omega_m^\varphi(f,\delta)_p = \sup_{0<h\leqq\delta} \left\| \Delta_{h\varphi(\cdot)}^m(f,\cdot)\right\|_p, \tag{1}$$

where $\varphi(x) := \sqrt{1-x^2}$, and

$$\Delta_\eta^m(f,x) := \begin{cases} \sum_{i=0}^m \binom{m}{i}(-1)^{m-i} f(x-\frac{m}{2}\eta + i\eta), & \text{if } x \pm \frac{m}{2}\eta \in [-1,1], \\ 0, & \text{otherwise}, \end{cases}$$

is the symmetric m-th difference. (Note that if we set $\varphi(x) \equiv 1$, then (1) becomes the definition of the usual m-th modulus of smoothness $\omega_m(f,\delta)_p := \omega_m^1(f,\delta)_p$.)

The following result on comonotone approximation of continuous functions in the sup-norm (i.e., in the case when $p=\infty$) is known.

THEOREM A. *Let* $f \in C[-1,1] \cap \Delta^1(Y_r)$, $r \geqq 1$. *Then*

$$E_n^{(1)}(f,Y_r)_\infty \leqq C^*\big(r,d(r)\big)\,\omega_2(f,n^{-1}), \quad n \geqq 1, \tag{2}$$

and

$$E_n^{(1)}(f,Y_r)_\infty \leqq C^{**}(r)\,\omega^\varphi(f,n^{-1}), \quad n \geqq 1, \tag{3}$$

where

$$d(r) := \min\{y_1+1, y_2-y_1, \dots, y_r-y_{r-1}, 1-y_r\}. \tag{4}$$

Estimate (3) was first proved by Leviatan [10], with a constant $C^{**}(r,d_0)$, where $d_0 := \min\{y_1+1, 1-y_r\}$. In its present form it appears in a recent paper by Leviatan and Shevchuk [11]. Estimate (2) is due to Shvedov [18] (see also Yu [20]). It was also shown by Shvedov [18] that the constant C^* in (2) cannot be replaced by one independent of $d(r)$ (if no extra conditions are put on n). Moreover, estimate (2) is exact in the sense that ω_2 cannot be replaced by ω_3 as follows immediately from a result of Zhou [21].

For other relevant results see the list of references.

The purpose of this paper is to prove the following generalization of Theorem A in $\mathbf{L}_p$, $0<p\leqq\infty$.

THEOREM 1. *Let* $f \in \mathbf{L}_p[-1,\, 1] \cap \Delta^1(Y_r)$, $0 < p \leqq \infty$. *Then, for each* $n > C(r)/d(r)$, *where* $d(r)$ *is defined in* (4),

$$E_n^{(1)}(f, Y_r)_p \leqq C(r)\omega_2^{\varphi}(f, n^{-1})_p. \tag{5}$$

The constant $C(r)$ *depends on* p *when* $p \to 0$.

We emphasize that the constant in (5) does not depend on Y_r. (This does not contradict the above mentioned negative result by Shvedov since (5) is valid only for $n > n(Y_r) = C(r)/d(r)$.)

PROOF. We are going to prove Theorem 1 in stages. We first approximate $f \in \mathbf{L}_p[-1,\, 1] \cap \Delta^1(Y_r)$ by a continuous piecewise-linear spline $s \in \Delta^1(Y_r)$ such that $\|f - s\|_p \leqq c\omega_2^{\varphi}(f, n^{-1})_p$. Then we will show how to approximate s by polynomials in $\Delta^1(Y_r)$.

We begin with the partition of the interval $[-1,\, 1]$ by the Chebyshev nodes $x_k := x_{kn} := \arccos \pi k/n$ which we augment with Y_r. Then we delete x_i and x_{i-1} for which there is a y_j, $j = 1, \ldots, r$ such that $x_i \leqq y_j < x_{i-1}$, and we end up with a new partition which we denote $Z_{r,n}$. Explicitly,

$$Z_{r,n} := Y_r \cup \left(\{x_k\}_{k=0}^{n} \setminus \{x_i, x_{i-1} \,:\, x_i \leqq y_j < x_{i-1} \ \text{ for some } \ j = 1, \ldots, r\} \right).$$

Now we have,

LEMMA 2. *Let a function* $f \in \mathbf{L}_p \cap \Delta^1(Y_r)$, $0 < p \leqq \infty$, *be given. Then for every* $n \geqq C(r)/d(r)$, *there exists a continuous piecewise-linear spline* $s \in \Delta^1(Y_r)$ *on the knot sequence* $Z_{r,n}$ *satisfying*

$$\|f - s\|_p \leqq C(r)\omega_2^{\varphi}(f, n^{-1})_p \tag{6}$$

and

$$\omega_2^{\varphi}(s, n^{-1})_p \leqq C(r)\omega_2^{\varphi}(f, n^{-1})_p, \tag{7}$$

where $C(r)$ *is as in Theorem* 1.

Note first that (7) is stated here only for convenient reference as it follows immediately by (6) since

$$\omega_2^{\varphi}(s, n^{-1})_p^p \leqq C\omega_2^{\varphi}(f, n^{-1})_p^p + C\|s - f\|_p^p.$$

Also, when $p = \infty$, then the construction of the spline s is trivial. Indeed, we can simply take s as the piecewise-linear interpolant to f on $Z_{r,n}$. For then $s \in \Delta^1(Y_r)$, and using Whitney's theorem (see [4, p. 183, Theorem 4.2]) we conclude that s satisfies (6). Therefore we will concentrate on proving (6) for the case $0 < p < \infty$.

PROOF. Assume $0 < p < \infty$. Then it is not difficult to construct a spline $\tilde{s}$ which satisfies all of the conditions of the theorem except that it may be discontinuous at $\{y_i\}_{i=1}^r$. Moreover, $\tilde{s}$ can be so chosen that, for every interval I of the partition $Z_{r,n}$, the restriction of $\tilde{s}$ to I is a near-best linear approximant to f in $\mathbf{L}_p(I)$. There are different ways to construct such a spline. In particular, it can be constructed by following the line of proof of [2, Theorem 3]. (The only difference is that in [2] a continuous piecewise-quadratic spline was constructed and this demanded a much more elaborate work.)

Our next step is to alter the spline $\tilde{s}$ in the neighborhood of the y_j's in order to obtain a continuous spline s satisfying all the requirements of the lemma. To simplify the notation we describe the construction of s in the neighborhood of a generic knot $\tilde{y}$, which will denote any of the y_j's, and under the assumption that $\tilde{s}$ is nondecreasing in some small neighborhood on the left of $\tilde{y}$ and nonincreasing in some small neighborhood on the right. Let $\tilde{y} \in [x_i, x_{i-1})$, say and let $p_1(x)$ be the linear piece of $\tilde{s}$ in $[x_{i+2}, x_{i+1}]$, while $p_2(x)$ be the linear piece of $\tilde{s}$ in $[x_{i-2}, x_{i-3}]$. Note that n should be sufficiently large so that x_{i+2} and x_{i-3} are well defined. In fact we assume it is so large that p_1 is nondecreasing and p_2 is nonincreasing. This is the first occurrence where the dependence of n on $d(r)$, comes in. Now, we define s in $[x_{i+2}, x_{i-3}]$ as the piecewise-linear continuous spline $s(x) = \tilde{s}(x)$ for $x \notin (x_{i+1}, x_{i-2})$, and $s(\tilde{y}) = p_1(\tilde{y})$, if $p_2(x_{i-2}) \leqq p_1(x_{i+1})$, or $s(\tilde{y}) = p_2(\tilde{y})$, if $p_2(x_{i-2}) > p_1(x_{i+1})$. Putting $s(x) := \tilde{s}(x)$ outside the neighborhoods of the y_j's, we obtain a continuous piecewise-linear spline $s \in \Delta^1(Y_r)$.

It remains to verify that s satisfies (6). To this end it suffices to show that the degree of local approximation of f (near $\tilde{y}$) by s is not worse than that by $\tilde{s}$. We consider the case where $p_2(x_{i-2}) \leqq p_1(x_{i+1})$, the other case is analogous. Since in this case $s|_{[x_{i+1},\tilde{y}]} = \tilde{s}|_{[x_{i+1},\tilde{y}]}$, all we have to show is that

$$(8) \qquad \|f - s\|_{\mathbf{L}_p[\tilde{y},x_{i-2}]} \leqq C\big(\|f - p_1\|_{[\mathbf{L}_p[\tilde{y},x_{i-2}]} + \|f - p_2\|_{[\mathbf{L}_p[\tilde{y},x_{i-2}]}\big).$$

Indeed,

$$\|s - p_2\|_{\mathbf{C}[\tilde{y},x_{i-2}]} = \big|s(\tilde{y}) - p_2(\tilde{y})\big| = \big|p_1(\tilde{y}) - p_2(\tilde{y})\big|$$

$$\leqq \|p_1 - p_2\|_{\mathbf{C}[\tilde{y},x_{i-2}]} \leqq C h_i^{-1/p} \|p_1 - p_2\|_{\mathbf{L}_p[\tilde{y},x_{i-2}]}.$$

Thus,

$$\|s - p_2\|_{\mathbf{L}_p[\tilde{y},x_{i-2}]} \leqq C h_i^{1/r} \|s - p_2\|_{\mathbf{C}[\tilde{y},x_{i-2}]} \leqq C \|p_1 - p_2\|_{\mathbf{L}_p[\tilde{y},x_{i-2}]},$$

and

$$\|f - s\|^p_{\mathbf{L}_p[\tilde{y},x_{i-2}]} \leqq C\big(\|f - p_2\|^p_{[\mathbf{L}_p[\tilde{y},x_{i-2}]} + \|s - p_2\|^p_{[\mathbf{L}_p[\tilde{y},x_{i-2}]}\big)$$

$$\leqq C\big(\|f-p_1\|^p_{[\mathbf{L}_p[\tilde{y},x_{i-2}]}+\|f-p_2\|^p_{[\mathbf{L}_p[\tilde{y},x_{i-2}]}\big). \qquad \square$$

Lemma 2 implies that from now on, we may assume that the function f in the statement of Theorem 1 is a continuous piecewise-linear spline on the knot sequence $Z_{r,n}$. Evidently, this assumption considerably simplifies all subsequent considerations. Furthermore, replacing f by $f-f(y_1)$, we may assume without loss of generality, that $f(y_1)=0$.

Hence, in the rest of the paper f is going to be a continuous piecewise-linear function on the knot sequence $Z_{r,n}$ which belongs to $\Delta^1(Y_r)$ and satisfies $f(y_1)=0$.

Let $y_1\in I_j:=[x_j,x_{j-1}]$ and set $h_j:=|I_j|=x_{j-1}-x_j$. We will show that

$$\|f\|_{\mathbf{L}_p[y_1-h_j/6,y_1+h_j/6]}\leqq C\omega_2(f,h_j,J_j)_p, \tag{9}$$

where $J_j=[x_{j+2},x_{j-2}]$.

Clearly, while $I_j\subset J_j$, we have $|J_j|\leqq C|I_j|=Ch_j$, and for $n\geqq C(r)/d(r)$ with a sufficiently large $C(r)$ we obtain

$$\omega_2(f,h_j,J_j)_p\leqq C\omega_2^{\varphi}(f,n^{-1})_p. \tag{10}$$

In order to prove (9), we take L to be the straight line such that $L|_{[y_1,x_{j-2}]}=f|_{[y_1,x_{j-2}]}$, and we get

$$\|f\|_{\mathbf{L}_p[y_1-h_j/6,y_1]}\leqq\|f-L\|_{\mathbf{L}_p[y_1-h_j/6,y_1]}$$
$$=\Big\|f(\cdot)-L(\cdot)+\big(L(\cdot)-2L(\cdot+h_j/6)+L(\cdot+h_j/3)\big)\Big\|_{\mathbf{L}_p[y_1-h_j/6,y_1]}$$
$$=\Big\|f(\cdot)-2f(\cdot+h_j/6)+f(\cdot+h_j/3)\Big\|_{\mathbf{L}_p[y_1-h_j/6,y_1]}$$
$$=\|\Delta^2_{h_j/6}f\|_{\mathbf{L}_p[y_1-h_j/6,y_1]}\leqq C\omega_2(f,h_j,J_j)_p,$$

where in the first inequality we used the fact that $f\leqq 0$ in $[y_1-h_j/6,y_1]$, since it is nondecreasing there and $f(y_1)=0$, while $L\geqq 0$ in that interval because it is a nonincreasing straight line and $L(y_1)=f(y_1)=0$.

Similarly, one can show that

$$\|f\|_{\mathbf{L}_p[y_1,y_1+h_j/6]}\leqq C\omega_2(f,h_j,J_j)_p,$$

and, hence, (9) is satisfied.

We now define the flipped function

$$\hat{f}(x):=\begin{cases}-f(x), & \text{if } x<y_1,\\ f(x), & \text{if } x\geqq y_1.\end{cases}$$

Then evidently, the function $\hat{f}$ is a continuous piecewise-linear spline from the class $\Delta^1(Y_r \setminus \{y_1\})$, and (10) implies that for $n \geqq C(r)/d(r)$,

$$\omega_2^{\varphi}(\hat{f}, n^{-1})_p \leqq C\omega_2^{\varphi}(f, n^{-1})_p.$$

We can now apply the method from [1] and [10] (see also [9]) and prove Theorem 1 by induction on r. For $r = 0$ Theorem 1 becomes a theorem on monotone polynomial approximation in $\mathbf{L}_p$, $0 < p \leqq \infty$, which was proved in [19] (see also [12]) for $1 \leqq p \leqq \infty$, and in [3] for $0 < p < 1$. To complete the proof it remains to show that if Theorem 1 is valid for $\hat{f} \in \Delta^1(Y_r \setminus \{y_1\})$, then it remains valid for $f \in \Delta^1(Y_r)$. We will use the construction from [9]. Namely, let $q_n \in \mathbf{P}_n$ be a polynomial which is comonotone with $\hat{f}$ and such that

$$\|\hat{f} - q_n\|_p \leqq C\omega_2^{\varphi}(\hat{f}, n^{-1})_p. \tag{11}$$

Similarly to [9] one can show that for sufficiently large $\mu = \mu(r)$ ($\mu = 15r$ will do), there exist polynomials $V_n(x)$ and $W_n(x)$ of degree $\leqq C(r)n$ such that the polynomial

$$p_n(x) := \big(q_n(x) - q_n(y_1)\big) V_n(x) + q_n(y_1)W_n(x)$$

is comonotone with f, and the following inequalities are satisfied:

$$\big|\operatorname{sgn}(x - y_1) - V_n(x)\big| \leqq C(r)\psi_j^{\mu}(x),$$

and

$$\big|\operatorname{sgn}(x - y_1) - W_n(x)\big| \leqq C(r)\psi_j^{\mu}(x),$$

where $\psi_j(x) := \frac{h_j}{|x-x_j|+h_j}$ (recall that $y_1 \in [x_j, x_{j-1})$).

To complete the proof it remains to show that

$$\|f - p_n\|_p \leqq C(r)\omega_2^{\varphi}(f, n^{-1})_p. \tag{12}$$

Now,

$$\begin{aligned}\|f - p_n\|_p^p = \Big\|(\hat{f} - q_n)\operatorname{sgn}(\cdot - y_1) + q_n\big(\operatorname{sgn}(\cdot - y_1) - V_n(\cdot)\big)\\ +q_n(y_1)(V_n - W_n)\Big\|_p^p\end{aligned}$$

$$\leqq C\omega_2^{\varphi}(\hat{f}, n^{-1})_p^p + C\int_{-1}^{1} |f(x)|^p \psi_j^{\mu p}(x)\,dx + C\int_{-1}^{1} |q_n(y_1)|^p \psi_j^{\mu p}(x)\,dx$$

$$=: I_1 + I_2 + I_3, \quad \text{say.}$$

To estimate I_3 we observe that $\int_{-1}^{1} \psi_j^{\mu p}(x)\,dx \leqq Ch_j$ and that q_n is monotone near y_1. Hence,

$$I_3 = \int_{-1}^{1} \left|q_n(y_1)\right|^p \psi_j^{\mu p}(x)\,dx \leqq Ch_j \left|q_n(y_1)\right|^p \leqq C\|q_n\|^p_{\mathrm{L}_p[y_1-h_j/6,y_1+h_j/6]}$$

$$\leqq C\|q_n - \hat{f}\|^p_{\mathrm{L}_p[y_1-h_j/6,y_1+h_j/6]} + C\|\hat{f}\|^p_{\mathrm{L}_p[y_1-h_j/6,y_1+h_j/6]} \leqq C\omega_2^{\varphi}(f,n^{-1})_p^p,$$

where we applied (9) and (10).

It remains to estimate I_2, for which we need one more lemma.

LEMMA 3. *Let $f \in \Delta^1(Y_r)$ be a continuous piecewise-linear spline on the knot sequence $Z_{r,n}$, $y_1 \in [x_j, x_{j-1})$, and $f(y_1) = 0$. Then for all $x \in [-1,1]$,*

$$(13) \qquad \left|f(x)\right| \leqq C\left(1 + \frac{|x-x_j|}{\delta_n(x,x_j)}\right)^2 \delta_n(x,x_j)^{-1/p}\omega_2^{\varphi}(f,n^{-1})_p,$$

where $\delta_n(x,x_j) = \min\left\{\Delta_n(x), \Delta_n(x_j)\right\}$.

PROOF. For the sake of convenience in notation set $Z_{r,n} = \{-1 = z_m < z_{m-1} < \ldots < z_1 < z_0 = 1\}$ and $\hat{h}_i := z_{i-1} - z_i$. Then $\hat{h}_i \leqq 12\hat{h}_{i\pm1}$. Fix $x > y_1$ (the case $x \leqq y_1$ is similar) and denote

$$Z_{r,n}(y_1,x) := \{i | z_i \in Z_{r,n} \,:\, y_1 \leqq z_i \leqq x\}.$$

Since f is piecewise linear, we have

$$\left|f(x)\right| = \left|f(x) - f(y_1)\right| \leqq \left|f'(\xi)\right|(x - y_1)$$

for some $\xi \in (y_1, x)$. Now,

$$\left|f'(\xi)\right| \leqq \left|f'(\xi) - f'(y_1+)\right| + \left|f'(y_1+)\right|$$

$$\leqq \left|f'(\xi) - f'(y_1+)\right| + \left|f'(y_1+) - f'(y_1-)\right|$$

$$\leqq \sum_{i\in Z_{r,n}(y_1,\xi)} \left|f'(z_i+) - f'(z_i-)\right| \leqq \sum_{i\in Z_{r,n}(y_1,x)} \left|f'(z_i+) - f'(z_i-)\right|$$

$$\leqq C\left(1 + \frac{|x-x_j|}{\delta_n(x,x_j)}\right) \max_{i\in Z_{r,n}(y_1,x)} \left|f'(z_i+) - f'(z_i-)\right|,$$

where in the second inequality we used the fact that $f'(y_1\pm)$ have opposite signs, and in the last inequality, that $\hat{h}_i \geqq \delta_n(x,x_j)$, for all $i \in Z_{r,n}(y_1,x)$. Therefore,

$$\left|f(x)\right| \leqq C\left(1 + \frac{|x-x_j|}{\delta_n(x,x_j)}\right)|x-x_j| \max_{i\in Z_{r,n}(y_1,x)} \left|f'(z_i+) - f'(z_i-)\right|.$$

Thus, to complete the proof we need an estimate of $\max_{i\in Z_{r,n}(y_1,x)} \left| f'(z_i+) - f'(z_i-)\right|$. To this end, let $i \in Z_{r,n}(y_1,x)$ be fixed, and let f_1 be the linear function defined by $f_1|_{[z_{i+1},z_i]} := f|_{[z_{i+1},z_i]}$. We observe that with $\tilde h := 1/100n$ and $\alpha = 1/1000$, say, the set

$$A := \{ x \,:\, z_{i+1} \leqq x - \tilde h\varphi(x) < x < z_i$$
$$< (1-\alpha)z_i + \alpha z_{i-1} \leqq x + \tilde h\varphi(x) < z_{i-1}\},$$

is of measure meas $A \sim \hat h_i$, and for every $x \in A$,

$$\left|\Delta^2_{\tilde h\varphi(x)}(f,x)\right| = \left| f_1\big(x+\tilde h\varphi(x)\big) - f\big(x+\tilde h\varphi(x)\big)\right|$$
$$\geqq \left| f_1\big((1-\alpha)z_i+\alpha z_{i-1}\big) - f\big((1-\alpha)z_i+\alpha z_{i-1}\big)\right|$$
$$= C\alpha\hat h_i \left| f'(z_i+) - f'(z_i-)\right|.$$

This in turn implies,

$$\omega_2^\varphi(f,n^{-1})_p^p - \sup_{0<h\leqq n^{-1}} \int_{-1}^1 \left|\Delta^2_{h\varphi(x)}(f,x)\right|^p dx \geqq \int_{-1}^1 \left|\Delta^2_{\tilde h\varphi(x)}(f,x)\right|^p dx$$
$$\geqq C \operatorname{meas} A\alpha^p \hat h_i^p \left| f'(z_i+) - f'(z_i-)\right|^p - C\hat h_i^{p+1}\left| f'(z_i+) - f'(z_i-)\right|^p.$$

Thus,

$$\left| f'(z_i+) - f'(z_i-)\right| \leqq C\hat h_i^{-(p+1)/p}\omega_2^\varphi(f,n^{-1})_p,$$

and since $\hat h_i \geqq \delta_n(x,x_j)$, $i \in Z_{r,n}(y_1,x)$, then

$$\left| f(x)\right| \leqq C\left(1+\frac{|x-x_j|}{\delta_n(x,x_j)}\right)|x-x_j|\delta_n(x,x_j)^{-(p+1)/p}\omega_2^\varphi(f,n^{-1})_p$$
$$\leqq C\left(1+\frac{|x-x_j|}{\delta_n(x,x_j)}\right)^2\delta_n(x,x_j)^{-1/p}\omega_2^\varphi(f,n^{-1})_p.$$

The proof is complete. □

We are ready to complete the proof of Theorem 1 by showing the proper estimate of I_2. By the same arguments as in the proof of Lemma 3.4 of [7], we see that

$$\psi_j(x)^2 \leqq C\frac{\delta_n(x,x_j)}{|x-x_j|+\delta_n(x,x_j)}.$$

Hence,

$$I_2 \leq C\omega_2^\varphi(f, n^{-1})_p^p \int_{-1}^1 \left(1 + \frac{|x - x_j|}{\delta_n(x, x_j)}\right)^{2p}$$

$$\cdot\delta_n(x, x_j)^{-1} \left(\frac{\delta_n(x, x_j)}{|x - x_j| + \delta_n(x, x_j)}\right)^{\mu p/2} dx \leqq C\omega_2^\varphi(f, n^{-1})_p^p.$$

References

[1] R. K. Beatson and D. Leviatan, On comonotone approximation, *Canad. Math. Bull.*, **26** (1983), 220–224.

[2] R. A. DeVore, Y. K. Hu and D. Leviatan, Convex polynomial and spline approximation in $\mathbf{L}_p$, $0 < p < \infty$, *Constr. Approx.*, **12** (1996), 409–422.

[3] R. A. DeVore, D. Leviatan and X. M. Yu, Polynomial approximation in L_p $(0 < p < 1)$ space, *Constr. Approx.*, **8** (1992), 187–201.

[4] R. A. DeVore and G. G. Lorentz, *Constructive Approximation*, Springer-Verlag (Berlin, 1993).

[5] R. A. DeVore and V. A. Popov, Interpolation of Besov spaces, *Trans. Am. Math. Soc.*, **305** (1988), 397–414.

[6] Z. Ditzian and V. Totik, *Moduli of Smoothness*, Springer-Verlag (Berlin, 1987).

[7] Y. K. Hu, K. Kopotun and X. M. Yu, On positive and copositive polynomial and spline approximation in $\mathbf{L}_p[-1, 1]$, $0 < p < \infty$, *J. Approx. Theory*, **86** (1996), 320–334.

[8] G. L. Iliev, Exact estimates for partially monotone approximation, *Anal. Math.*, **4** (1978), 181–197.

[9] K. A. Kopotun, Coconvex polynomial approximation of twice differentiable functions, *J. Approx. Theory*, **83** (1995), 141–156.

[10] D. Leviatan, Monotone and comonotone polynomial approximation revisited, *J. Approx. Theory*, **53** (1988), 1–16.

[11] D. Leviatan and I. A. Shevchuk, Some positive results and counter examples in comonotone approximation, *J. Approx. Theory*, **86** (1996), 195–206.

[12] D. Leviatan and X. M. Yu, Shape preserving approximation by polynomials in L^p (preprint) (1991).

[13] D. J. Newman, Efficient comonotone approximation, *J. Approx. Theory*, **25** (1979), 189–192.

[14] D. J. Newman, E. Passow and L. Raymon, Piecewise monotone polynomial approximation, *Trans. Am. Math. Soc.*, **172** (1972), 465–472.

[15] E. Passow and L. Raymon, Monotone and comonotone approximation, *Proc. Amer. Math. Soc.*, **42** (1974), 390–394.

[16] E. Passow , L. Raymon and J. A. Roulier, Comonotone polynomial approximation, *J. Approx. Theory*, **11** (1974), 221–224.

[17] A. S. Shvedov, Orders of coapproximation of functions by algebraic polynomials, *Mat. Zametki*, **29** (1981), 117–130. (English translation: *Math. Notes*, **30**, 63–70.)

[18] A. S. Shvedov, Coapproximation of piecewise-monotone functions by polynomials, *Mat. Zametki*, **30** (1981), 839–846. (English translation: *Math. Notes*, **30**, 920–924.)

[19] X. M. Yu, Monotone polynomial approximation in L_p spaces, *Acta Math. Sinica (New Series)*, **3** (1987), 315–326.

[20] X. M. Yu, Degree of comonotone polynomial approximation, *Approx. Theory Appl.*, **4** (1988), 73–78.

[21] S. P. Zhou, On comonotone approximation by polynomials in L^p space, *Analysis*, **13** (1993), 363–376.

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