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INTERPOLATORY POINTWISE ESTIMATES FOR CONVEX POLYNOMIAL APPROXIMATION

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Abstract. This paper deals with approximation of smooth convex functions f on an interval by convex algebraic polynomials which interpolate f and its derivatives at the endpoints of this interval. We call such estimates "interpolatory". One important corollary of our main theorem is the following result on approximation of $f \in \Delta^{(2)}$, the set of convex functions, from W^r, the space of functions on $[-1,1]$ for which $f^{(r-1)}$ is absolutely continuous and $\|f^{(r)}\|_\infty := \operatorname{ess\,sup}_{x\in[-1,1]} |f^{(r)}(x)| < \infty$:

For any $f \in W^r \cap \Delta^{(2)}$, $r \in \mathbb{N}$, there exists a number $\mathcal{N} = \mathcal{N}(f,r)$, such that for every $n \ge \mathcal{N}$, there is an algebraic polynomial of degree $\le n$ which is in $\Delta^{(2)}$ and such that

$$\Big\| \frac{f-P_n}{\varphi^r} \Big\|_\infty \le \frac{c(r)}{n^r} \|f^{(r)}\|_\infty,$$

where $\varphi(x) := \sqrt{1-x^2}$.

For $r=1$ and $r=2$, the above result holds with $\mathcal{N}=1$ and is well known. For $r \ge 3$, it is not true, in general, with $\mathcal{N}$ independent of f.

1. Introduction and main results

We start by recalling some standard notation. As usual, $C^r(I)$ denotes the space of r times continuously differentiable functions on a closed inter-

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val I, $C^0(I) := C(I)$ is the space of continuous functions on I, equipped with the uniform norm which will be denoted by $\|\cdot\|_I$. For $k \in \mathbb{N}$ and an interval I, $\Delta_u^k(f,x;I) := \sum_{i=0}^k (-1)^i \binom{k}{i} f(x+(k/2-i)u)$ if $x \pm ku/2 \in I$ and $:= 0$, otherwise, and $\omega_k(f,t;I) := \sup_{0<u\le t} \|\Delta_u^k(f,\cdot;I)\|_I$ is the kth modulus of smoothness of f on I. When dealing with $I = [-1,1]$, we suppress referring to the interval and use the notation $\|\cdot\| := \|\cdot\|_{[-1,1]}$, $\omega_k(f,t) := \omega_k(f,t;[-1,1])$, $C^r := C^r[-1,1]$, etc. We denote by $\Delta^{(q)}$ the class of all q-monotone functions on $[-1,1]$, i.e., continuous functions such that $\Delta_u^q(f,x) \ge 0$ for all $x \in [-1,1]$ and $u > 0$. In particular, $\Delta^{(1)}$ and $\Delta^{(2)}$ are the classes of all monotone and convex functions on $[-1,1]$, respectively. Also,

$$\varphi(x) := \sqrt{1-x^2} \quad \text{and} \quad \rho_n(x) := \varphi(x)n^{-1} + n^{-2},\ n \in \mathbb{N}, \tag{1.1}$$

$\rho_0(x) \equiv 1$, and Π_n denotes the space of algebraic polynomials of degree $\le n$.

The following classical Timan–Dzyadyk–Freud–Brudnyi direct theorem for the approximation by algebraic polynomials (see e.g. [3, Theorem 8.5.3]) shows that the order of approximation becomes significantly better near the endpoints of $[-1,1]$: if $k \in \mathbb{N}$, $r \in \mathbb{N}_0$ and $f \in C^r$, then for each $n \ge k+r-1$ there is a polynomial $P_n \in \Pi_n$ satisfying

$$|f(x) - P_n(x)| \le c(k,r)\rho_n^r(x)\omega_k(f^{(r)}, \rho_n(x)), \quad x \in [-1,1]. \tag{1.2}$$

Clearly, if we require that the approximating polynomials interpolate f as well as its derivatives at the endpoints, and we are successful, then the estimates should become even better.

Indeed, the following Telyakovskii–Gopengauz–type (i.e., "interpolatory"-type) theorem is an immediate consequence of [8, Corollary 2-3.4] (see e.g. [8] for the history of this problem).

THEOREM 1.1 [8, Corollary 2-3.4]. *Let $r \in \mathbb{N}_0$, $k \in \mathbb{N}$ and $f \in C^r$. Then for any $n \ge \max\{k+r-1, 2r+1\}$, there is a polynomial $P_n \in \Pi_n$ such that* (1.2) *is valid and, moreover,*

$$|f(x) - P_n(x)| \le c(r,k)\varphi^{2r}(x)\omega_k(f^{(r)}, \varphi^{2/k}(x)n^{-2(k-1)/k}), \tag{1.3}$$

if $1 - n^{-2} \le |x| \le 1$.

It follows from [8, Theorem 3] that, for any $\gamma \in \mathbb{R}$, the quantity $\varphi^{2/k}(x)n^{-2(k-1)/k}$ in (1.3) cannot be replaced by $\varphi^{2\beta}(x)n^{\gamma}$ with $\beta > 1/k$. Hence, the estimate (1.3) provides the optimal rate of approximation near the endpoints of $[-1,1]$.

It is a natural question if these estimates are valid if we approximate q-monotone functions by q-monotone polynomials. Of course, as is rather well known, (1.2) may not be valid in the q-monotone case for certain r and k

even if n is allowed to depend on the function f that is being approximated. For example, this is the case if (i) $1 \le q \le 3$, $0 \le r \le q-1$ and $r+k \ge q+2$ ([16] if $q=1$, [21] if $q=2$ or $q=3$), and (ii) $q \ge 4$ and $r+k \ge 3$ ([1]).

Moreover, for any $q, r, k, n \in \mathbb{N}$, there exists a function $f_n \in C^r \cap \Delta^{(q)}$ such that (1.3) is not valid for any polynomial $P_n \in \Pi_n \cap \Delta^{(q)}$ (the construction of such an f_n is the same as in [14], see also [7,13,18]). This means that, in the case $r \ge 1$, (1.3) cannot be true for all functions $f \in C^r \cap \Delta^{(q)}$ and all $n \ge \mathcal{N}(k, r, q)$. We emphasize that this does *not* mean that, for each fixed $f \in C^r \cap \Delta^{(q)}$, (1.3) is invalid for sufficiently large n, i.e., (1.3) may still be valid if $n \ge \mathcal{N}(f)$ (in fact, the proof of this fact in the case $q = k = 2$ is the main result of this paper).

If $r = 0$ and k is "small", then the situation is different: for any $q, n \in \mathbb{N}$, if $r = 0$ and $1 \le k \le 2$, then (1.2) and (1.3) are both valid for q-monotone approximation (it is possible to show that the case for $k = 1$ follows from that for $k = 2$). Indeed, the following interpolatory estimate follows from [4] ($q = 1$), [15,20] ($q = 2$) and [2] ($q \ge 3$): for any $q, n \in \mathbb{N}$ and $f \in C \cap \Delta^{(q)}$, there exists a polynomial $P_n \in \Pi_n \cap \Delta^{(q)}$ such that

$$|f(x) - P_n(x)| \le c(q)\omega_2(f, \varphi(x)/n), \quad x \in [-1, 1], \tag{1.4}$$

where $c(q)$ is an absolute constant. Additionally, (1.2) and (1.3) with $n \ge 2$ are valid for convex approximation (i.e., $q = 2$) if $r = 0$ and $k = 3$ ([8]), and the case $q = 3$, $r = 0$ and $k = 3$ or $k = 4$ is still unresolved (in fact, it is not even known if (1.2) holds if $(q, r, k) = (3, 0, 4)$).

Recently, we were able to show (see [13]) that (1.2) and (1.3) hold for monotone approximation ($q = 1$) if $r \in \mathbb{N}$, $k = 2$ and $n \ge \mathcal{N}(f, r)$, and the main purpose of this paper is to prove an analogous result for convex approximation ($q = 2$). In fact, we follow similar ideas and apply some of the construction in [13], but there are some additional rather significant technical difficulties that we have to overcome in this case (for example, proofs in the cases for $r = 1$ and $r \ge 2$ turn out to be completely different). Also, one of the important tools that we are using is our recent result [14] on convex approximation of $f \in C^r \cap \Delta^{(2)}$, by convex piecewise polynomials (see Theorem 8.1 below).

The following theorem is the main result in this manuscript.

THEOREM 1.2. *Given $r \in \mathbb{N}$, there is a constant $c = c(r)$ with the property that if $f \in C^r \cap \Delta^{(2)}$, then there exists a number $\mathcal{N} = \mathcal{N}(f, r)$, depending on f and r, such that for every $n \ge \mathcal{N}$, there is $P_n \in \Pi_n \cap \Delta^{(2)}$ satisfying*

$$|f(x) - P_n(x)| \le c(r)\, (\varphi(x)/n)^r\, \omega_2(f^{(r)}, \varphi(x)/n), \quad x \in [-1, 1]. \tag{1.5}$$

Moreover, for $x \in [-1, -1+n^{-2}] \cup [1-n^{-2}, 1]$ *the following stronger estimates are valid*:

$$|f(x) - P_n(x)| \le c(r)\varphi^{2r}(x)\omega_2(f^{(r)}, \varphi(x)/n) \tag{1.6}$$

and

$$|f(x) - P_n(x)| \le c(r)\varphi^{2r}(x)\omega_1(f^{(r)}, \varphi^2(x)). \tag{1.7}$$

REMARK 1.3. [14, Theorem 2.3] implies that Theorem 1.2 is *not* valid with $\mathcal{N}$ independent of f.

We now discuss some corollaries and applications of Theorem 1.2.

Recall that, given a number $\alpha > 0$, $\mathrm{Lip}^* \alpha$ denotes the class of all functions f on $[-1, 1]$ such that $\omega_2(f^{(\lceil\alpha\rceil-1)}, t) = O\left(t^{\alpha-\lceil\alpha\rceil+1}\right)$. Together with the classical inverse theorems (see e.g. [12, Theorem 5 and Corollary 6]), (1.2) implies that, if $\alpha > 0$, then a function f is in $\mathrm{Lip}^* \alpha$ if and only if

$$\inf_{P_n \in \Pi_n} \|\rho_n^{-\alpha}(f - P_n)\| = O(1). \tag{1.8}$$

COROLLARY 1.4. *If* $\alpha > 0$ *and* $f \in \mathrm{Lip}^* \alpha \cap \Delta^{(2)}$, *then there exists a constant* $C = C(\alpha)$ *such that, for all sufficiently large* n, *there are polynomials* $P_n \in \Pi_n \cap \Delta^{(2)}$ *satisfying*

$$|f(x) - P_n(x)| \le C\,(\varphi(x)/n)^{\alpha}, \quad x \in [-1, 1]. \tag{1.9}$$

For $0 < \alpha < 2$, (1.9) follows from (1.4) (and was stated in [15]).

In order to state another consequence of Theorem 1.2 we recall that W^r denotes the space of functions on $[-1, 1]$ for which $f^{(r-1)}$ is absolutely continuous and $\|f^{(r)}\|_\infty := \operatorname{ess\,sup}_{x \in [-1,1]} |f^{(r)}(x)| < \infty$.

COROLLARY 1.5. *For any* $f \in W^r \cap \Delta^{(2)}$, $r \in \mathbb{N}$, *there exists a number* $\mathcal{N} = \mathcal{N}(f, r)$, *such that*

$$\sup_{n \ge \mathcal{N}} \inf_{P_n \in \Pi_n \cap \Delta^{(2)}} \left\| \frac{f - P_n}{\varphi^r(\min\{1/n, \varphi\})^r} \right\|_\infty \le c(r)\|f^{(r)}\|_\infty. \tag{1.10}$$

In particular,

$$\sup_{n \ge \mathcal{N}} \inf_{P_n \in \Pi_n \cap \Delta^{(2)}} \left\| \frac{f - P_n}{\varphi^r} \right\|_\infty \le \frac{c(r)}{n^r}\|f^{(r)}\|_\infty. \tag{1.11}$$

It follows from [14, Theorem 2.3] that, if $r \ge 2$ and $r \ge 3$, then, respectively, inequalities (1.10) and (1.11) are not true, in general, with $\mathcal{N}$ independent of f. For all other $r \in \mathbb{N}$, these inequalities hold with $\mathcal{N} = 1$ which is a corollary of (1.4) with $q = 2$.

2. Notations and some inequalities for the Chebyshev partition

Most symbols used in this paper were introduced and discussed in [13]. For convenience, we list them in the following table which also includes symbols introduced in the previous section. Note that, in the proofs below (but not in definitions and statements), we often omit writing index "n" if it does not create any confusion (thus, we write "ρ" instead of "ρ_n", "x_j" instead of "$x_{j,n}$", etc.).

Chebyshev knots and Chebyshev partition	
$x_j := x_{j,n}$	$:= \cos(j\pi/n)$, $0 \le j \le n$; 1 for $j < 0$ and -1 for $j > n$ (Chebyshev knots)
T_n	$:= (x_j)_{j=0}^n$ (Chebyshev partition)
$I_j := I_{j,n}$	$:= [x_j, x_{j-1}]$
$h_j := h_{j,n}$	$:= \lvert I_{j,n}\rvert = x_{j-1} - x_j$
$I_{i,j}$	$:= \bigcup_{k=\min\{i,j\}}^{\max\{i,j\}} I_k = [x_{\max\{i,j\}}, x_{\min\{i,j\}-1}]$, $1 \le i, j \le n$ (the smallest interval containing both I_i and I_j)
$h_{i,j}$	$:= \lvert I_{i,j}\rvert = \sum_{k=\min\{i,j\}}^{\max\{i,j\}} h_k = x_{\min\{i,j\}-1} - x_{\max\{i,j\}}$
ψ_j	$:= \psi_j(x) := \lvert I_j\rvert/(\lvert x - x_j\rvert + \lvert I_j\rvert)$
$\varphi(x)$	$:= \sqrt{1-x^2}$
$\rho_n(x)$	$:= \varphi(x)n^{-1} + n^{-2}$, $n \in \mathbb{N}$, and $\rho_0(x) \equiv 1$
$\delta_n(x)$	$:= \min\{1, n\varphi(x)\}$

k-majorants	
Φ^k	$:= \big\{\, \psi \in C[0,\infty) \mid \psi\uparrow,\ \psi(0)=0, \text{ and } t_2^{-k}\psi(t_2) \le t_1^{-k}\psi(t_1)$ for $0 < t_1 \le t_2 \big\}$. Note: if $f \in C^r$, then $\phi(t) := t^r\omega_k(f^{(r)}, t)$ is equivalent to a function from Φ^{k+r}

Piecewise polynomials on Chebyshev partition	
$\Sigma_k := \Sigma_{k,n}$	the set of continuous piecewise polynomials of degree $\le k-1$ with knots at x_j, $1 \le j \le n-1$
$\Sigma_k^{(1)} := \Sigma_{k,n}^{(1)}$	the set of continuously differentiable piecewise polynomials of degree $\le k-1$ with knots at x_j, $1 \le j \le n-1$
$p_j := p_j(S)$	$:= S\vert_{I_j}$, $1 \le j \le n$ (polynomial piece of S on the interval I_j)
$b_{i,j}(S,\phi)$	$:= \frac{\lVert p_i - p_j\rVert_{I_i}}{\phi(h_j)}\left(\frac{h_j}{h_{i,j}}\right)^k$, where $\phi \in \Phi^k$, $\phi \not\equiv 0$ and $S \in \Sigma_k$
$b_k(S,\phi,A)$	$:= \max_{1\le i,j\le n}\{b_{i,j}(S,\phi) \mid I_i \subset A \text{ and } I_j \subset A\}$, where an interval $A \subseteq [-1,1]$ contains at least one interval I_ν
$b_k(S,\phi)$	$:= b_k(S,\phi,[-1,1]) = \max_{1\le i,j\le n} b_{i,j}(S,\phi)$

Constants	
$C(\gamma_1,\dots,\gamma_\mu)$	positive constants depending only on parameters $\gamma_1,\dots,\gamma_\mu$ that may be different on different occurences
c	positive constants that are either absolute or may only depend on the parameters k and r (if present)
C_i	positive constants that are fixed throughout this paper
$\overset{\gamma_1,\dots,\gamma_\mu}{\sim}$	$A \overset{\gamma_1,\dots,\gamma_\mu}{\sim} B$ iff $C^{-1}B \le A \le CB$, for some positive constant $C = C(\gamma_1,\dots,\gamma_\mu)$

Indicator functions and truncated powers	
$\chi_j(x)$	$:= \chi_{[x_j,1]}(x) := 1$, if $x_j \le x \le 1$, and $:= 0$, otherwise
$\Phi_j(x)$	$:= (x-x_j)_+ := (x-x_j)\chi_j(x) = \int_{-1}^x \chi_j(t)\,dt$

We now collect all facts and inequalities for the Chebyshev partition that we need throughout this paper. Many of them are checked by straightforward calculations (also, see e.g. [5,13,19] for references). Unless specified otherwise, it is assumed that $1 \le j \le n$, $x,y \in [-1,1]$.

$$n^{-1}\varphi(x) < \rho_n(x) < h_j < 5\rho_n(x), \quad x \in I_j \tag{2.1}$$

$$h_{j\pm1} < 3h_j \tag{2.2}$$

$$\rho_n^2(y) < 4\rho_n(x)(|x-y| + \rho_n(x)) \tag{2.3}$$

$$(|x-y| + \rho_n(x))/2 < (|x-y| + \rho_n(y)) < 2(|x-y| + \rho_n(x)) \tag{2.4}$$

$$\rho_n(x) \le |x-x_j|, \quad \text{for any } 0 \le j \le n \text{ and } x \notin (x_{j+1}, x_{j-1}) \tag{2.5}$$

$$\begin{cases} \delta_n(x) \le n\varphi(x) < \pi\delta_n(x), & \text{if } x \in [-1,x_{n-1}] \cup [x_1,1], \\ \delta_n(x) = 1, & \text{if } x \in [x_{n-1}, x_1] \end{cases} \tag{2.6}$$

$$\rho_n^2(x) < 8h_j(|x-x_j| + \rho_n(x)) \tag{2.7}$$

$$\Big(\frac{\rho_n(x)}{\rho_n(x) + |x-x_j|}\Big)^2 < c\psi_j(x) \tag{2.8}$$

$$\rho_n(x) + |x-x_j| \sim \rho_n(x) + \operatorname{dist}(x, I_j) \tag{2.9}$$

$$\sum_{j=1}^n \psi_j^2(x) \le c \tag{2.10}$$

$$\sum_{j=1}^n \Big(\frac{\rho_n(x)}{\rho_n(x) + \operatorname{dist}(x, I_j)}\Big)^4 \le c \tag{2.11}$$

$$c\psi_j^2(x)\delta_n^2(x) \le \frac{1-x^2}{(1+x_{j-1})(1-x_j)} \le c\delta_n^2(x)\psi_j^{-2}(x) \tag{2.12}$$

$$c\psi_j^2(x)\rho_n(x) \le h_j \le c\psi_j^{-1}(x)\rho_n(x) \tag{2.13}$$

$$c\psi_j^{2k}(x)\phi(\rho_n(x)) \le \phi(h_j) \le c\psi_j^{-k}(x)\phi(\rho_n(x)). \tag{2.14}$$

3. Auxiliary results on polynomial approximation of indicator functions and truncated powers

Recall the notation

$$t_j(x) := \Big(\frac{\cos 2n \arccos x}{x - x_j^0}\Big)^2 + \Big(\frac{\sin 2n \arccos x}{x - \bar{x}_j}\Big)^2, \tag{3.1}$$

where $\bar{x}_j := \cos((j-1/2)\pi/n)$ for $1 \le j \le n$, $x_j^0 := \cos((j-1/4)\pi/n)$ for $1 \le j < n/2$, $x_j^0 := \cos((j-3/4)\pi/n)$ for $n/2 \le j \le n$, and note that $t_j \in \Pi_{4n-2}$ and, for all $1 \le j \le n$,

$$t_j(x) \sim (|x - x_j| + h_j)^{-2}, \quad x \in [-1,1], \tag{3.2}$$

(see e.g. [19] or [10, (22), Proposition 5]).

For $\gamma_1, \gamma_2 \in \mathbb{N}_0$, $\xi, \mu \in \mathbb{N}$, and $1 \le j \le n$, we let

$$\begin{aligned} \mathcal{T}_j(x) := \mathcal{T}_{j,n}(x) &:= \mathcal{T}_{j,n}(x; \gamma_1, \gamma_2, \xi, \mu) \\ &:= d_j^{-1} \int_{-1}^{x} (y - x_j)^{\gamma_1} (x_{j-1} - y)^{\gamma_2} (1-y^2)^{\xi} t_j^{\mu}(y)\, dy, \end{aligned}$$

where $d_j := d_j(\gamma_1, \gamma_2, \xi, \mu)$ is the normalizing constant such that $\mathcal{T}_j(1) = 1$. Then, it is possible to show (see e.g. [9, Proposition 4]) that, for sufficiently large μ, the function $\mathcal{T}_j$ is well defined and is a polynomial of degree $\le c\mu n$ (with some absolute constant c), and

$$d_j \sim (1 + x_{j-1})^{\xi} (1 - x_j)^{\xi} h_j^{-2\mu+1+\gamma_1+\gamma_2}.$$

Also,

$$1 - x_{j-1} < \int_{-1}^{1} \mathcal{T}_j(t)\, dt < 1 - x_j, \quad 1 \le j \le n. \tag{3.3}$$

Indeed, denoting for convenience $\vartheta(y) := (y - x_j)^{\gamma_1}(x_{j-1} - y)^{\gamma_2}(1 - y^2)^{\xi} t_j^{\mu}(y)$, we have

$$\int_{-1}^{1} \mathcal{T}_j(t)\, dt < 1 - x_j \iff \int_{-1}^{1}\int_{-1}^{t} \vartheta(y)\, dy\, dt < (1 - x_j) \int_{-1}^{1} \vartheta(t)\, dt$$

$$\iff \int_{-1}^{1} (t - x_j)\vartheta(t)\,dt = d_j(\gamma_1 + 1, \gamma_2, \xi, \mu) > 0$$

(the other inequality is proved similarly).

Now, for the polynomials

$$\tau_j(x) := \mathcal{T}_{j,n}(x; 0, 0, \xi, \mu) \quad \text{and} \quad \widetilde{\tau}_j(x) := \mathcal{T}_{j,n}(x; 1, 1, \xi, \mu),$$

the following lemma was proved in [13].

LEMMA 3.1 [13, Lemmas 4.1 and 4.2]. *If $\alpha, \beta \geq 1$, then for sufficiently large ξ and μ depending only on α and β and for each $1 \leq j \leq n-1$, the polynomials τ_j and $\widetilde{\tau}_j$ of degree $\leq C(\alpha,\beta)n$ satisfy*

$$\tau_j'(x) \geq C(\alpha,\beta)|I_j|^{-1}\delta_n^{8\alpha}(x)\psi_j^{30(\alpha+\beta)}(x), \quad x \in [-1,1], \tag{3.4}$$

$$\widetilde{\tau}_j'(x) \leq 0, \quad \textit{for } x \in [-1, x_j] \cup [x_{j-1}, 1], \tag{3.5}$$

and for all $x \in [-1,1]$,

$$\max\bigl\{|\tau_j'(x)|, |\widetilde{\tau}_j'(x)|\bigr\} \leq C(\alpha,\beta)|I_j|^{-1}\delta_n^{\alpha}(x)\psi_j^{\beta}(x) \tag{3.6}$$

and

$$\max\bigl\{|\chi_j(x) - \tau_j(x)|, |\chi_j(x) - \widetilde{\tau}_j(x)|\bigr\} \leq C(\alpha,\beta)\delta_n^{\alpha}(x)\psi_j^{\beta}(x). \tag{3.7}$$

REMARK 3.2. The statement of this lemma is not valid if $j = n$ since $\chi_n \equiv 1$, $\tau_n(-1) = 0$ and $\delta_n(-1) = 0$.

Inequalities (3.3) imply that, for each $1 \leq j \leq n-1$, there exists a constant $0 < \lambda_j < 1$ such that the polynomial

$$\begin{aligned} \mathcal{Q}_j(x) &:= \mathcal{Q}_{j,n}(x) := \mathcal{Q}_{j,n}(x; \gamma_1, \gamma_2, \xi, \mu) \\ &:= \int_{-1}^{x} (\lambda_j \mathcal{T}_j(t) + (1 - \lambda_j)\mathcal{T}_{j+1}(t))\,dt \end{aligned} \tag{3.8}$$

satisfies $\mathcal{Q}_j(1) = 1 - x_j$. This implies that, if $\mathcal{T}_j$ is such that (3.7) is satisfied, then $\mathcal{Q}_j$ provides a "good" approximation of Φ_j, $1 \leq j \leq n-2$. The proof of this fact is rather standard. Indeed, first note that, for $1 \leq j \leq n-2$ and $x \in [-1,1]$,

$$|\chi_j(x) - \chi_{j+1}(x)| \leq |\chi_{I_{j+1}}(x)| \leq C\delta_n^{\alpha}(x)\psi_j^{\beta}(x) \quad \text{and} \quad \psi_j(x) \sim \psi_{j+1}(x).$$

Now, if $x \leq x_j$, then (assume that $\beta > 1$)

$$|\Phi_j(x) - \mathcal{Q}_j(x)| \leq \left| \int_{-1}^{x} \bigl(\lambda_j|\mathcal{T}_j(t) - \chi_j(t)| + (1 - \lambda_j)|\mathcal{T}_{j+1}(t) - \chi_j(t)|\bigr)\,dt \right|$$

$$\leq C \int_{-1}^{x} \delta_n^\alpha(t) \psi_j^\beta(t)\, dt \leq C \delta_n^\alpha(x) \int_{-\infty}^{x} |I_j|^\beta (x_j - t + |I_j|)^{-\beta}\, dt$$
$$\leq C |I_j| \delta_n^\alpha(x) \psi_j^{\beta-1}(x)$$

and, if $x > x_j$, then, similarly,

$$|\Phi_j(x) - \mathcal{Q}_j(x)| = \left| \int_x^1 \left(\chi_j(t) - \mathcal{Q}_j'(t) \right)\, dt \right|$$
$$\leq \left| \int_x^1 \left(\lambda_j |\mathcal{T}_j(t) - \chi_j(t)| + (1 - \lambda_j) |\mathcal{T}_{j+1}(t) - \chi_j(t)| \right) dt \right|$$
$$\leq C |I_j| \delta_n^\alpha(x) \psi_j^{\beta-1}(x).$$

Now, for $1 \leq j \leq n-1$, defining

$$\begin{cases} \mathcal{F}_j(x) := \mathcal{F}_{j,n}(x) := \mathcal{Q}_{2j,2n}(x; 0, 0, \xi, \mu), \\ \widetilde{\mathcal{F}}_j(x) := \widetilde{\mathcal{F}}_{j,n}(x) := \mathcal{Q}_{2j-1,2n}(x; 1, 1, \xi, \mu), \end{cases} \tag{3.9}$$

and noting that $x_{j,n} = x_{2j,2n}$, $h_{j,n} \sim h_{2j,2n} \sim h_{2j-1,2n}$, $\psi_{j,n} \sim \psi_{2j,2n} \sim \psi_{2j-1,2n}$, $\delta_n \sim \delta_{2n}$, we have the following result which follows from Lemma 3.1.

LEMMA 3.3. *If $\alpha, \beta \geq 1$, then for sufficiently large ξ and μ depending only on α and β and for each $1 \leq j \leq n-1$, polynomials $\mathcal{F}_j$ and $\widetilde{\mathcal{F}}_j$ of degree $\leq C(\alpha,\beta) n$ defined in* (3.9) *satisfy*

$$\mathcal{F}_j''(x) \geq C(\alpha,\beta) |I_j|^{-1} \delta_n^{8\alpha}(x) \psi_j^{30(\alpha+\beta)}(x), \quad x \in [-1,1], \tag{3.10}$$

$$\widetilde{\mathcal{F}}_j''(x) \leq 0, \quad \text{for } x \in [-1, x_j] \cup [x_{j-1}, 1], \tag{3.11}$$

and for all $x \in [-1,1]$,

$$\max\left\{ |\mathcal{F}_j''(x)|, |\widetilde{\mathcal{F}}_j''(x)| \right\} \leq C(\alpha,\beta) |I_j|^{-1} \delta_n^\alpha(x) \psi_j^\beta(x) \tag{3.12}$$

$$\max\left\{ |\chi_j(x) - \mathcal{F}_j'(x)|, |\chi_j(x) - \widetilde{\mathcal{F}}_j'(x)| \right\} \leq C(\alpha,\beta) \delta_n^\alpha(x) \psi_j^\beta(x) \tag{3.13}$$

and

$$\max\left\{ |\Phi_j(x) - \mathcal{F}_j(x)|, |\Phi_j(x) - \widetilde{\mathcal{F}}_j(x)| \right\} \leq C(\alpha,\beta) |I_j| \delta_n^\alpha(x) \psi_j^{\beta-1}(x). \tag{3.14}$$

4. Auxiliary results on properties of piecewise polynomials

LEMMA 4.1 [13, Lemma 5.1]. *Let $k \in \mathbb{N}$, $\phi \in \Phi^k$, $f \in C[-1,1]$ and $S \in \Sigma_{k,n}$. If $\omega_k(f,t) \leq \phi(t)$ and $|f(x) - S(x)| \leq \phi(\rho_n(x))$, $x \in [-1,1]$, then $b_k(S,\phi) \leq c(k)$.*

LEMMA 4.2 [6, Lemma 2.1]. *Let $k \ge 3$, $\phi \in \Phi^k$ and $S \in \Sigma_{k,n}^{(1)}$. Then $b_k(S,\phi) \le c(k)\|\rho_n^2\phi^{-1}(\rho_n)S''\|_\infty$.*

The following lemma on simultaneous polynomial approximation of piecewise polynomials and their derivatives is an immediate corollary of [13, Lemma 8.1] (with $q = r = 2$ and $k \ge 2$).

LEMMA 4.3 [13, Lemma 8.1]. *Let $\gamma > 0$, $k \in \mathbb{N}$, $\phi \in \Phi^k$, and let $n, n_1 \in \mathbb{N}$ be such that n_1 is divisible by n. If $S \in \Sigma_{k,n}$, then there exists a polynomial $D_{n_1}(\cdot, S)$ of degree $\le Cn_1$ such that*

$$|S(x) - D_{n_1}(x,S)| \le C\delta_n^\gamma(x)\phi(\rho_n(x))b_k(S,\phi). \tag{4.1}$$

Moreover, if $S \in C^1$ and $A := [x_{\mu^*}, x_{\mu_*}]$, $0 \le \mu_* < \mu^* \le n$, then for all $x \in A \setminus \{x_j\}_{j=1}^{n-1}$, we have

$$\begin{aligned} \big|S''(x) - D''_{n_1}(x,S)\big| &\le C\delta_n^\gamma(x)\frac{\phi(\rho_n(x))}{\rho_n^2(x)} \\ &\times \Big(b_k(S,\phi,A) + b_k(S,\phi)\,\frac{n}{n_1}\Big(\frac{\rho_n(x)}{\operatorname{dist}(x,[-1,1]\setminus A)}\Big)^{\gamma+1}\Big). \end{aligned} \tag{4.2}$$

All constants C may depend only on k and γ and are independent of the ratio n_1/n.

5. Convex polynomial approximation of piecewise polynomials with "small" derivatives

LEMMA 5.1. *Let $\alpha > 0$, $k \in \mathbb{N}$ and $\phi \in \Phi^k$, be given. If $S \in \Sigma_{k,n} \cap \Delta^{(2)}$ is such that*

$$|S''(x)| \le \frac{\phi(\rho_n(x))}{\rho_n^2(x)}, \quad x \in [x_{n-1}, x_1] \setminus \{x_j\}_{j=1}^{n-1}, \tag{5.1}$$

$$0 \le S'(x_j+) - S'(x_j-) \le \frac{\phi(\rho_n(x_j))}{\rho_n(x_j)}, \quad 1 \le j \le n-1, \tag{5.2}$$

and

$$S''(x) = 0, \quad x \in [-1, x_{n-1}) \cup (x_1, 1], \tag{5.3}$$

then there is a polynomial $P \in \Delta^{(2)} \cap \Pi_{Cn}$, $C = C(k,\alpha)$ such that

$$|S(x) - P(x)| \le C(k,\alpha)\delta_n^\alpha(x)\,\phi\left(\rho_n(x)\right), \quad x \in [-1,1]. \tag{5.4}$$

Proof. Denote by S_1 the piecewise linear continuous function interpolating S at the points x_j, $0 \le j \le n$, and let $l_j := S_1\big|_{I_j}$. Then $S_1 \in \Delta^{(2)}$,

$$S_1(x) = S(x), \quad x \in I_1 \cup I_n, \tag{5.5}$$

and, for $x \in I_j$, $1 \le j \le n$, we have by Whitney's inequality and (2.1)

$$|S(x) - S_1(x)| \le c\omega_2(S, h_j; I_j) \le ch_j^2 \|S''\|_{L_\infty(I_j)} \le c\phi(h_j),$$

which can be rewritten as

$$|S(x) - S_1(x)| \le c\phi(\rho_n(x)), \quad x \in [-1,1]. \tag{5.6}$$

We now write S_1 as

$$S_1(x) = S_1(-1) + S_1'(-1)(x+1) + \sum_{j=1}^{n-1} \alpha_j \Phi_j(x), \quad \alpha_j := S_1'(x_j+) - S_1'(x_j-),$$

and note that, by Markov and Whitney inequalities,

$$\begin{aligned} 0 \le \alpha_j = l_j'(x_j) - l_{j+1}'(x_j) &\le ch_j^{-1}\|l_j - l_{j+1}\|_{I_j \cup I_{j+1}} \\ \le ch_j^{-1}\omega_2(S, h_j; I_j \cup I_{j+1}) &\le ch_j\big(\|S''\|_{L_\infty(I_j)} + \|S''\|_{L_\infty(I_{j+1})}\big) \\ + c\big(S'(x_j+) - S'(x_j-)\big) &\le ch_j^{-1}\phi(h_j), \quad 1 \le j \le n-1. \end{aligned}$$

Now, if

$$P(x) := S_1(-1) + S_1'(-1)(x+1) + \sum_{j=1}^{n-1} \alpha_j \mathcal{F}_j(x),$$

then P is a convex polynomial of degree $\le Cn$ and, in view of (5.5) and (5.6), we only need to estimate $|S_1(x) - P(x)|$. Note that (2.13) implies, for all $1 \le j \le n$ and $x \in [-1,1]$,

$$\phi(h_j) \le \phi(c\psi_j^{-1}(x)\rho_n(x)) \le C\,\psi_j^{-k}(x)\phi(\rho_n(x)).$$

Hence, by Lemma 3.3 and (2.10), we have

$$\begin{aligned} |S_1(x) - P(x)| &\le \sum_{j=1}^{n-1} \alpha_j |\Phi_j(x) - \mathcal{F}_j(x)| \le C \sum_{j=1}^{n-1} \phi(h_j)\delta_n^\alpha(x)\psi_j^\beta(x) \\ &\le C\delta_n^\alpha(x)\phi(\rho_n(x)) \sum_{j=1}^{n-1} \psi_j^{\beta-k}(x) \le C\delta_n^\alpha(x)\phi(\rho_n(x)), \end{aligned}$$

provided $\beta \ge k+2$. $\square$

6. One particular polynomial with controlled second derivative

All constants C in this section may depend on k, α and β.

We start with the following auxiliary lemma.

Lemma 6.1 [17, Lemma 9]. *Let $A := \{j_0, \dots, j_0 + l_0\}$ and let $A_1, A_2 \subset A$ be such that $\# A_1 = 2l_1$ and $\# A_2 = l_2$. Then, there exist $2l_1$ constants a_i, $i \in A_1$, such that $|a_i| \le (l_0/l_1)^2$ and*

$$\frac{1}{l_2}\sum_{j\in A_2}(x-x_j) + \frac{1}{l_1}\sum_{j\in A_1} a_j(x-x_j) \equiv 0.$$

Lemma 6.2. *Let $\alpha > 0$, $k \in \mathbb{N}$, $k \ge 2$, $\beta > 0$ be sufficiently large ($\beta \ge k+7$ will do) and let $\phi \in \Phi^k$ be of the form $\phi(t) := t\psi(t)$, $\psi \in \Phi^{k-1}$. Also, let $E \subset [-1,1]$ be a closed interval which is the union of $m_E \ge 100$ of the intervals I_j, and let a set $J \subset E$ consist of m_J intervals I_j, where $1 \le m_J < m_E/4$. Then there exists a polynomial $Q_n(x) = Q_n(x, E, J)$ of degree $\le Cn$, satisfying*

$$Q_n''(x) \ge C\,\frac{m_E}{m_J}\,\delta_n^{\alpha_1}(x)\,\frac{\phi(\rho_n(x))}{\rho_n^2(x)}\Big(\frac{\rho_n(x)}{\max\{\rho_n(x), \operatorname{dist}(x,E)\}}\Big)^{\beta_1}, \tag{6.1}$$

where $x \in J \cup ([-1,1]\setminus E)$,

$$Q_n''(x) \ge -\delta_n^{\alpha}(x)\frac{\phi(\rho_n(x))}{\rho_n^2(x)}, \quad x \in E \setminus J, \tag{6.2}$$

and

$$|Q_n(x)| \le C\, m_E^{k_1}\delta_n^{\alpha}(x)\rho_n(x)\phi(\rho_n(x))\sum_{j:I_j\subset E}\frac{h_j}{(|x-x_j|+\rho_n(x))^2}, \; x \in [-1,1], \tag{6.3}$$

where $\alpha_1 = 8\alpha$, $\beta_1 = 60(\alpha+\beta)+k+1$ and $k_1 = k+6$.

Proof. As in the proof of [13, Lemma 9.1], we may assume that $I_n \not\subset E$ provided that the condition $m_J < m_E/4$ is replaced by $m_J \le m_E/4$. Also, we use the same notation that was used in [13]: $\rho := \rho_n(x)$, $\delta := \delta_n(x)$, $\psi_j := \psi_j(x)$,

$$\mathcal{E} := \{1 \le j \le n \mid I_j \subset E\}, \quad \mathcal{J} := \{1 \le j \le n \mid I_j \subset J\},$$
$$j_* := \min\{j \mid j \in \mathcal{E}\}, \quad j^* := \max\{j \mid j \in \mathcal{E}\},$$
$$\mathcal{A} := \mathcal{J} \cup \{j_*, j^*\} \quad \text{and} \quad \mathcal{B} := \mathcal{E} \setminus \mathcal{A}.$$

Now, let $\widetilde{E} \subset E$ be the subinterval of E such that

(i) $\widetilde{E}$ is a union of $\lfloor m_E/3 \rfloor$ intervals I_j, and

(ii) $\widetilde{E}$ is centered at 0 as much as E allows it, i.e., among all subintervals of E consisting of $\lfloor m_E/3 \rfloor$ intervals I_j, the center of $\widetilde{E}$ is closest to 0.

Then (see [13]),

$$\text{if } I_j \subset \widetilde{E} \text{ and } I_i \subset E \setminus \widetilde{E}, \quad \text{then} \quad |I_j| \ge |I_i|, \tag{6.4}$$

$$|I_j| \sim \frac{|\widetilde{E}|}{m_E}, \quad \text{for all } I_j \subset \widetilde{E}, \tag{6.5}$$

and, with $\widetilde{\mathcal{E}} := \{1 \le j \le n \mid I_j \subset \widetilde{E}\}$ and $\widetilde{\mathcal{B}} := \mathcal{B} \cap \widetilde{\mathcal{E}} = \widetilde{\mathcal{E}} \setminus \mathcal{A}$,

$$\#\widetilde{\mathcal{B}} \ge m_E/20. \tag{6.6}$$

Note that index $j = n$ is in none of the sets $\mathcal{A}$, $\mathcal{B}$ and $\widetilde{\mathcal{B}}$.

It follows from Lemma 6.1 ($l_0 \sim m_E$, $l_1 \sim \lfloor \#\widetilde{\mathcal{B}}/2 \rfloor \sim m_E$, $l_2 \sim m_J$) that there exist constants λ_i, $i \in \widetilde{\mathcal{B}}$, such that $|\lambda_i| \le c$, $i \in \widetilde{\mathcal{B}}$, and

$$\frac{m_E}{m_J} \sum_{j \in \mathcal{A}} (x - x_j) + \sum_{j \in \widetilde{\mathcal{B}}} \lambda_i (x - x_j) \equiv 0. \tag{6.7}$$

We now let i_* be such that I_{i^*} is the largest interval in $\widetilde{E}$ and $h_* := h_{i^*} = |I_{i^*}|$, and

$$Q_n(x) := \kappa \frac{\phi(h_*)}{h_*} \Bigg(\frac{m_E}{m_J} \sum_{j \in \mathcal{A}} \mathcal{F}_j(x) + \sum_{j \in \widetilde{\mathcal{B}}} \lambda_j \overline{\mathcal{F}}_j(x) \Bigg),$$

where κ is a sufficiently small absolute constant to be prescribed and

$$\overline{\mathcal{F}}_j := \begin{cases} \widetilde{\mathcal{F}}_j, & \text{if } \lambda_j < 0, \\ \mathcal{F}_j, & \text{if } \lambda_j \ge 0. \end{cases}$$

It follows from (6.4) that

$$h_j \le h_*, \quad j \in \mathcal{E},$$

and so $\rho \le h_*$ and $\phi(\rho)/\rho = \psi(\rho) \le \psi(h_*) = \phi(h_*)/h_*$, for all $x \in E$ as well as all $x \notin E$ such that $h_* \ge \rho$. If $x \notin E$ and $h_* < \rho$, then by (2.1), (2.3) and (2.4)

$$\frac{\phi(h_*)}{h_*} \ge \frac{\phi(\rho) h_*^{k-1}}{\rho^k} \ge \frac{\phi(\rho)}{\rho^k} \max\{h_{j_*}^{k-1}, h_{j^*}^{k-1}\}$$

$$\geq c\frac{\phi(\rho)}{\rho^k}\cdot\frac{\rho^{2k-2}}{(\min\{|x-x_{j_*}|,|x-x_{j^*}|\}+\rho)^{k-1}}$$

$$\geq c\phi(\rho)\,\frac{\rho^{k-2}}{(\max\{\rho,\operatorname{dist}(x,E)\})^{k-1}}\,.$$

Hence,

$$\frac{\phi(h_*)}{h_*}\geq c\frac{\phi(\rho)}{\rho}\Big(\frac{\rho}{\max\{\rho,\operatorname{dist}(x,E)\}}\Big)^{k-1},\quad\text{for all } x\in[-1,1]. \tag{6.8}$$

We now note that $\lambda_i\overline{\mathcal{F}}''_j(x)\geq 0$ if $j\in\mathcal{B}$ and $x\in J\cup([-1,1]\setminus E)$ (as well as for any $x\in I_{j_*}\cup I_{j^*}$). Hence, for these x, using Lemma 3.3, (2.13), (2.8) and (6.8) we have

$$Q''_n(x)\geq\kappa\frac{\phi(h_*)}{h_*}\cdot\frac{m_E}{m_J}\sum_{j\in\mathcal{A}}\mathcal{F}''_j(x)\geq C\kappa\delta^{8\alpha}(x)\frac{\phi(h_*)}{h_*}\cdot\frac{m_E}{m_J}\sum_{j\in\mathcal{A}}h_j^{-1}\psi_j^{30(\alpha+\beta)}$$

$$\geq C\kappa\delta^{8\alpha}(x)\frac{\phi(h_*)}{\rho h_*}\cdot\frac{m_E}{m_J}\sum_{j\in\mathcal{A}}\psi_j^{30(\alpha+\beta)+1}$$

$$\geq C\kappa\delta^{8\alpha}(x)\frac{\phi(h_*)}{\rho h_*}\cdot\frac{m_E}{m_J}\sum_{j\in\mathcal{A}}\left(\frac{\rho}{\rho+|x-x_j|}\right)^{60(\alpha+\beta)+2}$$

$$\geq C\kappa\delta^{8\alpha}(x)\frac{m_E}{m_J}\cdot\frac{\phi(\rho)}{\rho^2}\left(\frac{\rho}{\max\{\rho,\operatorname{dist}(x,E)\}}\right)^{60(\alpha+\beta)+k+1},$$

since, for $x\notin E$, $\max\{\rho,\operatorname{dist}(x,E)\}\sim\min\{|x-x_{j^*}|,|x-x_{j_*}|\}+\rho$, and if $x\in J$, then $x\in I_j$ for some $j\in\mathcal{A}$, so that $\rho/(|x-x_j|+\rho)\sim 1$ for that j.

If $x\in E\setminus J$ and $x\notin I_{j_*}\cup I_{j^*}$, then there exists $j_0\in\mathcal{B}$ such that $x\in I_{j_0}$. If $j_0\notin\widetilde{\mathcal{B}}$, or if $j_0\in\widetilde{\mathcal{B}}$ and $\lambda_{j_0}\geq 0$, then, clearly, $Q''_n(x)\geq 0$. Otherwise, since $h_*\sim h_{j_0}$ by (6.5), we have using (3.12)

$$Q''_n(x)\geq\kappa\lambda_{j_0}\phi(h_*)h_*^{-1}\widetilde{\mathcal{F}}''_{j_0}(x)\geq -C\kappa\phi(h_{j_0})h_{j_0}^{-2}\delta^\alpha\psi_{j_0}^\beta$$

$$\geq -C\kappa\,\frac{\phi(\rho)}{\rho^2}\,\delta^\alpha\geq-\frac{\phi(\rho)}{\rho^2}\,\delta^\alpha,$$

for sufficiently small κ.

We now estimate $|Q_n(x)|$. Let

$$L(x):=\kappa\frac{\phi(h_*)}{h_*}\Big(\frac{m_E}{m_J}\sum_{j\in\mathcal{A}}\Phi_j(x)+\sum_{j\in\widetilde{\mathcal{B}}}\lambda_j\Phi_j(x)\Big),$$

It follows from [13, (9.8)] that, for any $j \in \mathcal{E}$, $cm_E \le |E|/h_j \le m_E^2$. This implies that $h_* \le c|E|/m_E \le cm_E h_j$, $j \in \mathcal{E}$, and so $\phi(h_*) \le cm_E^k \phi(h_j)$, $j \in \mathcal{E}$. Hence, using (3.14) as well as the estimate (see [13, pp. 1282-1283])

$$\sum_{j\in\mathcal{E}} \phi(h_j)\psi_j^{\beta-1} \le C\phi(\rho)\sum_{j\in\mathcal{E}} \frac{h_j\rho}{(|x-x_j|+\rho)^2}$$

which is true if $\beta \ge k+7$, we have

$$|Q_n(x)-L(x)| = \kappa\frac{\phi(h_*)}{h_*}\left|\frac{m_E}{m_J}\sum_{j\in\mathcal{A}}(\mathcal{F}_j(x)-\Phi_j(x)) + \sum_{j\in\widetilde{\mathcal{B}}}\lambda_j(\overline{\mathcal{F}}_j(x)-\Phi_j(x))\right|$$

$$\le Cm_E\delta^\alpha\frac{\phi(h_*)}{h_*}\sum_{j\in\mathcal{E}} h_j\psi_j^{\beta-1} \le Cm_E^{k+1}\delta^\alpha\sum_{j\in\mathcal{E}}\phi(h_j)\psi_j^{\beta-1}$$

$$\le Cm_E^{k+1}\delta^\alpha\phi(\rho)\sum_{j\in\mathcal{E}}\frac{h_j\rho}{(|x-x_j|+\rho)^2}.$$

It remains to estimate $|L(x)|$. First assume that $x \notin E$. If $x \le x_{j^*}$, then $\Phi_j(x) = 0$, $j \in \mathcal{A}\cup\widetilde{\mathcal{B}}$, and $L(x)=0$. If, on the other hand, $x > x_{j_*}$, then $\Phi_j(x) = x - x_j$, $j \in \mathcal{A}\cup\widetilde{\mathcal{B}}$, so that (6.7) implies that $L(x)=0$. Hence, in particular, $L(x)=0$ for $x \in I_1\cup I_n$.

Suppose now that $x \in E\setminus I_1$ (recall that we already assumed that E does not contain I_n). Then, as above, $h_* \le c|E|/m_E \le c\rho m_E$ and so $\phi(h_*) \le cm_E^k\phi(\rho)$. Also, $h_* \ge |E|/m_E^2$. Hence, since $\delta = 1$ on $[x_{n-1}, x_1]$,

$$|L(x)| \le C\frac{\phi(h_*)}{h_*}\left(\frac{m_E}{m_J}\sum_{j\in\mathcal{A}}|x-x_j| + c\sum_{j\in\widetilde{\mathcal{B}}}|x-x_j|\right)$$

$$\le Cm_E^{k+3}\frac{\phi(\rho)}{|E|}\sum_{j\in\mathcal{E}}|x-x_j| \le Cm_E^{k+3}\frac{\phi(\rho)}{|E|}\sum_{j\in\mathcal{E}}|E| \le Cm_E^{k+4}\delta^\alpha\phi(\rho).$$

It remains to note that

$$1 = |E|\sum_{j\in\mathcal{E}}\frac{h_j}{|E|^2} \le c|E|\sum_{j\in\mathcal{E}}\frac{h_j}{(|x-x_j|+\rho)^2} \le cm_E^2\sum_{j\in\mathcal{E}}\frac{\rho h_j}{(|x-x_j|+\rho)^2},$$

and the proof is complete. □

7. Convex polynomial approximation of piecewise polynomials

LEMMA 7.1 [6, Lemma 4.3]. *Let $k \ge 3$, $\phi \in \Phi^k$ and $S \in \Sigma_{k,n}^{(1)}$ be such that $b_k(S,\phi) \le 1$. If $1 \le \mu,\nu \le n$ are such that the interval $I_{\mu,\nu}$ contains at least*

$2k-5$ intervals I_i and points $x_i^* \in (x_i, x_{i-1})$ so that

$$\rho_n^2(x_i^*)\phi^{-1}(\rho_n(x_i^*))|S''(x_i^*)| \le 1,$$

then, for every $1 \le j \le n$, we have

$$\|\rho_n^2\phi^{-1}(\rho_n)S''\|_{L_\infty(I_j)} \le c(k)\big[(j-\mu)^{4k} + (j-\nu)^{4k}\big].$$

THEOREM 7.2. *Let $k, r \in \mathbb{N}$, $r \ge 2$, $k \ge r+1$, and let $\phi \in \Phi^k$ be of the form $\phi(t) := t^r\psi(t)$, $\psi \in \Phi^{k-r}$. Also, let $d_+ \ge 0$, $d_- \ge 0$ and $\alpha \ge 0$ be given. Then there is a number $\mathcal{N} = \mathcal{N}(k, r, \phi, d_+, d_-, \alpha)$ satisfying the following assertion. If $n \ge \mathcal{N}$ and $S \in \Sigma_{k,n}^{(1)} \cap \Delta^{(2)}$ is such that*

$$b_k(S, \phi) \le 1, \tag{7.1}$$

and, additionally,

$$\text{if } d_+ > 0, \text{ then } d_+|I_2|^{r-2} \le \min_{x\in I_2} S''(x), \tag{7.2}$$

$$\text{if } d_+ = 0, \text{ then } S^{(i)}(1) = 0, \text{ for all } 2 \le i \le k-2, \tag{7.3}$$

$$\text{if } d_- > 0, \text{ then } d_-|I_{n-1}|^{r-2} \le \min_{x\in I_{n-1}} S''(x), \tag{7.4}$$

$$\text{if } d_- = 0, \text{ then } S^{(i)}(-1) = 0, \text{ for all } 2 \le i \le k-2, \tag{7.5}$$

then there exists a polynomial $P \in \Delta^{(2)} \cap \Pi_{Cn}$, $C = C(k, \alpha)$, satisfying, for all $x \in [-1, 1]$,

$$|S(x) - P(x)| \le C(k, \alpha)\, \delta_n^\alpha(x)\phi(\rho_n(x)), \quad \text{if } d_+ > 0 \text{ and } d_- > 0, \tag{7.6}$$

(7.7)
$$|S(x) - P(x)| \le C(k, \alpha)\, \delta_n^{\min\{\alpha, 2k-2\}}(x)\phi(\rho_n(x)), \quad \text{if } \min\{d_+, d_-\} = 0.$$

The proof of Theorem 7.2 is quite long and technical and is similar (with some rather significant changes) to that of [13, Theorem 10.2]. It is given in the last section of this paper.

8. Convex approximation by smooth piecewise polynomials

THEOREM 8.1 [14, Theorem 2.1]. *Given $r \in \mathbb{N}$, there is a constant $c = c(r)$ such that if $f \in C^r[-1, 1]$ is convex, then there is a number $\mathcal{N} = \mathcal{N}(f, r)$, depending on f and r, such that for $n \ge \mathcal{N}$, there are convex piecewise polynomials S of degree $r+1$ with knots at the Chebyshev partition T_n (i.e., $S \in \Sigma_{r+2,n} \cap \Delta^{(2)}$), satisfying*

$$|f(x) - S(x)| \le c(r)(\varphi(x)/n)^r\omega_2(f^{(r)}, \varphi(x)/n), \quad x \in [-1, 1], \tag{8.1}$$

and, moreover, for $x \in [-1,-1+n^{-2}] \cup [1-n^{-2},1]$,

$$|f(x) - S(x)| \le c(r)\varphi^{2r}(x)\omega_2(f^{(r)}, \varphi(x)/n) \tag{8.2}$$

and

$$|f(x) - S(x)| \le c(r)\varphi^{2r}(x)\omega_1(f^{(r)}, \varphi^2(x)). \tag{8.3}$$

As was shown in [14], $\mathcal{N}$ in the statement of Theorem 8.1, in general, cannot be independent of f.

We will now show that the following "smooth analog" of this result also holds.

THEOREM 8.2. *Given* $r \in \mathbb{N}$, *there is a constant* $c = c(r)$ *such that if* $f \in C^r[-1,1]$ *is convex, then there is a number* $\mathcal{N} = \mathcal{N}(f,r)$, *depending on* f *and* r, *such that for* $n \ge \mathcal{N}$, *there are continuously differentiable convex piecewise polynomials* S *of degree* $r+1$ *with knots at the Chebyshev partition* T_n (*i.e.*, $S \in \Sigma^{(1)}_{r+2,n} \cap \Delta^{(2)}$), *satisfying* (8.1), (8.2) *and* (8.3).

Let $\mathcal{S}_r(\mathbf{z}_m)$ denote the space of all piecewise polynomial functions (ppf) of degree $r-1$ (order r) with the knots $\mathbf{z}_m := (z_i)_{i=0}^m$, $a =: z_0 < z_1 < \cdots < z_{m-1} < z_m := b$. Also, the scale of the partition $\mathbf{z}_m$ is denoted by

$$\vartheta(\mathbf{z}_m) := \max_{0 \le j \le m-1} \frac{|J_{j\pm1}|}{|J_j|}, \tag{8.4}$$

where $J_j := [z_j, z_{j+1}]$.

In order to prove Theorem 8.2 we need the following lemma which is an immediate corollary of a more general result in [11].

LEMMA 8.3 [11, Lemma 3.8]. *Let* $r \in \mathbb{N}$, $\mathbf{z}_m := (z_i)_{i=0}^m$, $a =: z_0 < z_1 < \cdots < z_{m-1} < z_m := b$ *be a partition of* $[a,b]$, *and let* $s \in \Delta^{(2)} \cap \mathcal{S}_{r+2}(\mathbf{z}_m)$. *Then, there exists* $\tilde{s} \in \Delta^{(2)} \cap \mathcal{S}_{r+2}(\mathbf{z}_m) \cap C^1[a,b]$ *such that, for any* $1 \le j \le m-1$,

$$\|s - \tilde{s}\|_{[z_{j-1},z_{j+1}]} \le c(r, \vartheta(\mathbf{z}_m))\omega_{r+2}(s, z_{j+2} - z_{j-2}; [z_{j-2}, z_{j+2}]), \tag{8.5}$$

where $z_j := z_0$, $j < 0$ *and* $z_j := z_m$, $j > m$. *Moreover,*

$$\tilde{s}^{(\nu)}(a) = s^{(\nu)}(a) \quad \text{and} \quad \tilde{s}^{(\nu)}(b) = s^{(\nu)}(b), \quad \nu = 0, 1. \tag{8.6}$$

PROOF OF THEOREM 8.2. Let n be a sufficiently large fixed number, and let $S_0 \in \Sigma_{r+2,n} \cap \Delta^{(2)}$ be a piecewise polynomial from the statement of Theorem 8.1 for which estimates (8.1)–(8.3) hold. Let $a := x_{2n-1,2n}$, $b := x_{1,2n}$ and let $\mathbf{z}_n := (z_i)_{i=0}^n$ be such that $z_0 := a$, $z_n := b$ and $z_i := x_{n-i}$, $1 \le i \le n-1$ (note that $\mathbf{z}_n \subset T_{2n}$). Clearly, $S_0 \in \mathcal{S}_{r+2}(\mathbf{z}_n)$, $\vartheta(\mathbf{z}_n) \sim 1$, and

Lemma 8.3 implies that there exists $\tilde S_0 \in \Delta^{(2)} \cap \mathcal{S}_{r+2}(\mathbf{z}_n) \cap C^1[a,b]$ such that, for any $1 \le j \le n$,

$$\|S_0 - \tilde S_0\|_{\tilde I_j} \le c(r)\omega_{r+2}(S_0, h_j; \mathcal{J}_j)\,, \tag{8.7}$$

where $\tilde I_j := I_j \cap [a,b]$ and $\mathcal{J}_j := [x_{j+2}, x_{j-2}] \cap [a,b]$, and

$$\tilde S_0^{(\nu)}(a) = S_0^{(\nu)}(a) \quad \text{and} \quad \tilde S_0^{(\nu)}(b) = S_0^{(\nu)}(b)\,, \quad \nu = 0,1\,. \tag{8.8}$$

We now define

$$S(x) := \begin{cases} S_0(x), & \text{if } x \in [-1,1] \setminus [a,b], \\ \tilde S_0(x), & \text{if } x \in [a,b]. \end{cases}$$

Clearly, $S \in \Sigma^{(1)}_{r+2,2n} \cap \Delta^{(2)}$, estimates (8.2) and (8.3) hold (with n replaced by $2n$), and (8.1) also holds (clearly, it does not matter if we use n or $2n$ there) since $\varphi(x)/n \sim h_j$, for any $x \in \mathcal{J}_j$, $1 \le j \le n$. Thus, for $x \in \tilde I_j$, $1 \le j \le n$,

$$\begin{aligned} |f(x) - S(x)| &\le |f(x) - S_0(x)| + \|S_0 - \tilde S_0\|_{\tilde I_j} \\ &\le c\|f - S_0\|_{\mathcal{J}_j} + c\omega_{r+2}(f, h_j; \mathcal{J}_j) \le c h_j^r \omega_2(f^{(r)}, h_j) \\ &\le c(\varphi(x)/n)^r \omega_2(f^{(r)}, \varphi(x)/n). \quad \square \end{aligned}$$

REMARK 8.4. It follows from [11, Corollary 3.7] that Theorem 8.2 is, in fact, valid for splines of minimal defect, i.e., for $S \in \Sigma_{r+2,n} \cap C^r \cap \Delta^{(2)}$.

9. Proof of Theorem 1.2

9.1. The case for $r \ge 2$. Let S be the piecewise polynomial from the statement of Theorem 8.2. Without loss of generality, we can assume that S does not have knots at x_1 and x_{n-1} (it is sufficient to treat S as a piecewise polynomial with knots at the Chebyshev partition T_{2n}). Then,

$$\begin{aligned} l_1(x) &:= S(x)\big|_{I_1 \cup I_2} \\ &= f(1) + \frac{f'(1)}{1!}(x-1) + \dots + \frac{f^{(r)}(1)}{r!}(x-1)^r + a_+(n;f)(x-1)^{r+1} \end{aligned}$$

and

$$\begin{aligned} l_n(x) &:= S(x)\big|_{I_n \cup I_{n-1}} \\ &= f(-1) + \frac{f'(-1)}{1!}(x+1) + \dots + \frac{f^{(r)}(1)}{r!}(x+1)^r + a_-(n;f)(x+1)^{r+1}, \end{aligned}$$

where $a_+(n,f)$ and $a_-(n,f)$ are some constants that depend only on n and f.

We will now show that

$$n^{-2}\max\{|a_+(n,f)|,|a_-(n,f)|\} \to 0 \quad \text{as } n\to\infty. \tag{9.1}$$

Indeed, it follows from (8.3) that, for all $x\in I_1\cup I_2$,

$$\begin{aligned}
|a_+(n,f)(1-x)| &\le \frac{|l_1(x)-f(x)|}{(1-x)^r}\\
&+\frac{1}{(1-x)^r}\left|f(x)-f(1)-\frac{f'(1)}{1!}(x-1)-\dots-\frac{f^{(r)}(1)}{r!}(x-1)^r\right|\\
&\le c\omega_1(f^{(r)},1-x)+\frac{1}{(r-1)!\,(1-x)^r}\left|\int_x^1\big(f^{(r)}(t)-f^{(r)}(1)\big)(t-x)^{r-1}\,dt\right|\\
&\le c\omega_1(f^{(r)},1-x),
\end{aligned}$$

and, in particular, $n^{-2}|a_+(n,f)|\le c\omega_1(f^{(r)},n^{-2})\to 0$ as $n\to\infty$. Analogously, one draws a similar conclusion for $|a_-(n,f)|$.

For $f\in C^r$, $r\ge 2$, let $i_+\ge 2$, be the smallest integer $2\le i\le r$, if it exists, such that $f^{(i)}(1)\ne 0$, and denote

$$D_+(r,f):=\begin{cases}(2r!)^{-1}|f^{(i_+)}(1)|, & \text{if } i_+ \text{ exists},\\ 0, & \text{otherwise}.\end{cases}$$

Similarly, let $i_-\ge 2$, be the smallest integer $2\le i\le r$, if it exists, such that $f^{(i)}(-1)\ne 0$, and denote

$$D_-(r,f):=\begin{cases}(2r!)^{-1}|f^{(i_-)}(-1)|, & \text{if } i_- \text{ exists},\\ 0, & \text{otherwise}.\end{cases}$$

Hence, if n is sufficiently large, then

$$S''(x)\ge D_+(r,f)(1-x)^{r-2},\quad x\in(x_2,1], \tag{9.2}$$

and

$$S''(x)\ge D_-(r,f)(x+1)^{r-2},\quad x\in[-1,x_{n-2}). \tag{9.3}$$

PROOF OF THEOREM 1.2 IN THE CASE $r\ge 2$. Given $r\in\mathbb{N}$, $r\ge 2$, and a convex $f\in C^r$, let $\psi\in\Phi^2$ be such that $\omega_2(f^{(r)},t)\sim\psi(t)$, denote $\phi(t):=t^r\psi(t)$, and note that $\phi\in\Phi^{r+2}$.

For a sufficiently large $\mathcal{N} \in \mathbb{N}$ and each $n \ge \mathcal{N}$, we take the piecewise polynomial $S \in \Sigma_{r+2,n}$ of Theorem 8.2 satisfying (9.2) and (9.3), and observe that

$$\omega_{r+2}(f,t) \le t^r \omega_2(f^{(r)},t) \sim \phi(t),$$

so that by Lemma 4.1 with $k = r+2$, we conclude that

$$b_{r+2}(S,\phi) \le c.$$

Now, it follows from (9.2) and (2.2) that

$$\min_{x\in I_2} S''(x) \ge D_+(r,f)|I_1|^{r-2} \ge 3^{-r+2} D_+(r,f)|I_2|^{r-2}$$

and, similarly, (9.3) yields

$$\min_{x\in I_{n-1}} S''(x) \ge 3^{-r+2} D_-(r,f)|I_{n-1}|^{r-2}.$$

Hence, using Theorem 7.2 with $k=r+2$, $d_+ := 3^{-r+2}D_+(r,f)$, $d_- := 3^{-r+2}D_-(r,f)$ and $\alpha = 2k-2 = 2r+2$, we conclude that there exists a polynomial $P \in \Pi_{cn} \cap \Delta^{(2)}$ such that

$$|S(x)-P(x)| \le c\delta_n^{2r+2}(x)\rho_n^r(x)\psi(\rho_n(x)), \quad x\in[-1,1]. \tag{9.4}$$

In particular, for $x \in I_1 \cup I_n$, $x \ne -1, 1$, using the fact that $\rho_n(x) \sim n^{-2}$ for these x, and $t^{-2}\psi(t)$ is nonincreasing we have

$$\begin{aligned} |S(x)-P(x)| &\le c(n\varphi(x))^{2r+2}\rho_n^r(x)\psi(\rho_n(x)) \qquad (9.5)\\ &\le cn^2\varphi^{2r+2}(x)\Big(\frac{n\rho_n(x)}{\varphi(x)}\Big)^2\psi\Big(\frac{\varphi(x)}{n}\Big) \le c\varphi^{2r}(x)\omega_2\Big(f^{(r)},\frac{\varphi(x)}{n}\Big). \end{aligned}$$

In turn, this implies for $x \in I_1 \cup I_n$, that

$$|S(x)-P(x)| \le c\Big(\frac{\varphi(x)}{n}\Big)^r\omega_2\Big(f^{(r)},\frac{\varphi(x)}{n}\Big),$$

which combined with (9.4) implies

$$|S(x)-P(x)| \le c\Big(\frac{\varphi(x)}{n}\Big)^r\omega_2\Big(f^{(r)},\frac{\varphi(x)}{n}\Big), \quad x\in[-1,1]. \tag{9.6}$$

Now, (9.6) together with (8.1) yield (1.5), and (9.5) together with (8.2) yield (1.6). In order to prove (1.7), using the fact that $t^{-1}\omega_1(f^{(r)},t)$ is nonincreasing we have, for $x \in I_1 \cup I_n$, $x \ne -1, 1$,

$$|S(x)-P(x)| \le c(n\varphi(x))^{2r+2}\rho_n^r(x)\omega_1(f^{(r)},\rho_n(x)) \tag{9.7}$$

$$\leq cn^2\varphi^{2r+2}(x)\frac{\rho_n(x)}{\varphi^2(x)}\omega_1(f^{(r)},\varphi^2(x)) \leq c\varphi^{2r}(x)\omega_1(f^{(r)},\varphi^2(x)),$$

which together with (8.3) completes the proof of Theorem 1.2. □

9.2. The case for $r = 1$. It is possible to show that, in order to prove Theorem 1.2 in the case $r = 1$, it is sufficient to construct a convex polynomial P_n that approximates the quadratic spline S from Theorem 8.1 (with $r = 1$) so that

$$|S(x) - P_n(x)| \leq c\omega_3(f, \rho_n(x)),$$

and

$$P_n(\pm 1) = S(\pm 1) \quad \text{and} \quad P_n'(\pm 1) = S'(\pm 1). \tag{9.8}$$

In order to construct such a polynomial, one can use exactly the same method as in [8] (using S instead of f and $2n$ instead of n) with the only difference that extra factors $(1 - y^2)$ should appear inside integrals in the definitions of Q_j, $\overline{Q}_j$ and T_j (see [8, p. 158]). All estimates are then still valid and either follow from the statements in Section 3 or are proved similarly. This change is needed in order to guarantee that (9.8) holds because addition of these factors implies that the first derivatives of these modified polynomials Q_j, $\overline{Q}_j$ and T_j are 0 at ± 1 which, in turn, implies that the first derivatives of other auxiliary polynomials σ_j, R_j and $\overline{R}_j$ at ± 1 are "correct". We omit details.

Appendix A. Proof of Theorem 7.2

Throughout the proof, we fix $\beta := k + 7$ and $\gamma := \beta_1 - 1 = 60(\alpha + \beta) + k$. Hence, the constants $C_1, \ldots, C_6$ (defined below) as well as the constants C, may depend only on k and α. We also note that S does not have to be twice differentiable at the Chebyshev knots x_j. Hence, when we write $S''(x)$ (or $S_i''(x)$, $1 \leq i \leq 4$) everywhere in this proof, we implicitly assume that $x \neq x_j$, $1 \leq j \leq n - 1$.

Let $C_1 := C$, where the constant C is taken from (6.1) (without loss of generality we assume that $C_1 \leq 1$), and let $C_2 := C$ with C taken from (4.2). We also fix an integer C_3 such that

$$C_3 \geq 8k/C_1. \tag{A.1}$$

Without loss of generality, we may assume that n is divisible by C_3, and put $n_0 := n/C_3$.

We divide $[-1, 1]$ into n_0 intervals

$$E_q := [x_{qC_3}, x_{(q-1)C_3}] = I_{qC_3} \cup \cdots \cup I_{(q-1)C_3+1}, \quad 1 \leq q \leq n_0,$$

consisting of C_3 intervals I_i each (i.e., $m_{E_q} = C_3$, for all $1 \le q \le n_0$).

We write "$j \in UC$" (where "UC" stands for "Under Control") if there is $x_j^* \in (x_j, x_{j-1})$ such that

$$S''(x_j^*) \le \frac{5C_2\phi(\rho_n(x_j^*))}{\rho_n^2(x_j^*)}. \tag{A.2}$$

We say that $q \in G$ (for "Good"), if the interval E_q contains at least $2k-5$ intervals I_j with $j \in UC$.

Then, (A.2) and Lemma 7.1 imply that,

$$S''(x) \le \frac{C\phi(\rho)}{\rho^2}, \quad x \in E_q,\ q \in G. \tag{A.3}$$

Set $E := \bigcup_{q \notin G} E_q$, and decompose S into a "small" part and a "big" one, by setting

$$s_1(x) := \begin{cases} S''(x), & \text{if } x \notin E, \\ 0, & \text{otherwise}, \end{cases}.$$

and

$$s_2(x) := S''(x) - s_1(x) = \begin{cases} 0, & \text{if } x \notin E, \\ S''(x), & \text{otherwise}, \end{cases}$$

and putting

$$S_1(x) := S(-1) + (x+1)S'(-1) + \int_{-1}^{x} (x-u)s_1(u)\,du,$$

$$S_2(x) := \int_{-1}^{x} (x-u)s_2(u)\,du.$$

(Note that s_1 and s_2 are well defined for $x \ne x_j$, $1 \le j \le n-1$, so that S_1 and S_2 are well defined everywhere and possess second derivatives for $x \ne x_j$, $1 \le j \le n-1$.)

Evidently, $S_1, S_2 \in \Sigma_{k,n}^{(1)}$, and

$$S_1''(x) \ge 0 \quad \text{and} \quad S_2''(x) \ge 0, \quad x \in [-1,1].$$

Now, (A.3) implies that

$$S_1''(x) \le \frac{C\phi(\rho)}{\rho^2}, \quad x \in [-1,1],$$

which, in turn, yields by Lemma 4.2, $b_k(S_1,\phi)\le C$. Together with (7.1), we obtain

$$b_k(S_2,\phi)\le b_k(S_1,\phi)+b_k(S,\phi)\le C+1\le\lceil C+1\rceil =: C_4. \tag{A.4}$$

The set E is a union of disjoint intervals $F_p=[a_p,b_p]$, between any two of which, all intervals E_q are with $q\in G$. We may assume that $n>C_3C_4$, and write $p\in AG$ (for "Almost Good"), if F_p consists of no more than C_4 intervals E_q, that is, it consists of no more than C_3C_4 intervals I_j. Hence, by Lemma 7.1,

$$S_2''(x)\le\frac{C\,\phi(\rho)}{\rho^2},\quad x\in F_p,\ p\in AG. \tag{A.5}$$

One may think of intervals F_p, $p\notin AG$, as "long" intervals where S'' is "large" on many subintervals I_i and rarely dips down to 0. Intervals F_p, $p\in AG$, as well as all intervals E_q which are not contained in any F_p's (i.e., all "good" intervals E_q) are where S'' is "small' in the sense that the inequality $S''(x)\le C\phi(\rho)/\rho^2$ is valid there.

Set $F:=\bigcup_{p\notin AG}F_p$, note that $E=\bigcup_{p\in AG}F_p\cup F$, and decompose S again by setting

$$s_4:=\begin{cases}S''(x), & \text{if } x\in F,\\ 0, & \text{otherwise,}\end{cases}$$

and

$$s_3(x):=S''(x)-s_4(x)=\begin{cases}0, & \text{if } x\in F,\\ S''(x), & \text{otherwise,}\end{cases}$$

and putting

$$S_3(x):=S(-1)+(x+1)S'(-1)+\int_{-1}^x(x-u)s_3(u)\,du\,, \tag{A.6}$$

$$S_4(x):=\int_{-1}^x(x-u)s_4(u)\,du.$$

Then, evidently,

$$S_3,S_4\in\Sigma_{k,n}^{(1)},\quad S_3+S_4=S, \tag{A.7}$$

and

$$S_3''(x)\ge 0\ \text{ and }\ S_4''(x)\ge 0,\quad x\in[-1,1]. \tag{A.8}$$

We remark that, if $x \notin \bigcup_{p\in AG} F_p$, then $s_1(x) = s_3(x)$ and $s_2(x) = s_4(x)$. If $x \in \bigcup_{p\in AG} F_p$, then $s_1(x) = s_4(x) = 0$ and $s_2(x) = s_3(x) = S''(x)$.

For $x \in \bigcup_{p\in AG} F_p$, (A.5) implies that

$$S_3''(x) = S_2''(x) \le \frac{C\,\phi(\rho)}{\rho^2}.$$

For all other x's,

$$S_3''(x) = S_1''(x) \le \frac{C\,\phi(\rho)}{\rho^2}.$$

We conclude that

$$S_3''(x) \le \frac{C_5\,\phi(\rho)}{\rho^2}, \quad x \in [-1,1], \tag{A.9}$$

which by virtue of Lemma 4.2, yields that $b_k(S_3,\phi) \le C$. As above, we obtain

$$b_k(S_4,\phi) \le b_k(S_3,\phi) + b_k(S,\phi) \le C + 1 \le \lceil C+1 \rceil =: C_6. \tag{A.10}$$

We will approximate S_3 and S_4 by convex polynomials that achieve the required degree of pointwise approximation.

A.1. Approximation of S_3. If $d_+ > 0$, then there exists $\mathcal{N}^* \in \mathbb{N}$, $\mathcal{N}^* = \mathcal{N}^*(d_+,\psi)$, such that, for $n > \mathcal{N}^*$,

$$\frac{\phi(\rho)}{\rho^2} = \rho^{r-2}\psi(\rho) < \frac{d_+|I_2|^{r-2}}{C_5} \le C_5^{-1}\,S''(x), \quad x \in I_2,$$

where the first inequality follows since $\psi(\rho) \le \psi(2/n) \to 0$ as $n \to \infty$, and the second inequality follows by (7.2). Hence, by (A.9), if $n > \mathcal{N}^*$, then $s_3(x) \ne S''(x)$ for $x \in I_2$. Therefore, since $s_3(x) = S''(x)$, for all $x \notin F$, we conclude that $I_2 \subset F$, and so $E_1 \subset F$, and $s_3(x) = 0$, $x \in E_1$. In particular, $s_3(x) \equiv 0$, $x \in I_1$.

Similarly, if $d_- > 0$, then using (7.4) we conclude that there exists $\mathcal{N}^{**} \in \mathbb{N}$, $\mathcal{N}^{**} = \mathcal{N}^{**}(d_-,\psi)$, such that, if $n > \mathcal{N}^{**}$, then $s_3(x) \equiv 0$ for all $x \in I_n$.

Thus, we conclude that for $n \ge \max\{\mathcal{N}^*, \mathcal{N}^{**}\}$, we have

$$s_3(x) = 0, \quad \text{for all } x \in I_1 \cup I_n. \tag{A.11}$$

Therefore, in view of (A.7) and (A.8), it follows by Lemma 5.1 combined with (A.9) that, in the case $d_+ > 0$ and $d_- > 0$, there exists a convex polynomial $r_n \in \Pi_{Cn}$ such that

$$|S_3(x) - r_n(x)| \le C\,\delta^\alpha \phi(\rho), \quad x \in [-1,1]. \tag{A.12}$$

Suppose now that $d_+ = 0$ and $d_- > 0$. First, proceeding as above, we conclude that $s_3 \equiv 0$ on I_n. Additionally, if $E_1 \subset F$, then, as above, $s_3 \equiv 0$ on I_1 as well. Hence, (A.11) holds which, in turn, implies (A.12).

If $E_1 \not\subset F$, then $s_3(x) = S''(x)$, $x \in I_1$, and so it follows from (7.3) that, for some constant $A_1 \geq 0$,

$$s_3(x) = S''(x) = A_1(1-x)^{k-3}, \quad x \in I_1.$$

Note that A_1 may depend on n, but by (7.16) we conclude that,

$$A_1 \leq C_5 \frac{\phi(\rho_n(x_1))}{(1-x_1)^{k-3}\rho^2(x_1)} \sim n^{2k-2}\phi(n^{-2}).$$

Hence, for $x \in I_1$,

$$S_3''(x) = s_3(x) = A_2 n^{2k-2}\phi(n^{-2})(1-x)^{k-3}, \tag{A.13}$$

where A_2 is a nonnegative constant that may depend on n but $A_2 \leq C$.

We now construct $\widetilde{S}_3 \in \Sigma_{k,2n}$ which satisfies all conditions of Lemma 5.1 (with $2n$ instead of n). Note that $x_j := x_{j,n} = x_{2j,2n}$, denote $\xi := x_{1,2n}$ and define

$$\widetilde{S}_3(x) := \begin{cases} S_3(x), & \text{if } x < x_1, \\ S_3(1) + (x-1)S_3'(1) =: L(x), & \text{if } \xi < x \leq 1, \\ \ell(x), & \text{if } x \in [x_1, \xi], \end{cases}$$

where $\ell(x)$ is the linear polynomial chosen so that $\widetilde{S}_3$ is continuous on $[-1,1]$, i.e., $\ell(x_1) = S_3(x_1)$ and $\ell(\xi) = S_3(1) + (\xi - 1)S_3'(1) = L(\xi)$. Clearly, $\widetilde{S}_3 \in C[-1,1]$ (and, in fact, is in $C^1[-1,x_1)$), $\widetilde{S}_3''(x) \leq C\rho^{-2}\phi(\rho)$, $x \notin \{x_j\}_{j=1}^{n-1} \cup \{\xi\}$, and $\widetilde{S}_3'' \equiv 0$ on $I_{1,2n} \cup I_n$. Note that $\widetilde{S}_3'$ may be discontinuous at x_1 and ξ, but, evidently, the slope of L is no less than the slope of ℓ, so that $\widetilde{S}_3$ is convex in $[x_1, 1]$.

Denote

$$S_L := S_3 - L, \quad \widetilde{S}_L := \widetilde{S}_3 - L \quad \text{and} \quad \ell_L := \ell - L,$$

and note that

$$\widetilde{S}_L(x) = \begin{cases} 0, & x \in [\xi, 1], \\ \ell_L(x), & x \in [x_1, \xi]. \end{cases}$$

Also, $\ell_L(x_1) = S_L(x_1)$ and $\ell_L(\xi) = \widetilde{S}_L(\xi) = 0$, and in view of (A.13),

$$0 \leq S_L(x) = S_3(x) - S_3(1) - (x-1)S_3'(1) \tag{A.14}$$

$$= \int_x^1 (u-x)s_3(u)\,du = A_3 n^{2k-2}\phi(n^{-2})(1-x)^{k-1}, \quad x \in I_1,$$

where $A_3 \le C$. Now, the tangent line to S_L at $x = x_1$ is

$$y(x) = S_L(x_1) + (x - x_1)S_L'(x_1)$$
$$= A_3 n^{2k-2}\phi(n^{-2})(1-x_1)^{k-2}\big(1 - x_1 - (k-1)(x - x_1)\big),$$

which intersects the x axis at

$$x_1 + \frac{1-x_1}{k-1} \le \frac{1+x_1}{2} < x_{1,2n} = \xi.$$

Hence, the slope of ℓ_L is no less than the slope of that tangent and, in turn, we conclude that the slope of ℓ is no less than $S_3'(x_1)$, so that $\widetilde{S}_3$ is convex in $[-1,1]$.

Further, we have,

$$\text{(A.15)} \qquad |S_3(x) - \widetilde{S}_3(x)| = |S_L(x) - \widetilde{S}_L(x)| \le S_L(x) + \widetilde{S}_L(x)$$
$$\le S_L(x_1) + \widetilde{S}_L(x_1) = 2S_L(x_1) \le C\phi(n^{-2}), \quad x \in [x_1, \xi],$$

and

$$\text{(A.16)} \qquad |\widetilde{S}_3(x) - S_3(x)| = |L(x) - S_3(x)| = S_L(x)$$
$$= A_3\phi(n^{-2})n^{2k-2}(1-x)^{k-1} \le C\delta^{2k-2}\phi(n^{-2}), \quad x \in [\xi, 1].$$

Note that $\widetilde{S}_3'$ may have (nonnegative) jumps at x_1 and ξ. However,

$$\text{(A.17)} \qquad \widetilde{S}_3'(\xi+) - \widetilde{S}_3'(\xi-) + \widetilde{S}_3'(x_1+) - \widetilde{S}_3'(x_1-)$$
$$= S_3'(1) - S_3'(x_1) = -S_L'(x_1) \le Cn^2\phi(n^{-2}),$$

so that Lemma 5.1 implies that there exists a convex polynomial $r_n \in \Pi_{Cn}$ such that,

$$|\widetilde{S}_3(x) - r_n(x)| \le C\,\delta^\alpha\phi(\rho), \quad x \in [-1,1].$$

Observing that $\widetilde{S}_3 \equiv S_3$ on $[-1, x_1]$, and combining with (A.15) and (A.16) (recalling that $n^{-2} \le \rho$), we conclude that

$$|\widetilde{S}_3(x) - S_3(x)| \le C\delta^{2k-2}\phi(\rho), \quad x \in [-1,1],$$

so that

$$\text{(A.18)} \qquad |S_3(x) - r_n(x)| \le C\,\delta^{\min\{\alpha, 2k-2\}}\phi(\rho), \quad x \in [-1,1].$$

Finally, if $d_- = 0$ and $d_+ > 0$, then the considerations are completely analogous and, if $d_- = 0$ and $d_+ = 0$, then $\widetilde{S}_3$ can be modified further on I_n using (7.5) and the above argument.

Hence, we've constructed a convex polynomial $r_n \in \Pi_{Cn}$ such that, in the case when both d_+ and d_- are strictly positive, (A.12) holds, and (A.18) is valid if at least one of these numbers is 0.

A.2. Approximation of S_4. Given a set $A \subset [-1, 1]$, denote

$$A^e := \bigcup_{I_j \cap A \neq \emptyset} I_j \quad \text{and} \quad A^{2e} := (A^e)^e,$$

where $I_0 = \emptyset$ and $I_{n+1} = \emptyset$. For example, $[x_7, x_3]^e = [x_8, x_2]$, $I_1^e = I_1 \cup I_2$, etc.

Also, given subinterval $I \subset [-1, 1]$ with its endpoints at the Chebyshev knots, we refer to the right-most and the left-most intervals I_i contained in I as $EP_+(I)$ and $EP_-(I)$, respectively (for the "End Point" intervals). More precisely, if $1 \le \mu < \nu \le n$ and

$$I = \bigcup_{i=\mu}^{\nu} I_i,$$

then $EP_+(I) := I_\mu$, $EP_-(I) := I_\nu$ and $EP(I) := EP_+(I) \cup EP_-(I) = I_\mu \cup I_\nu$. For example, $EP_+[-1, 1] := I_1$, $EP_-[-1, 1] := I_n$, $EP_+[x_7, x_3] = [x_4, x_3] = I_4$, $EP_-[x_7, x_3] = [x_7, x_6] = I_7$, $EP[x_7, x_3] = I_4 \cup I_7$, etc. Here, we simplified the notation by using $EP_\pm[a, b] := EP_\pm([a, b])$ and $EP[a, b] := EP([a, b])$.

In order to approximate S_4, we observe that for $p \notin AG$,

$$S_4''(x) = S_2''(x), \quad x \in F_p^{2e},$$

so that by virtue of (A.4), we conclude that

$$b_k(S_4, \phi, F_p^{2e}) = b_k(S_2, \phi, F_p^{2e}) \le b_k(S_2, \phi) \le C_4. \tag{A.19}$$

(Note that, for $p \in AG$, S_4 is linear in F_p^{2e} and so $b_k(S_4, \phi, F_p^{2e}) = 0$.)

We will approximate S_4 using the polynomial $D_{n_1}(\cdot, S_4) \in \Pi_{Cn_1}$ defined in Lemma 4.3 (with $n_1 := C_6 n$), and then we construct two "correcting" polynomials $\overline{Q}_n, M_n \in \Pi_{Cn}$ (using Lemma 6.2) in order to make sure that the resulting approximating polynomial is convex.

We begin with $\overline{Q}_n$. For each q for which $E_q \subset F$, let J_q be the union of all intervals $I_j \subset E_q$ with $j \in UC$ with the union of both intervals $I_j \subset E_q$ at the endpoints of E_q. In other words,

$$J_q := \bigcup_j \{\, I_j \mid j \in UC \text{ and } I_j \subset E_q \,\} \cup EP(E_q).$$

Since $E_q \subset F$, then $q \notin G$ and so the number of intervals $I_j \subset E_q$ with $j \in UC$ is at most $2k-6$. Hence, by (A.1),

$$m_{J_q} \le 2k-4 < 2k \le \frac{C_1 C_3}{4} \le \frac{C_3}{4},$$

Recalling that the total number m_{E_q} of intervals I_j in E_q is C_3 we conclude that Lemma 6.2 can be used with $E := E_q$ and $J := J_q$. Thus, set

$$\overline{Q}_n := \sum_{q\,:\,E_q\subset F} Q_n(\cdot, E_q, J_q),$$

where Q_n are polynomials from Lemma 6.2, and denote

$$J := \bigcup_{q\,:\,E_q\subset F} J_q.$$

Then, (6.1) through (6.3) imply that $\overline{Q}_n$ satisfies

$$\text{(A.20)}\qquad \begin{cases} \text{(a)} \quad \overline{Q}_n''(x) \ge 0, \quad x \in [-1,1] \setminus F, \\ \text{(b)} \quad \overline{Q}_n''(x) \ge -\dfrac{\phi(\rho)}{\rho^2} \quad x \in F \setminus J, \\ \text{(c)} \quad \overline{Q}_n''(x) \ge 4\dfrac{\phi(\rho)}{\rho^2}\delta^{8\alpha}, \quad x \in J. \end{cases}$$

Note that the inequalities in (A.20) are valid since, for any given x, all relevant $Q_n''(x, E_q, J_q)$, except perhaps one, are nonnegative, and

$$C_1 \frac{m_{E_q}}{m_{J_q}} \ge \frac{C_1 C_3}{2k} \ge 4.$$

Also, it follows from (6.3) that, for any $x \in [-1,1]$,

$$\text{(A.21)}\qquad |\overline{Q}_n(x)| \le C\delta^\alpha \rho\phi(\rho) \sum_{q:E_q\subset F}\ \sum_{j:I_j\subset E_q} \frac{h_j}{(|x-x_j|+\rho)^2}$$

$$\le C\delta^\alpha \rho\phi(\rho) \sum_{j=1}^{n} \frac{h_j}{(|x-x_j|+\rho)^2} \le C\delta^\alpha \rho\phi(\rho) \int_0^\infty \frac{du}{(u+\rho)^2} = C\delta^\alpha \phi(\rho).$$

Next, we define the polynomial M_n. For each F_p with $p \notin AG$, let J_p^- denote the union of the two intervals on the left side of F_p^e (or just the interval I_n if $-1 \in F_p$), and let J_p^+ denote the union of the two intervals on the right side of F_p^e (or just one interval I_1 if $1 \in F_p$), i.e.,

$$J_p^- = EP_-(F_p^e) \cup EP_-(F_p) \quad \text{and} \quad J_p^+ = EP_+(F_p^e) \cup EP_+(F_p).$$

Also, let F_p^- and F_p^+ be the closed intervals each consisting of $m_{F_p^\pm} := C_3C_4$ intervals I_j and such that $J_p^- \subset F_p^- \subset F_p^e$ and $J_p^+ \subset F_p^+ \subset F_p^e$, and put

$$J_p^* := J_p^- \cup J_p^+ \quad \text{and} \quad J^* := \bigcup_{p \notin AG} J_p^*.$$

Now, we set

$$M_n := \sum_{p \notin AG} \left(Q_n(\cdot, F_p^+, J_p^+) + Q_n(\cdot, F_p^-, J_p^-) \right).$$

Since $m_{F_p^+} = m_{F_p^-} = C_3C_4$ and $m_{J_p^+}, m_{J_p^-} \le 2$, it follows from (A.1) that

$$C_1 \min\left\{ \frac{m_{F_p^+}}{m_{J_p^+}}, \frac{m_{F_p^-}}{m_{J_p^-}} \right\} \ge \frac{C_1C_3C_4}{2} \ge 2C_4.$$

Then Lemma 6.2 implies

$$|M_n(x)| \le C\, \delta^\alpha \phi(\rho) \tag{A.22}$$

(this follows from (6.3) using the same sequence of inequalities that was used to prove (A.21) above), and

$$\text{(A.23)} \quad \begin{cases} \text{(a)} \quad M_n''(x) \ge -2\dfrac{\phi(\rho)}{\rho^2}, \quad x \in F \setminus J^*, \\[2ex] \text{(b)} \quad M_n''(x) \ge 2C_4\, \delta^{8\alpha} \dfrac{\phi(\rho)}{\rho^2}, \quad x \in J^*, \\[2ex] \text{(c)} \quad M_n''(x) \ge 2C_4\, \delta^{8\alpha} \dfrac{\phi(\rho)}{\rho^2} \left(\dfrac{\rho}{\operatorname{dist}(x, F)} \right)^{\gamma+1}, \quad x \in [-1,1] \setminus F^e, \end{cases}$$

where in the last inequality we used the fact that

$$\max\{\rho, \operatorname{dist}(x, F^e)\} \le \operatorname{dist}(x, F), \quad x \in [-1,1] \setminus F^e,$$

which follows from (2.5).

The third auxiliary polynomial is $D_{n_1} := D_{n_1}(\cdot, S_4)$ with $n_1 = C_6 n$ from Lemma 4.3. By (A.10), (4.1) yields

$$|S_4(x) - D_{n_1}(x)| \le C\, \delta^\gamma \phi(\rho) \le C\, \delta^\alpha \phi(\rho), \quad x \in [-1,1], \tag{A.24}$$

since $\gamma > \alpha$, and (4.2) implies that, for any interval $A \subset [-1,1]$ having Chebyshev knots as endpoints,

$$|S_4''(x) - D_{n_1}''(x)| \le C_2\, \delta^\gamma \frac{\phi(\rho)}{\rho^2}\, b_k(S_4, \phi, A) + C_2 C_6\, \delta^\gamma \frac{\phi(\rho)}{\rho^2} \frac{n}{n_1} \Big(\frac{\rho}{\operatorname{dist}(x, [-1,1] \setminus A)} \Big)^{\gamma+1}, \quad x \in A. \tag{A.25}$$

We now define

$$R_n := D_{n_1} + C_2 \overline{Q}_n + C_2 M_n. \tag{A.26}$$

By virtue of (A.21), (A.22), and (A.24) we obtain

$$|S_4(x) - R_n(x)| \le C\, \delta^\alpha \phi(\rho), \quad x \in [-1,1],$$

which combined with (A.12) and (A.18), proves (7.6) and (7.7) for $P := R_n + r_n$.

Thus, in order to conclude the proof of Theorem 7.2, we should prove that P is convex. We recall that r_n is convex, so it is sufficient to show that R_n is convex as well.

Note that (A.26) implies

$$R_n''(x) \ge C_2 \overline{Q}_n''(x) + C_2 M_n''(x) - |S_4''(x) - D_{n_1}''(x)| + S_4''(x), \quad x \in [-1,1],$$

(this inequality is extensively used in the three cases below), and that (A.25) holds for *any* interval A with Chebyshev knots as the endpoints, and so we can use different intervals A for different points $x \in [-1,1]$. We consider three cases depending on whether (i) $x \in F \setminus J^*$, or (ii) $x \in J^*$, or (iii) $x \in [-1,1] \setminus F^e$.

Case (i): $x \in F \setminus J^*$. In this case, for some $p \notin AG$, $x \in F_p \setminus J_p^*$, and so we take $A := F_p$. Then, the quotient inside the parentheses in (A.25) is bounded above by 1 (this follows from (2.5)). Also, since $s_4(x) = S''(x)$, $x \in F$, it follows that $b_k(S_4, \phi, F_p) = b_k(S, \phi, F_p) \le 1$. Hence,

$$|S_4''(x) - D_{n_1}''(x)| \le C_2 \frac{\phi(\rho)}{\rho^2}\, b_k(S_4, \phi, F_p) + C_2 C_6 \frac{\phi(\rho)}{\rho^2} \frac{n}{n_1} \le 2C_2 \frac{\phi(\rho)}{\rho^2}, \quad x \in F \setminus J^*. \tag{A.27}$$

Note that $x \notin I_1 \cup I_n$ (since $F \setminus J^*$ does not contain any intervals in $EP(F_p)$, $p \notin AG$), and so $\delta = 1$.

It now follows by (A.20)(c), (A.23)(a), (A.27) and (A.8), that

$$R_n''(x) \ge C_2 \frac{\phi(\rho)}{\rho^2}(4-2-2) = 0, \quad x \in J \setminus J^*.$$

If $x \in F \setminus (J \cup J^*)$, then (A.2) is violated and so

$$S_4''(x) = S''(x) > \frac{5C_2\phi(\rho)}{\rho^2}.$$

Hence, by virtue of (A.20)(b), (A.23)(a) and (A.27), we get

$$R_n''(x) \ge C_2 \frac{\phi(\rho)}{\rho^2}(-1-2-2+5) = 0, \quad x \in F \setminus (J \cup J^*).$$

Case (ii): $x \in J^*$. In this case, $x \in J_p^*$, for some $p \notin AG$, and we take $A := F_p^{2e}$. Then, (A.19) and (A.25) imply (again, (2.5) is used to estimate the quotient inside the parentheses in (A.25)),

$$\text{(A.28)} \qquad |S_4''(x) - D_{n_1}''(x)| \le C_2\,\delta^\gamma \frac{\phi(\rho)}{\rho^2} b_k(S_4, \phi, F_p^{2e}) + C_2\,C_6\,\delta^\gamma \frac{\phi(\rho)}{\rho^2}\frac{n}{n_1}$$
$$\le 2C_2\,C_4\,\delta^\gamma \frac{\phi(\rho)}{\rho^2}, \quad x \in J^*.$$

Now, we note that $EP(F_p) \subset J$, for all $p \notin AG$, and so $F \cap J^* \subset J$. Hence, using (A.20)(a,c), (A.23)(b), (A.28) and (A.8), we obtain

$$R_n''(x) \ge 2C_2\,C_4\,\delta^{8\alpha} \frac{\phi(\rho)}{\rho^2} - 2C_2\,C_4\,\delta^\gamma \frac{\phi(\rho)}{\rho^2} \ge 0,$$

since $\gamma > 8\alpha$, and so $\delta^\gamma \le \delta^{8\alpha}$.

Case (iii): $x \in [-1,1] \setminus F^e$. In this case we take A to be the connected component of $[-1,1] \setminus F$ that contains x. Then by (A.25),

$$\text{(A.29)} \qquad |S_4''(x) - D_{n_1}''(x)| \le C_2\,\delta^\gamma \frac{\phi(\rho)}{\rho^2} b_k(S_4, \phi, A)$$
$$+ C_2\,C_6\,\delta^\gamma \frac{\phi(\rho)}{\rho^2}\frac{n}{n_1}\Big(\frac{\rho}{\operatorname{dist}(x, [-1,1] \setminus A)}\Big)^{\gamma+1}$$
$$= C_2\,\delta^\gamma \frac{\phi(\rho)}{\rho^2}\Big(\frac{\rho}{\operatorname{dist}(x, F)}\Big)^{\gamma+1}, \quad x \in [-1,1] \setminus F^e,$$

where we used the fact that S_4 is linear in A, and so $b_k(S_4, \phi, A) = 0$.

Now, (A.20)(a), (A.23)(c), (A.29) and (A.8) imply

$$R_n''(x) \ge \frac{\phi(\rho)}{\rho^2}\Big(\frac{\rho}{\operatorname{dist}(x,F)}\Big)^{\gamma+1}(2C_2C_4\delta^{8\alpha} - C_2\delta^{\gamma}) \ge 0,$$

since $C_4 \ge 1$ and $\gamma > 8\alpha$.

Thus, $R_n''(x) \ge 0$ for all $x \in [-1,1]$, and so we have constructed a convex polynomial P, satisfying (7.6) and (7.7), for each $n \ge \mathcal{N}$. This completes the proof of Theorem 7.2.

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