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Lecture Notes on Rheological Models

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Contents

Preface

We wrote the notes basically for students attending the course on continuum mechanics at the Faculty of Mathematics, Physics and Informatics, Comenius University in Bratislava. The notes are intended as a companion text to continuum mechanics and rheology. This version is far from a complete coverage of the topic and will be a subject of further extension, modification, and improvement.

Due to a short time available for preparation of the notes, the material partly reflects the fact that our research topic is theory and numerical modeling of elastic and seismic wave propagation.

In addition to the literature explicitly referred to in this version of the text we also included books by Ranalli (1995) and Dahlen & Tromp (1998) in the list of references as references relevant for the topic. The text of the lecture notes is mainly based on material given in the books by Moczo, Kristek & Halada (2004), Carcione (2001) and Dahlen & Tromp (1998).

Anelastic Materials

Elastic media do not have losses of internal energy. Real materials do not behave in this way. Materials that behave differently from the elastic media are called **anelastic**. The deviation from the elastic behavior of materials is **anelasticity**.

The simplest rheological model of an anelastic material is a linear **viscoelastic** body which combines two extreme behaviors – linear elasticity and linear **viscosity**. A material is linearly viscoelastic if the stress tensor is linearly related to the strain tensor, and the strain response to a linear combination of applied stresses is the same linear combination of strain responses to individual applied stresses.

A sudden application of a constant stress (Heaviside unit step function in stress) causes **creep** – a slow continuous increase of strain. A sudden removal of stress possibly yields **recovery** of material. A simple classification of creep and recovery is given in Table 1.

<table>
<tr><td>application of constant stress</td><td>→</td><td>creep – slow continuous increase of strain</td><td colspan="4"></td></tr>
<tr><td colspan="7"></td></tr>
<tr><td rowspan="4">removal of the stress</td><td rowspan="4">→</td><td rowspan="4">recovery – gradual decrease of strain</td><td>complete</td><td>elastic creep</td><td></td><td></td></tr>
<tr><td>partial</td><td>elastic flow</td><td></td><td></td></tr>
<tr><td rowspan="2">no</td><td rowspan="2">flow</td><td>linear strain rate</td><td>viscous flow</td></tr>
<tr><td>nonlinear strain rate</td><td>plastic flow</td></tr>
<tr><td colspan="7"></td></tr>
<tr><td colspan="7">creep with increasing rate can terminate in rupture</td></tr>
<tr><td colspan="7"></td></tr>
<tr><td colspan="7">Table 1. Classification of creep (adapted from Ben-Menahem & Singh 1981)</td></tr>
</table>

A sudden application of a constant strain (Heaviside unit step function in strain) causes **relaxation** – a gradual decrease of stress. A simple classification of relaxation is given in Table 2.

<table>
<tr><td rowspan="2">application of constant strain</td><td rowspan="2">→</td><td rowspan="2">relaxation – gradual decrease of stress</td><td>in material characterized by elastic flow</td><td>relaxation to nonzero stress</td></tr>
<tr><td>in material characterized by elastic creep</td><td>relaxation to zero stress</td></tr>
<tr><td colspan="5"></td></tr>
<tr><td colspan="5">Table 2. Classification of relaxation (adapted from Ben-Menahem & Singh 1981)</td></tr>
</table>

Linear Elastic Body

Linear elastic body**, Hooke body** (Hooke model, Hooke element, elastic spring), represents behavior of a perfectly elastic (lossless) solid material. Stress is proportional to strain:

$$\sigma(t) = M \cdot \varepsilon(t) . \tag{1}$$

Here $\sigma(t)$ is the stress as a function of time t, $\varepsilon(t)$ strain, and M the time-independent **elastic modulus**. An application of a load yields an instantaneous deformation. A removal of the load yields instantaneous and total recovery. Hooke body does not have a memory: stress at a given time only depends on the deformation at the same time. Hooke body is shown in Fig. 1. The strain-time diagram for a constant stress applied at time t_0 and removed at time t_1 is shown in Fig. 2, left, the stress-strain diagram in Fig. 2, right.

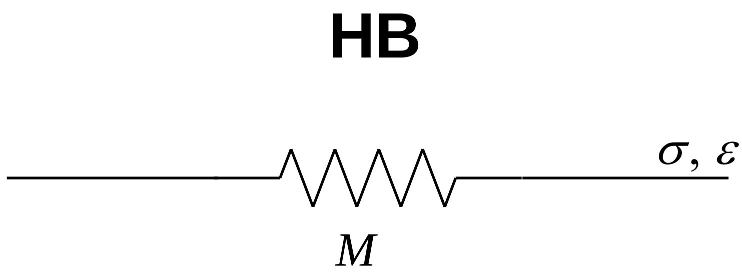


Fig. 1. Hooke body

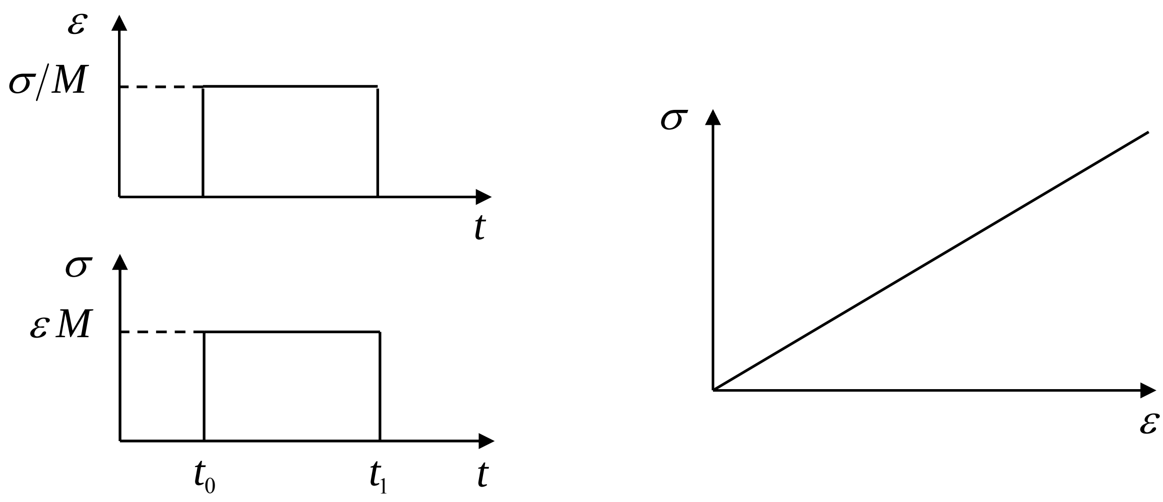


Fig. 2. Left: The strain-time diagram for a constant stress applied at time t_0 and removed at time t_1. Right: The stress-strain diagram.

Hereafter we will use symbol $\mathcal{F}$ for the direct and $\mathcal{F}^{-1}$ for the inverse Fourier transforms

$$\mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)\exp(-i\omega t)\,dt \ , \quad \mathcal{F}^{-1}\{X(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\exp(i\omega t)\,d\omega \ .$$

ω is the angular frequency. An application of the Fourier transform to equation (1) gives

$$\sigma(\omega) = M \cdot \varepsilon(\omega) \ . \tag{2}$$

An equivalent to eq. (1) is

$$\varepsilon(t) = C \cdot \sigma(t) \ , \tag{3}$$

where $C = 1/M$ is the **compliance**.

Linear Viscous Body

Linear viscous body, **Stokes body** (Stokes model, Stokes element, Stokes dashpot; also Newton model, Newton element, viscous dashpot) represents the other extreme behavior in the variety of linear rheological bodies, the behavior of the viscous fluid. Stress is proportional to strain rate:

$$\sigma(t) = \eta \cdot \dot{\varepsilon}(t) \ . \tag{4}$$

Here η is the time-independent **viscosity**. An application of a load yields non-instantaneous linearly increasing deformation. A removal of the load does not yield removal of deformation – there is no recovery. Stokes body has extreme memory. Stokes body is shown in Fig. 3. The strain-time diagram for a constant stress applied at time t_0 and removed at time t_1 is shown in Fig. 4 (left), the stress – strain-rate diagram in Fig. 4 (right).

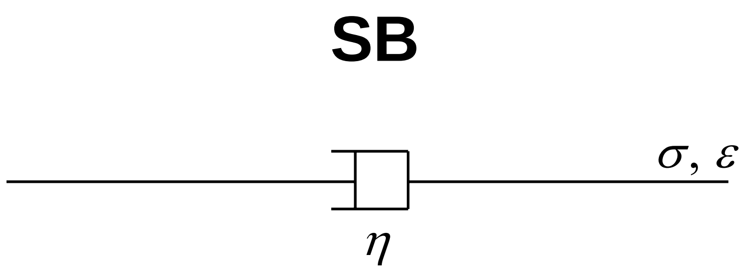


Fig. 3. Stokes body

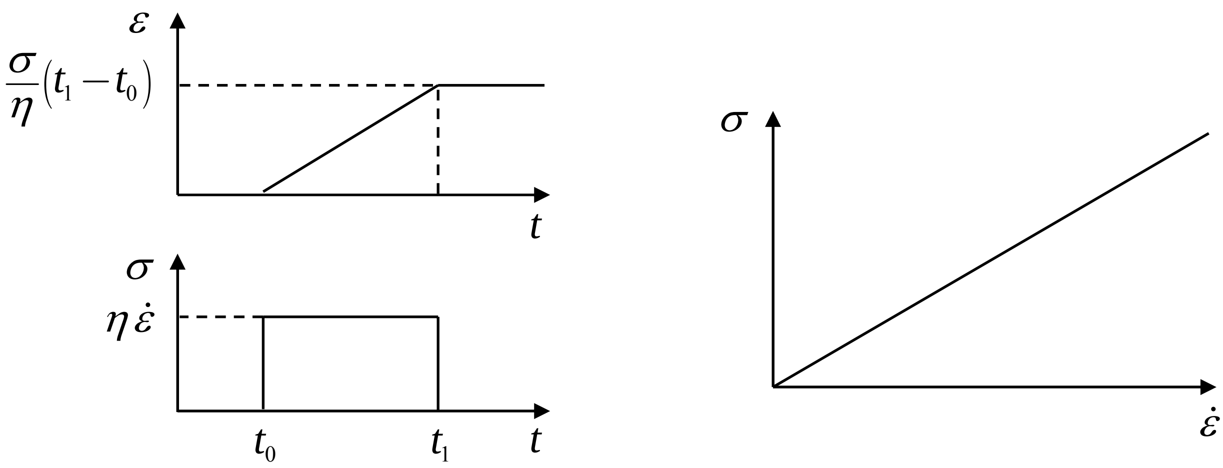


Fig. 4. Left: The strain-time diagram for a constant stress applied at time t_0 and removed at time t_1. Right: The stress – strain-rate diagram.

An application of the Fourier transform to equation (4) gives

$$\sigma(\omega) = i\,\omega\eta \cdot \varepsilon(\omega) \ . \tag{5}$$

Stress-Strain Relation in Viscoelastic Medium

Many real materials combine behaviors of both, elastic solids and viscous fluids. As a consequence, these materials remember their past, that is, the stress-strain relation also depends on time. We can approximate such a behavior using **viscoelastic** models of medium.

For a linear isotropic viscoelastic material the stress-strain relation is given by **Boltzmann superposition and causality principle**. In a simple scalar notation it is

$$\sigma(t) = \int_{-\infty}^{t} \psi(t-\tau)\,\dot{\varepsilon}(\tau)\,d\tau\,, \tag{6}$$

where $\sigma(t)$ is stress, $\dot{\varepsilon}(t)$ time derivative of strain, and $\psi(t)$ **stress relaxation function** - a stress response to Heaviside unit step function in strain. According to equation (6), the stress at a given time t is determined by the entire history of the strain until time t. The upper integration limit ensures the causality. Mathematically, the integral in equation (6), also called the hereditary integral, represents a time convolution of the relaxation function and strain rate. We can use symbol $*$ for the convolution. Equation (6) then can be written as

$$\sigma(t) = \psi(t) * \dot{\varepsilon}(t) \ . \tag{7}$$

Due to properties of convolution,

$$\sigma(t) = \dot{\psi}(t) * \varepsilon(t) \ . \tag{8}$$

Since $\psi(t)$ is the stress response to a unit step function in strain, its time derivative,

$$M(t) = \dot{\psi}(t) \tag{9}$$

is the stress response to the Dirac δ-function in strain. Equation (8) can be written as

$$\sigma(t) = M(t) * \varepsilon(t) \ . \tag{10}$$

We can compare eq. (10) with eq. (1): whereas the stress-strain relation for the elastic body is a simple linear relation with a constant elastic modulus, the stress-strain relation for the viscoelastic body has a convolutory form as a consequence of the time-dependent modulus $M(t)$.

An application of the Fourier transform to equation (10) gives

$$\sigma(\omega) = M(\omega) \cdot \varepsilon(\omega) \ , \tag{11}$$

where

$$M(\omega) = \mathcal{F}\left\{M(t)\right\} = \mathcal{F}\left\{\dot{\psi}(t)\right\} \tag{12}$$

is the complex, frequency-dependent **viscoelastic modulus**. An application of the inverse Fourier transform to eq. (12) gives

$$\dot{\psi}(t) = \mathcal{F}^{-1}\left\{M(\omega)\right\} \tag{13}$$

and, due to properties of the Fourier transform,

$$\psi(t) = \mathcal{F}^{-1}\left\{\frac{M(\omega)}{i\omega}\right\} \ . \tag{14}$$

Equation (11) indicates that the incorporation of the linear viscoelasticity and consequently attenuation into the frequency-domain computations is much easier than those in the time-domain computations – real frequency-independent moduli are simply replaced by complex, frequency-dependent quantities (**the correspondence principle in the linear theory of viscoelasticity**).

The time derivative of the stress is, see equation (8),

$$\dot{\sigma}(t) = \dot{\psi}(t) * \dot{\varepsilon}(t) \tag{15}$$

or, due to equation (9),

$$\dot{\sigma}(t) = M(t) * \dot{\varepsilon}(t) \ . \tag{16}$$

Consider eq. (12):

$$M(\omega) = \mathcal{F}\left\{\dot{\psi}(t)\right\} = \int_{-\infty}^{\infty} \dot{\psi}(t) \exp(-i\omega t) \, dt \ . \tag{17}$$

Because $\psi(t)$ is the stress response to Heaviside unit step function in strain,

$$\psi(t) = \tilde{\psi}(t) \, H(t) \ : \ \tilde{\psi}(0) = \psi(0^+) \, , \ \ \tilde{\psi}(t) = \psi(t); \ t > 0 \ . \tag{18}$$

Equivalently, $\tilde{\psi}(t) = \psi(t); \ t \geq 0$. Then

$$\dot{\psi}(t) = \dot{\tilde{\psi}}(t)\, H(t) + \tilde{\psi}(t)\, \delta(t) \tag{19}$$

and

$$\begin{aligned}
M(\omega) &= \int_{-\infty}^{\infty} \left[\dot{\tilde{\psi}}(t)\, H(t) + \tilde{\psi}(t)\, \delta(t) \right] \exp(-i\omega t)\, dt \\
&= \psi(0) + \int_0^{\infty} \dot{\psi}(t) \exp(-i\omega t)\, dt \\
&= \psi(\infty) + \psi(0) - \psi(\infty) + \int_0^{\infty} \dot{\psi}(t) \exp(-i\omega t)\, dt \\
&= \psi(\infty) + \left[\psi(0) - \psi(\infty) \right] \exp(-i\omega 0) + \int_0^{\infty} \dot{\psi}(t) \exp(-i\omega t)\, dt \\
&= \psi(\infty) - \left[\psi(\infty) - \psi(\infty) \right] \exp(-i\omega\infty) + \left[\psi(0) - \psi(\infty) \right] \exp(-i\omega 0) \\
&+ \int_0^{\infty} \dot{\psi}(t) \exp(-i\omega t)\, dt \\
&= \psi(\infty) - \left[\psi(t) - \psi(\infty) \right] \exp(-i\omega t) \Big|_0^{\infty} + \int_0^{\infty} \dot{\psi}(t) \exp(-i\omega t)\, dt \\
&= \psi(\infty) - \int_0^{\infty} \left[\psi(t) - \psi(\infty) \right] \frac{d}{dt} \exp(-i\omega t)\, dt \\
&= \psi(\infty) + i\omega \int_0^{\infty} \left[\psi(t) - \psi(\infty) \right] \exp(-i\omega t)\, dt \ .
\end{aligned}$$

We found that

$$M(\omega) = \psi(\infty) + i\omega \int_0^{\infty} \left[\psi(t) - \psi(\infty) \right] \exp(-i\omega t)\, dt \ . \tag{20}$$

It follows from eq. (20) that

$$M(\omega = 0) = \psi(t = \infty) \ . \tag{21}$$

Because $i\omega \mathcal{F} \left\{ \varphi(t) \right\} = \varphi(t = 0)$ for $\omega \to \infty$,

$$M(\omega = \infty) = \psi(t = 0) \ . \tag{22}$$

Having found relations (21) and (22), we can define the following characteristics: An instantaneous elastic response of the viscoelastic material is given by the so-called **unrelaxed modulus** M_U, a long-term equilibrium response is given by the **relaxed modulus** M_R

$$M_U = \lim_{t \to 0} \psi(t) \quad , \quad M_R = \lim_{t \to \infty} \psi(t) \ . \tag{23}$$

In the frequency domain

$$M_U = \lim_{\omega \to \infty} M(\omega) \quad , \quad M_R = \lim_{\omega \to 0} M(\omega) \ . \tag{24}$$

The **modulus defect** or **relaxation of modulus** is

$$\delta M = M_U - M_R \ . \tag{25}$$

An application of a unit-step strain, $\varepsilon(t) = H(t)$, causes decrease of $\psi(t)$, that is, **relaxation**, from the unrelaxed state with $\psi(0) = M_U$ to the relaxed state with $\psi(\infty) = M_R$.

Given the viscoelastic modulus, the **quality factor** $Q(\omega)$ is

$$Q(\omega) = \operatorname{Re} M(\omega) / \operatorname{Im} M(\omega) \ . \tag{26}$$

It can be shown that $1/Q(\omega)$ is a measure of internal friction in a linear viscoelastic body.

It is obvious that a numerical integration of the stress-strain relation (6) is practically intractable due to the large computer time and memory requirements. This led many modelers to incorporate only oversimplified $Q(\omega)$ laws in the time-domain computations.

An alternative to the stress-strain relation (6) is the strain-stress relation. The strain at a given time t is determined by the entire history of the stress until time t:

$$\varepsilon(t) = \int_{-\infty}^{t} \chi(t-\tau)\, \dot{\sigma}(\tau)\, d\tau \tag{27}$$

or

$$\varepsilon(t) = \chi(t) * \dot{\sigma}(t) \ . \tag{28}$$

Here $\chi(t)$ is the **creep function** - a strain response to Heaviside unit step function in stress. Due to properties of the convolution, eq. (28) can be rewritten as

$$\varepsilon(t) = \dot{\chi}(t) * \sigma(t) \ . \tag{29}$$

Since $\chi(t)$ is the strain response to a unit step function in stress, its time derivative,

$$C(t) = \dot{\chi}(t) \tag{30}$$

is the strain response to the Dirac δ-function in stress. Equation (29) can be written as

$$\varepsilon(t) = C(t) * \sigma(t) . \tag{31}$$

An application of the Fourier transform to eq. (31) yields

$$\varepsilon(\omega) = C(\omega)\, \sigma(\omega) , \tag{32}$$

where

$$C(\omega) = \mathcal{F}\left\{C(t)\right\} = \mathcal{F}\left\{\dot{\chi}(t)\right\} \tag{33}$$

is the complex, frequency-dependent **creep compliance**. An application of the inverse Fourier transform to eq. (33) gives

$$\dot{\chi}(t) = \mathcal{F}^{-1}\left\{C(\omega)\right\} \tag{34}$$

and, due to properties of the Fourier transform,

$$\chi(t) = \mathcal{F}^{-1}\left\{\frac{C(\omega)}{i\omega}\right\} . \tag{35}$$

Relations

$$C_U = \lim_{t\to 0}\chi(t) \quad , \quad C_R = \lim_{t\to\infty}\chi(t) \tag{36}$$

define the **unrelaxed compliance** C_U and **relaxed compliance** C_R. **Relaxation of compliance** is defined as

$$\delta C = C_R - C_U . \tag{37}$$

An application of a unit-step stress, $\sigma(t) = H(t)$, causes increase of $\chi(t)$, that is, **creep**, from the unrelaxed state with $\chi(0) = C_U$ to the relaxed state with $\chi(\infty) = C_R$.

Using eqs. (8) and (29), and properties of convolution we can write

$$\begin{aligned} \sigma(t) &= \dot{\psi}(t) * \varepsilon(t) \\ &= \dot{\psi}(t) * \left[\, \dot{\chi}(t) * \sigma(t) \,\right] \\ &= \left[\, \dot{\psi}(t) * \dot{\chi}(t) \,\right] * \sigma(t). \end{aligned} \tag{38}$$

It follows from eq. (38) that

$$\dot{\psi}(t) * \dot{\chi}(t) = \delta(t) \tag{39}$$

and, consequently,

$$M(\omega)\, C(\omega) = 1 \ . \tag{40}$$

For the unrelaxed and relaxed states it follows that

$$C_U = \frac{1}{M_U} \quad , \quad C_R = \frac{1}{M_R} \ . \tag{41}$$

Conversion of the Convolutory Stress-Strain Relation into a Differential Form

Consider $M(\omega)$ as a rational function

$$M(\omega) = \frac{P_m(i\omega)}{Q_n(i\omega)} \tag{42}$$

with

$$P_m(i\omega) = \sum_{l=1}^{m} p_l\, (i\omega)^l \quad , \quad Q_n(i\omega) = \sum_{l=1}^{n} q_l\, (i\omega)^l \ . \tag{43}$$

An application of the inverse Fourier transform to equation (11) with $M(\omega)$ given by equation (42) leads to

$$\sum_{l=1}^{n} q_l \frac{d^l \sigma(t)}{dt^l} = \sum_{l=1}^{m} p_l \frac{d^l \varepsilon(t)}{dt^l} \ , \tag{44}$$

the n^{th}-order differential equation for $\sigma(t)$, which can be eventually numerically solved much more easily than the convolution integral. In other words, the convolution integral in equation (6) can be converted into a differential form if $M(\omega)$ is a rational function of $i\omega$.

Day and Minster (1984) assumed that, in general, the viscoelastic modulus is not a rational function. Therefore they suggested approximating a viscoelastic modulus by an n^{th}-order rational function and determining its coefficients by the Padé approximant method. They obtained *n* ordinary differential equations for *n* additional **internal variables**, which replace the convolution integral. The sum of the internal variables multiplied by the unrelaxed modulus gives an additional viscoelastic term to the elastic stress. The revolutionary work of Day and Minster not only developed one particular approach but, in fact, indirectly suggested the future evolution – a direct use of the rheological models whose $M(\omega)$ is a rational function of $i\omega$.

Emmerich and Korn (1987) realized that an acceptable relaxation function corresponds to rheology of what they defined as the **generalized Maxwell body** – *n* Maxwell bodies and one Hooke element (elastic spring) connected in parallel; see Figure 8. Note that the generalized Maxwell body in the literature on rheology is defined without the additional single Hooke element. Therefore, we denote the model considered by Emmerich and Korn (1987) by **GMB-EK**.

Because, in fact, any model consisting of linear springs and dashpots (Stokes elements) connected in series or parallel has its viscoelastic modulus in form of a rational function of $i\omega$, the GMB-EK allowed replacing the convolution integral by a differential form. Emmerich and Korn (1987) obtained for the new variables similar differential equations as Day and Minster (1984). In order to fit an arbitrary $Q(\omega)$ law they chose the relaxation frequencies logarithmically equidistant over a desired frequency range and used the least-square method to determine weight factors of the relaxation mechanisms (classical Maxwell bodies). Emmerich and Korn (1987) demonstrated that their approach is better than the approach based on the Padé approximant method in both accuracy and computational efficiency.

Independently, Carcione et al. (1988a,b), in accordance with the approach of Liu et al. (1976), assumed the **generalized Zener body** (**GZB**) - *n* Zener bodies, that is, *n* standard linear bodies, connected in parallel; see Figure 9. Carcione et al. developed a theory for the GZB and introduced term memory variables for the obtained additional variables.

We will briefly review the GMB-EK and GZB presented in papers by Emmerich and Korn (1987) and Carcione et al. (1988a,b), respectively. It is, however, useful first to remind basics of the simple rheological models.

Rules for Linear Rheological Models

Models which quite well approximate rheological properties and behavior of the real Earth's material can be constructed by connecting the simplest rheological elements, Hooke and Stokes elements, in parallel or series. The properties of the models can be analyzed in the time and frequency domains. There are relatively simple rules in both domains that allow obtaining mathematical representations of the models. The **time-domain** and **frequency-domain rules for linear rheological models** are given in Table 3.

element	**stress-strain relation**
time domain	
Hooke (spring)	$\sigma(t) = M \cdot \varepsilon(t)$, M - elastic modulus
Stokes (dashpot)	$\sigma(t) = \eta \cdot \dot{\varepsilon}(t)$, η - viscosity
frequency domain	
Hooke (spring)	$\sigma(\omega) = M \cdot \varepsilon(\omega)$, M - elastic modulus
Stokes (dashpot)	$\sigma(\omega) = i\omega\eta \cdot \varepsilon(\omega)$, η - viscosity

connection	σ	ε
in series	equal	additive
in parallel	additive	equal

Table 3. Time-domain and frequency-domain rules for linear rheological models

Maxwell Body

One of the simplest viscoelastic models is Maxwell body (Fig. 5, top panel). We can easily derive the basic characteristics of this rheological model. An application of the frequency-domain rules leads to:

HB:

$$\sigma_{HB}(\omega) = M\,\varepsilon_{HB}(\omega) \tag{45}$$

SB:

$$\sigma_{SB}(\omega) = i\omega\eta\,\varepsilon_{SB}(\omega) \tag{46}$$

MB = HB – s – SB (here – s – means connection in series):

$$\sigma = \sigma_{HB} = \sigma_{SB} \quad , \quad \varepsilon = \varepsilon_{HB} + \varepsilon_{SB} \tag{47}$$

$$\varepsilon(\omega) = \frac{\sigma(\omega)}{M} + \frac{\sigma(\omega)}{i\omega\eta} \tag{48}$$

$$\sigma(\omega) = \left(\frac{i\omega\eta M}{M + i\omega\eta}\right)\varepsilon(\omega) \tag{49}$$

$$\sigma(\omega) = M(\omega)\varepsilon(\omega) \ ; \quad M(\omega) = \frac{i\omega M}{\omega_r + i\omega} \tag{50}$$

Here,

$$\omega_r = \frac{M}{\eta} . \tag{51}$$

From the frequency-dependent modulus we easily obtain the relaxed and unrelaxed moduli

$$M_R = \lim_{\omega \to 0} M(\omega) = 0 \ , \tag{52}$$

and

$$M_U = \lim_{\omega \to \infty} M(\omega) = M \ . \tag{53}$$

Relations (52) and (53) mean that Maxwell body under the application of a unit-step strain relaxes from value M_U (at the time of application of the unit-step strain) down to a zero stress. Because, eq. (41),

$$C_R = \frac{1}{M_R} , \tag{54}$$

Maxwell body creeps forever under the application of a unit-step stress.

Find now the stress relaxation function. Using eqs. (14) and (50) we have

$$\psi(t) = \mathcal{F}^{-1}\left\{\frac{M(\omega)}{i\omega}\right\} = \mathcal{F}^{-1}\left\{\frac{M}{\omega_r + i\omega}\right\}$$

and

$$\psi(t) = M\exp(-\omega_r t)H(t) = M\exp(-t/\tau_\sigma)H(t) . \tag{55}$$

Here,

$$\tau_\sigma = \frac{1}{\omega_r} = \frac{\eta}{M} \tag{56}$$

is the stress relaxation time (also Maxwell relaxation time). Then ω_r can be called the relaxation frequency. The relaxation time τ_σ characterizes time during which stress falls down by a characteristic value. Using eq. (9) we can also find the time-dependent modulus

$$M(t) = \dot{\psi}(t) = M\exp(-\omega_r t)\left[\,\delta(t) - \omega_r H(t)\,\right] . \tag{57}$$

An application of the time-domain rules leads to:

HB:

$$\sigma_{HB}(t) = M\,\varepsilon_{HB}(t) \tag{58}$$

SB:

$$\sigma_{SB}(t) = \eta\,\dot{\varepsilon}_{SB}(t) \tag{59}$$

MB = HB – s – SB:

$$\sigma = \sigma_{HB} = \sigma_{SB} \quad , \quad \varepsilon = \varepsilon_{HB} + \varepsilon_{SB} \tag{60}$$

$$\dot{\varepsilon}(t) = \frac{\dot{\sigma}(t)}{M} + \frac{\sigma(t)}{\eta} \tag{61}$$

$$\sigma(t) + \tau_r\,\dot{\sigma}(t) = \tau_r\,M\,\dot{\varepsilon}(t) \tag{62}$$

The use of eqs. (16) and (57) yields

$$\begin{aligned}\sigma(t) &= M(t) * \varepsilon(t)\\ &= \int_{-\infty}^{t} M \exp\left(-\omega_r (t-\tau)\right) \left[\, \delta(t-\tau) - \omega_r\, H(t-\tau) \,\right] \varepsilon(\tau)\, d\tau \\ &= M\varepsilon(t) - \int_{0}^{t} M \exp\left(-\omega_r (t-\tau)\right) \omega_r\, \varepsilon(\tau)\, d\tau\end{aligned}$$

and

$$\sigma(t) = M\varepsilon(t) - \omega_r\, M \int_{0}^{t} \exp\left(-\omega_r (t-\tau)\right) \varepsilon(\tau)\, d\tau \ . \tag{63}$$

Assume $\sigma(t) = H(t)$ in eq. (61) and integrate the equation with respect to time in the interval $\langle\, 0,\ t \,\rangle$:

$$\begin{aligned}\chi(t) = \varepsilon(t) &= \frac{1}{M}\int_{0}^{t} \delta(\xi)\, d\xi + \frac{1}{\eta}\int_{0}^{t} H(\xi)\, d\xi \\ &= \frac{1}{M}\left(1 + \frac{M}{\eta}\, t \right).\end{aligned}$$

Then, using definition (56),

$$\chi(t) = \frac{1}{M}\left(1 + \frac{t}{\tau_\sigma} \right); \quad t \geq 0 \ . \tag{64}$$

The first term on the right-hand side of eq. (64) represents the elastic deformation that appears instantaneously at the time of application of the unit-step stress. This deformation is instantaneously removed upon removal of the stress. The second term represents viscous deformation that grows with time and that will remain after the stress is removed. The behavior of Maxwell body is illustrated in Figs. 6 and 7.

Kelvin-Voigt Body

Another simplest viscoelastic models is Kelvin-Voigt body (Fig. 5, middle panel). We can easily derive the basic characteristics of this rheological model. An application of the frequency-domain rules leads to:

HB:

$$\sigma_{HB}(\omega) = M\,\varepsilon_{HB}(\omega) \tag{65}$$

SB:

$$\sigma_{SB}(\omega) = i\omega\eta\,\varepsilon_{SB}(\omega) \tag{66}$$

KVB = HB – p – SB (here – p – means connection in parallel):

$$\sigma = \sigma_{HB} + \sigma_{SB} \quad , \quad \varepsilon = \varepsilon_{HB} = \varepsilon_{SB} \tag{67}$$

$$\sigma(\omega) = \left(M + i\omega\eta \right) \varepsilon(\omega)$$

$$\sigma(\omega) = M(\omega)\,\varepsilon(\omega) \; ; \quad M(\omega) = M + i\omega\eta \tag{68}$$

$$M_R = \lim_{\omega\to 0} M(\omega) = M \tag{69}$$

$$\dot{\psi}(t) = M(t) = \mathcal{F}^{-1}\left\{M(\omega)\right\} = \mathcal{F}^{-1}\left\{M + i\omega\eta\right\}$$

$$M(t) = M\,\delta(t) + \eta\,\dot{\delta}(t) \tag{70}$$

$$\psi(t) = M\,H(t) + \eta\,\delta(t) \tag{71}$$

$$M_U = \lim_{t\to 0} \psi(t) = M + \eta\,\delta(0) \tag{72}$$

An application of the time-domain rules leads to:

HB:

$$\sigma_{HB}(t) = M\,\varepsilon_{HB}(t) \tag{73}$$

SB:

$$\sigma_{SB}(t) = \eta\,\dot{\varepsilon}_{SB}(t) \tag{74}$$

KVB = HB – p – SB:

$$\sigma = \sigma_{HB} + \sigma_{SB} \quad , \quad \varepsilon = \varepsilon_{HB} = \varepsilon_{SB} \tag{75}$$

$$\sigma(t) = M\varepsilon(t) + \eta\,\dot{\varepsilon}(t) \tag{76}$$

We would obtain eq. (76) also by using eqs. (10) and (70).

It follows from eq. (40) that the compliance is

$$C(\omega) = \frac{1}{M + i\omega\eta} \tag{77}$$

and from eq. (34) that the time derivative of the creep function is

$$\dot{\chi}(t) = \mathcal{F}^{-1}\left\{ \frac{1}{M + i\omega\eta} \right\} = \frac{1}{\eta}\exp\left(-M\,t/\eta\right) H(t) .$$

Then the creep function is obtained by the time integration

$$\chi(t) = \int_0^t \frac{1}{\eta}\exp\left(-M\,\vartheta/\eta\right) H(\vartheta)\,d\vartheta$$

that gives

$$\chi(t) = \frac{1}{M}\left[1 - \exp\left(-t/\tau_\varepsilon\right) \right] ; \quad t \geq 0 \ . \tag{78}$$

Here,

$$\tau_\varepsilon = \frac{\eta}{M} \tag{79}$$

is the strain relaxation time (also called retardation time). This terminology comes from the exponential character of increase of the creep function. It follows from eqs. (36) and (78) that

$$C_U = 0 . \tag{80}$$

The latter result means that the Kelvin-Voigt body has zero creep (zero strain) at the time of the application of the unit-step stress. At the same time, as

$$M_U = \frac{1}{C_U} = \infty \ , \tag{81}$$

the instantaneous stress response at the time of the application of the unit-step strain is singular. The behavior of the Kelvin-Voigt body is illustrated in Figs. 6 and 7.

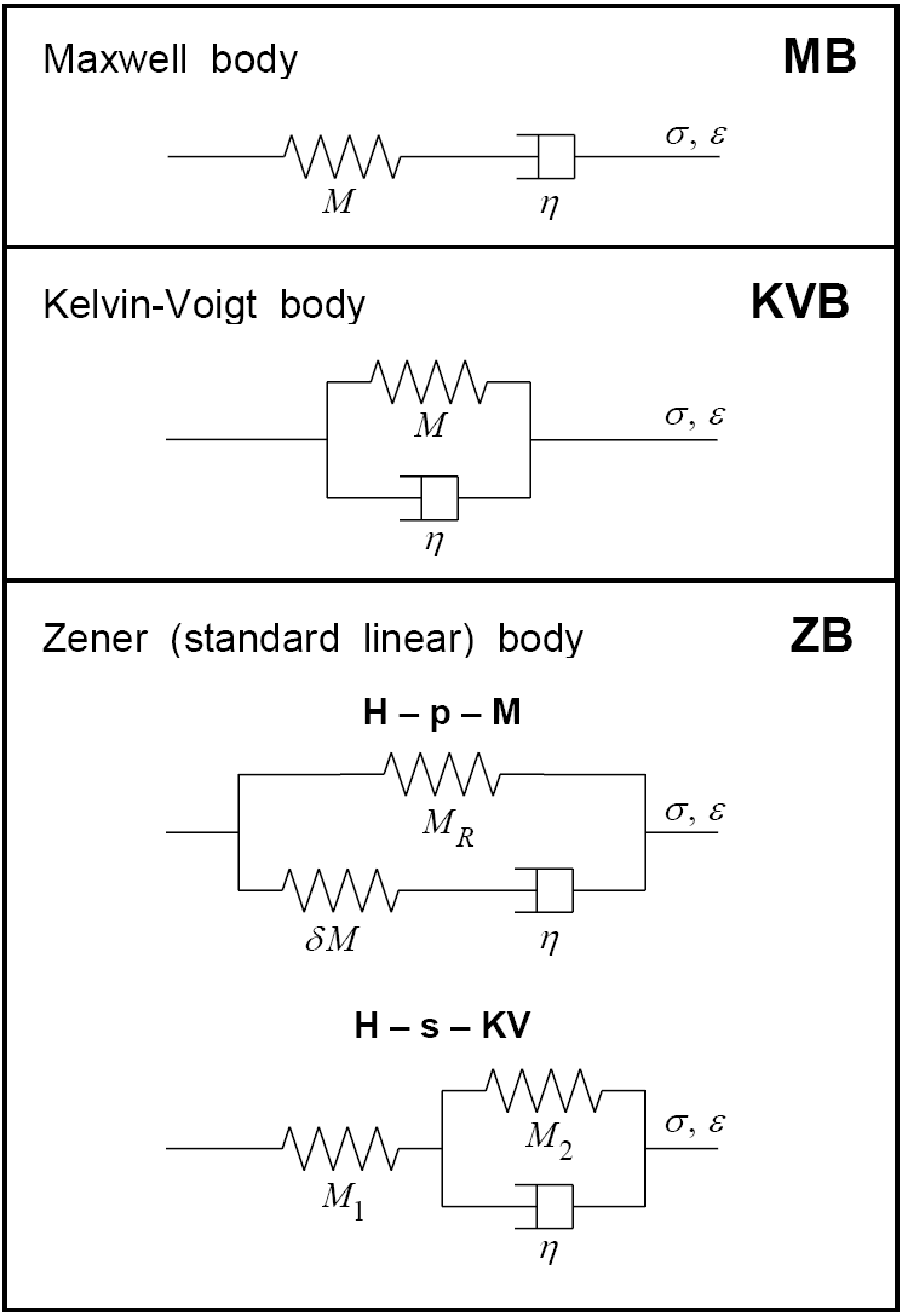


Fig. 5. The simplest rheological models of viscoelastic materials.

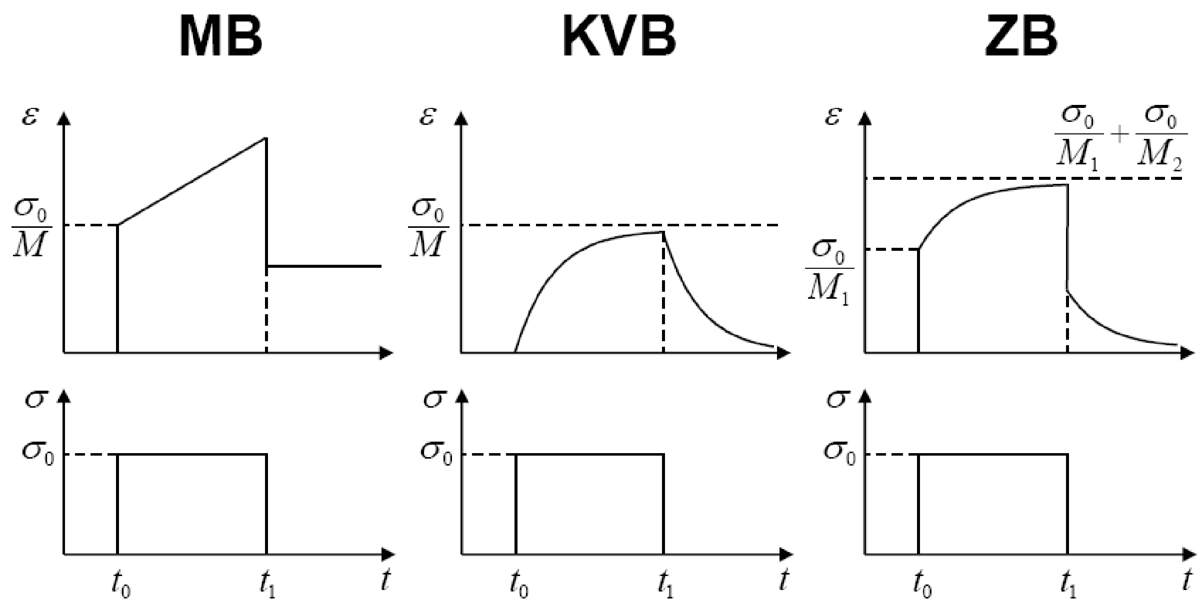


Fig. 6. Creep in Maxwell, Kelvin-Voigt and Zener (standard linear) bodies: strain-time diagrams for a constant stress applied at time t_0 and removed at time t_1.

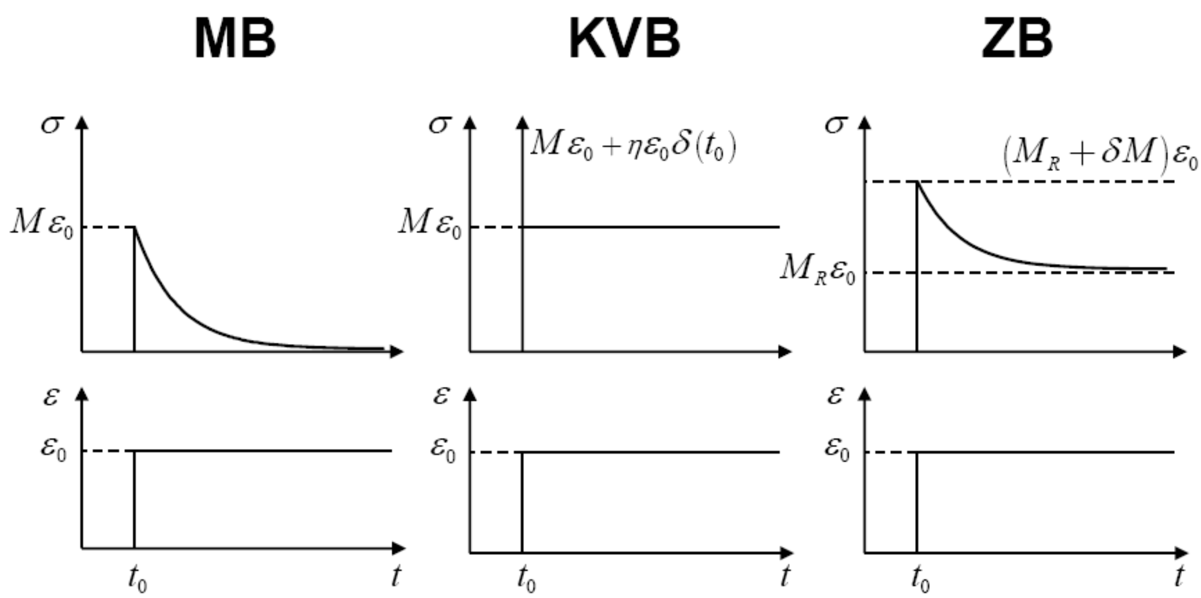


Fig. 7. Stress relaxation in Maxwell, Kelvin-Voigt and Zener (standard linear) bodies: stress-time diagrams for a constant strain applied at time t_0.

Zener (Standard Linear) Body

A more general than Maxwell and Kelvin-Voigt bodies is still relatively simple viscoelastic Zener (also standard linear) body (Fig. 5, bottom panel). There are two equivalent models: H – p – M (Hooke body connected in parallel with Maxwell body) and H – s – KV (Hooke body connected in series with Kelvin-Voigt body). It is easier to see the meaning of the elastic moduli in the H – p – M model. At the time of the application of the unit-step strain the instantaneous, i.e., unrelaxed, stress will be given by the sum of moduli of the two elastic springs, $M_U = M_R + \delta M$. At the same time deformation of the dashpot will start to grow from zero. The growth of the viscous deformation will gradually release stress of the spring connected in series with the dashpot (i.e., spring in Maxwell body). In the limit, the relaxed stress, M_R, will be only in the spring connected in parallel with Maxwell body. We can easily derive the basic characteristics of this rheological model. An application of the frequency-domain rules leads to:

HB:

$$\sigma_{HB}(\omega) = M_R \, \varepsilon_{HB}(\omega) \tag{82}$$

MB, eq. (49):

$$\sigma_{MB}(\omega) = \left(\frac{i\omega\eta\,\delta M}{\delta M + i\omega\eta} \right) \varepsilon_{MB}(\omega) \tag{83}$$

ZB = H – p – MB:

$$\sigma = \sigma_{HB} + \sigma_{MB} \quad , \quad \varepsilon = \varepsilon_{HB} = \varepsilon_{MB} \tag{84}$$

$$\sigma(\omega) = M_R \, \varepsilon(\omega) + \frac{i\omega\eta\,\delta M}{\delta M + i\omega\eta} \varepsilon(\omega)$$

$$= M_R \frac{1 + i\omega \dfrac{\eta}{\delta M} \dfrac{M_U}{M_R}}{1 + i\omega \dfrac{\eta}{\delta M}} \, \varepsilon(\omega)$$

Define stress and strain relaxation times, τ_σ and τ_ε,

$$\tau_\sigma = \frac{\eta}{\delta M} \quad , \quad \tau_\varepsilon = \frac{\eta}{\delta M} \frac{M_U}{M_R} \ . \tag{85}$$

Then

$$\sigma(\omega) = M(\omega) \, \varepsilon(\omega) \ ; \quad M(\omega) = M_R \frac{1 + i\omega\tau_\varepsilon}{1 + i\omega\tau_\sigma} . \tag{86}$$

Taking limits of $M(\omega)$ we verify our interpretation of the meaning of the elastic moduli:

$$\lim_{\omega \to \infty} M(\omega) = M_U = M_R + \delta M \quad , \quad \lim_{\omega \to 0} M(\omega) = M_R \ . \tag{87}$$

From eqs. (85) we have the simple relation between the unrelaxed and relaxed moduli:

$$M_U = M_R \frac{\tau_\varepsilon}{\tau_\sigma} \ . \tag{88}$$

We can now determine the stress relaxation function using eqs. (14) and (86):

$$\psi(t) = \mathcal{F}^{-1}\left\{ \frac{M(\omega)}{i\omega} \right\} = \mathcal{F}^{-1}\left\{ M_R \left[\frac{-i}{\omega} + \frac{i\tau_\varepsilon}{i - \tau_\sigma \omega} - \frac{i\tau_\sigma}{i - \tau_\sigma \omega} \right] \right\} .$$

It is now easy to find

$$\psi(t) = M_R \left[1 - \left(1 - \frac{\tau_\varepsilon}{\tau_\sigma} \right) \exp\left(-t/\tau_\sigma \right) \right] H(t) \ . \tag{89}$$

It is also easy to obtain the creep function of Zener body as

$$\chi(t) = \frac{1}{M_R} \left[1 - \left(1 - \frac{\tau_\sigma}{\tau_\varepsilon} \right) \exp\left(-t/\tau_\varepsilon \right) \right] H(t) \ . \tag{90}$$

The behavior of Zener body is illustrated in Figs. 6 and 7.

The GZB and GMB-EK Rheological Models

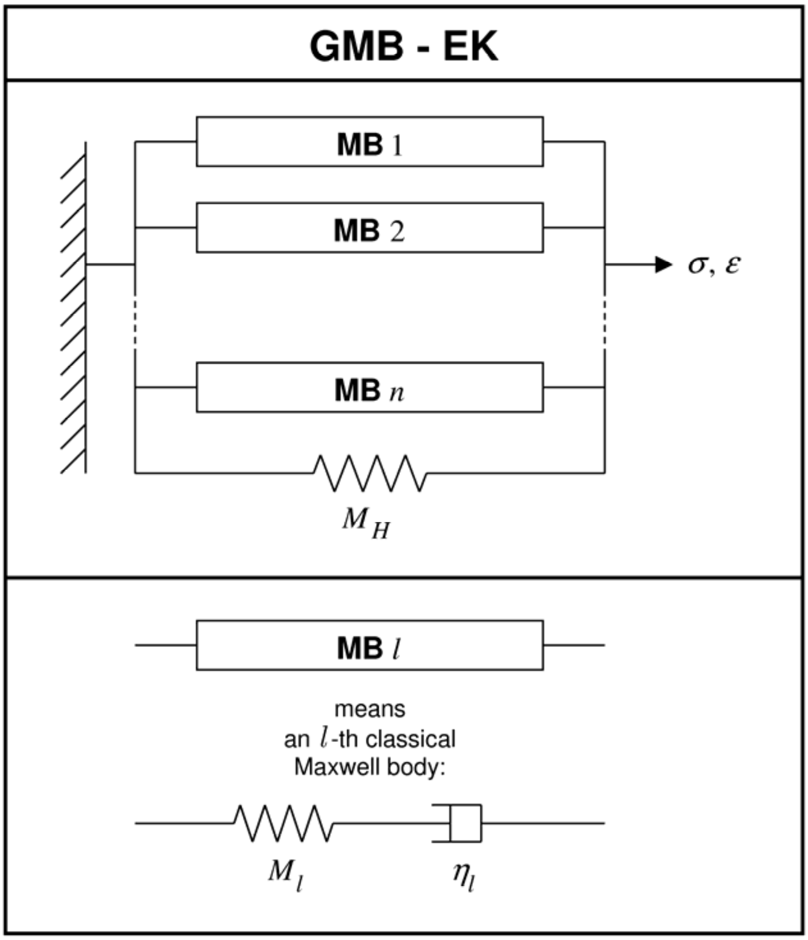


Figure 8. Rheological model of the Generalized Maxwell Body (GMB-EK) defined by Emmerich and Korn (1987). M_H and M_l denote elastic moduli, η_l viscosity.

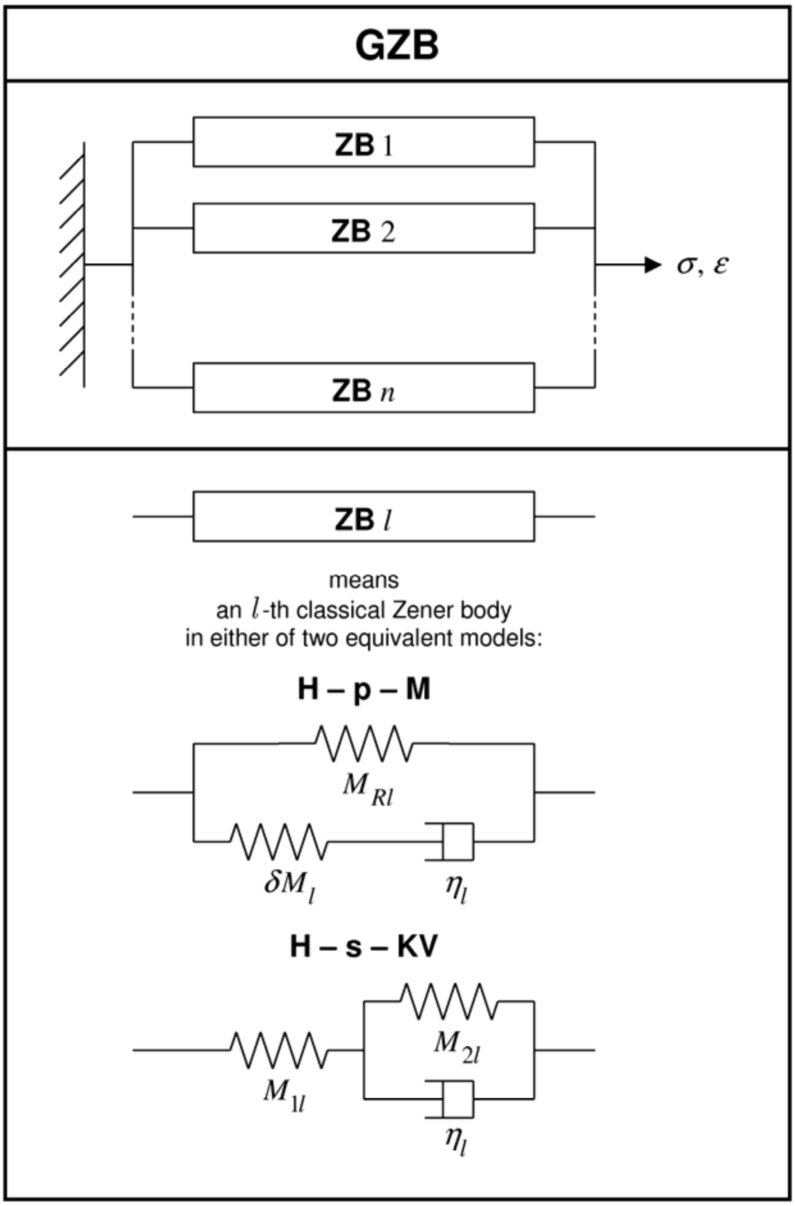


Figure 9. Rheological model of the Generalized Zener Body (GZB). For a classical Zener body (standard linear body) there are two equivalent models: H-p-M , that is, Hooke element connected in parallel with Maxwell body, and H-s-KV, that is, Hooke element connected in series with Kelvin-Voigt body. In the H-p-M model it is easier to recognize the relaxed modulus M_{Rl} and modulus defect δM_l. M_{1l} and M_{2l} in the H-s-KV model denote elastic moduli. In both models η_l stands for viscosity.

<u>GMB-EK.</u> For the GMB-EK we easily find

$$M(\omega) = M_H + \sum_{l=1}^{n} \frac{i M_l \omega}{\omega_l + i\omega} \tag{91}$$

with **relaxation frequencies**

$$\omega_l = \frac{M_l}{\eta_l} ; \quad l = 1, \ldots, n \ . \tag{92}$$

We find relaxed and unrelaxed moduli

$$M_R \equiv \lim_{\omega \to 0} M(\omega) = M_H \quad , \quad M_U \equiv \lim_{\omega \to \infty} M(\omega) = M_R + \sum_{l=1}^{n} M_l \ . \tag{93}$$

Since $M_U = M_R + \delta M$,

$$M_l = \delta M_l \ . \tag{94}$$

Without any simplification we can consider

$$\delta M_l = a_l \, \delta M \quad ; \quad \sum_{l=1}^{n} a_l = 1 \ . \tag{95}$$

Then

$$M(\omega) = M_R + \delta M \sum_{l=1}^{n} \frac{i a_l \omega}{\omega_l + i\omega} \ . \tag{96}$$

Using relation (14) we easily obtain the relaxation function

$$\psi(t) = \left[M_R + \delta M \sum_{l=1}^{n} a_l \, e^{-\omega_l t} \right] \cdot H(t) \ , \tag{97}$$

where $H(t)$ is the Heaviside unit step function. The above formulas were presented by Emmerich and Korn (1987).

<u>GZB.</u> From the two equivalent models of the GZB (see Figure 9) we choose the one in which a single ZB is of the H-p-M type (Hooke element in parallel with Maxwell body). This is because we can immediately see the meaning ($M_{Rl}, \delta M_l$) of the elastic moduli of both Hooke elements in each ZB. For the GZB we easily obtain a well-known

$$M(\omega) = \sum_{l=1}^{n} M_{Rl} \frac{1 + i\,\tau_{\varepsilon l}\,\omega}{1 + i\,\tau_{\sigma l}\,\omega} \tag{98}$$

with **relaxation times**

$$\tau_{\varepsilon l} = \frac{\eta_l}{\delta M_l} \frac{M_{Ul}}{M_{Rl}} \quad , \quad \tau_{\sigma l} = \frac{\eta_l}{\delta M_l} \quad , \quad \frac{\tau_{\varepsilon l}}{\tau_{\sigma l}} = \frac{M_{Ul}}{M_{Rl}} \tag{99}$$

and

$$M_{Ul} = M_{Rl} + \delta M_l \ . \tag{100}$$

The unrelaxed and relaxed moduli are

$$\begin{aligned} M_R &\equiv \lim_{\omega \to 0} M(\omega) = \sum_{l=1}^{n} M_{Rl} \\ M_U &\equiv \lim_{\omega \to \infty} M(\omega) = \sum_{l=1}^{n} M_{Rl} \frac{\tau_{\varepsilon l}}{\tau_{\sigma l}} = M_R + \sum_{l=1}^{n} \delta M_l \ . \end{aligned} \tag{101}$$

Using relation (14) we easily obtain the relaxation function

$$\psi(t) = \left\{ \sum_{l=1}^{n} M_{Rl} \left[1 - \left(1 - \frac{\tau_{\varepsilon l}}{\tau_{\sigma l}} \right) \exp\left(-t / \tau_{\sigma l} \right) \right] \right\} \cdot H(t) \tag{102}$$

Assuming simplification (Carcione, 2001)

$$M_{Rl} = \frac{1}{n} M_R \tag{103}$$

we get

$$\begin{aligned} M(\omega) &= \frac{M_R}{n} \sum_{l=1}^{n} \frac{1 + i\tau_{\varepsilon l}\omega}{1 + i\tau_{\sigma l}\omega} \ , \\ \psi(t) &= M_R \left[1 - \frac{1}{n} \sum_{l=1}^{n} \left(1 - \frac{\tau_{\varepsilon l}}{\tau_{\sigma l}} \right) \exp\left(-t / \tau_{\sigma l} \right) \right] \cdot H(t) \end{aligned} \tag{104}$$

Formulas (103) and (104) were presented by Carcione (2001). As far as we know, papers dealing with the incorporation of the attenuation based on the GZB, starting from Liu et al. (1976), had the same error – the missing factor $\frac{1}{n}$ in the viscoelastic modulus and relaxation function ($\frac{1}{L}$ in most of the papers, L being the number of classical Zener bodies, that is, the number of relaxation mechanisms).

The Relation between the GZB and GMB-EK

After papers by Emmerich and Korn (1987) and Carcione et al. (1988a,b) different authors decided either for the GMB-EK or GZB.

The GMB-EK formulas were used by Emmerich (1992), Fäh 1992, Moczo and Bard (1993), and in many other papers. Moczo et al. (1997) applied the approach also in the finite-element method and hybrid finite-difference – finite-element method. An important aspect was that in the papers one memory variable was defined for one displacement component. Later Xu and McMechan (1995) introduced term composite memory variables which, however, did not differ from the variables used from the very beginning in the above papers.

Robertsson et al. (1994) implemented the memory variables based on the GZB rheology into the staggered-grid velocity-stress finite-difference scheme. Their numerical results do not suffer from the missing factor $1/n$ because they were performed for $n = 1$. Blanch et al. (1995) suggested an approximate single-parameter method, τ -method, to approximate constant $Q(\omega)$ law. Xu and McMechan (1998) used simulated annealing for determining a best combination of relaxation mechanisms to approximate a desired $Q(\omega)$ law. In the two latter papers the factor $1/n$ was missing in the relaxation functions.

As far as we know, in many following papers the authors using the GZB did not comment on the rheology of the GMB-EK and the corresponding time-domain algorithms, and the authors using the GMB-EK did not comment those for the GZB. Thus, two parallel sets of papers and algorithms had been developed during years.

Therefore, following Moczo and Kristek (2005), look at the relation between the GZB and GMB-EK rheologies. Consider again the ZB (H-p-M) model. The application of the frequency-domain rules (Table 3) to the *l*-th ZB, that is to (H-p-M), gives

$$\sigma_l(\omega)\cdot\left(\frac{1}{\delta M_l}+\frac{1}{i\eta_l\,\omega}\right)=\left(1+\frac{M_{Rl}}{\delta M_l}+\frac{M_{Rl}}{i\eta_l\,\omega}\right)\cdot\varepsilon(\omega)\ . \tag{105}$$

Defining

$$\omega_l=\frac{\delta M_l}{\eta_l} \tag{106}$$

and rearranging equation (105) we get

$$\sigma_l(\omega)=M_l(\omega)\cdot\varepsilon(\omega)\ \ ;\quad M_l(\omega)=M_{Rl}+\frac{i\,\delta M_l\,\omega}{\omega_l+i\,\omega}\ . \tag{107}$$

For *n* ZB (H-p-M) connected in parallel, that is, for the GZB (Figure 9), the stress is

$$\sigma(\omega)\ =\ \sum_{l=1}^{n}\sigma_l(\omega)\ =\ \left[\sum_{l=1}^{n}M_l(\omega)\right]\cdot\varepsilon(\omega) \tag{108}$$

and thus

$$M(\omega) = \sum_{l=1}^{n} M_{Rl} + \sum_{l=1}^{n} \frac{i\,\delta M_l\,\omega}{\omega_l + i\,\omega} \,. \tag{109}$$

Since

$$M_R = \sum_{l=1}^{n} M_{Rl} \quad , \quad M_U = M_R + \sum_{l=1}^{n} \delta M_l \quad , \quad M_U = M_R + \delta M \;, \tag{110}$$

without loss of generality we can consider

$$\delta M_l = a_l\,\delta M \;\; ; \quad \sum_{l=1}^{n} a_l = 1 \tag{111}$$

and get

$$M(\omega) = M_R + \delta M \sum_{l=1}^{n} \frac{i\,a_l\,\omega}{\omega_l + i\omega} \,. \tag{112}$$

We see that for the GZB (H-p-M), Figure 9, we obtained exactly the same $M(\omega)$ as it has been obtained by Emmerich and Korn (1987) for their GMB-EK (Figure 8). It is also easy to get the same for the GZB (H-s-KV) or to rewrite non-simplified $\psi(t)$ for the GZB, equation (102), into the form of $\psi(t)$ for the GMB-EK, equation (97), without any simplification. In other words, the rheology of the GMB-EK and GZB is one and the same. As a consequence, we can continue with the GMB-EK and its simpler-form relations compared to those developed in papers on the GZB with two relaxation times. Also note that there is no need for a simplification (103) in equations (104).

Introduction of the Anelastic Functions

We will use term anelastic functions instead of memory variables. It is easy to rewrite the viscoelastic modulus (112) and relaxation function (97) using the unrelaxed modulus,

$$M(\omega) = M_U - \delta M \sum_{l=1}^{n} \frac{a_l\,\omega_l}{\omega_l + i\,\omega} \tag{113}$$

and

$$\psi(t) = \left[M_U - \delta M \sum_{l=1}^{n} a_l \left(1 - e^{-\omega_l t}\right) \right] \cdot H(t) \;, \tag{114}$$

and obtain the time derivative of the relaxation function

$$\begin{aligned} M(t) &= \dot{\psi}(t) \\ &= -\delta M \sum_{l=1}^{n} a_l\, \omega_l\, e^{-\omega_l t} \cdot H(t) \;+\; \left[M_U - \delta M \sum_{l=1}^{n} a_l \left(1 - e^{-\omega_l t}\right) \right] \cdot \delta(t) . \end{aligned} \tag{115}$$

Inserting equation (115) into equation (10) gives

$$\begin{aligned} \sigma(t) = & -\int_{-\infty}^{t} \delta M \sum_{l=1}^{n} a_l \omega_l e^{-\omega_l (t-\tau)} \cdot H(t-\tau) \cdot \varepsilon(\tau)\, d\tau \\ & + \int_{-\infty}^{t} M_U \cdot \delta(t-\tau) \cdot \varepsilon(\tau)\, d\tau \\ & - \int_{-\infty}^{t} \delta M \sum_{l=1}^{n} a_l \left(1 - e^{-\omega_l (t-\tau)}\right) \cdot \delta(t-\tau) \cdot \varepsilon(\tau)\, d\tau \end{aligned} \tag{116}$$

and

$$\sigma(t) = M_U \cdot \varepsilon(t) - \delta M \sum_{l=1}^{n} a_l \omega_l \int_{-\infty}^{t} \varepsilon(\tau) \cdot e^{-\omega_l (t-\tau)}\, d\tau \;. \tag{117}$$

Now it is possible to replace the convolution integral by additional functions **(anelastic functions, internal variables**, **new variables**, **memory variables**). While Day and Minster (1984), Emmerich and Korn (1987) and Carcione et al. (1988a,b) defined the additional functions as dependent also on the material properties, for an important reason that will be explained later, Kristek and Moczo (2003) defined their anelastic functions as independent of the material properties. Here we follow Kristek and Moczo (2003). Defining an anelastic function

$$\zeta_l(t) = \omega_l \int_{-\infty}^{t} \varepsilon(\tau) \cdot e^{-\omega_l (t-\tau)}\, d\tau \quad , \quad l = 1, \dots, n \tag{118}$$

we get the stress-strain relation in the form

$$\sigma(t) = M_U \cdot \varepsilon(t) \;-\; \sum_{l=1}^{n} \delta M\, a_l\, \zeta_l(t) \;. \tag{119}$$

Applying time derivative to equation (118) we get

$$\begin{aligned} \dot{\zeta}_l(t) &= \omega_l \frac{d}{dt} \int_{-\infty}^{t} \varepsilon(\tau) \cdot e^{-\omega_l (t-\tau)}\, d\tau \\ &= \omega_l \left[-\omega_l \int_{-\infty}^{t} \varepsilon(\tau) \cdot e^{-\omega_l (t-\tau)}\, d\tau + \varepsilon(t) \right] \\ &= \omega_l \left[-\zeta_l(t) + \varepsilon(t) \right] \end{aligned} \tag{120}$$

and

$$\dot{\zeta}_l(t) + \omega_l \, \zeta_l(t) = \omega_l \, \varepsilon(t) \quad ; \quad l = 1,...,n . \tag{121}$$

Equations (119) and (121) define the time-domain **stress-strain relation for the viscoelastic medium** whose rheology corresponds to rheology of the GMB-EK (and to its equivalent – the GZB).

If the staggered-grid velocity-stress finite-difference scheme is to be used, then the time derivative of the stress is needed. In such a case, $M(t)$ given by equation (115) is inserted into relation (16) and the above procedure of obtaining the anelastic functions and stress-strain relation can be followed with time derivatives of the stress and strain instead of the stress and strain themselves. An alternative procedure is to apply time derivatives to equations (119) and (121), and define the anelastic function as the time derivative of the anelastic function (118). In either case we obtain

$$\dot{\sigma}(t) = M_U \cdot \dot{\varepsilon}(t) \; - \; \sum_{l=1}^{n} \delta M \; a_l \; \xi_l(t) \tag{122}$$

and

$$\dot{\xi}_l(t) + \omega_l \, \xi_l(t) = \omega_l \, \dot{\varepsilon}(t) \quad ; \quad l = 1,...,n \; . \tag{123}$$

It is useful to define **anelastic coefficients**

$$Y_l = a_l \frac{\delta M}{M_U} \quad ; \quad l = 1,...,n \; . \tag{124}$$

Then the stress-strain relations (119) and (122) become

$$\sigma(t) = M_U \cdot \varepsilon(t) \; - \; \sum_{l=1}^{n} M_U \, Y_l \, \zeta_l(t) \tag{125}$$

and

$$\dot{\sigma}(t) = M_U \cdot \dot{\varepsilon}(t) \; - \; \sum_{l=1}^{n} M_U \, Y_l \, \xi_l(t) \; . \tag{126}$$

The related equations (121) and (123) are unchanged. It is clear that the stress or its time derivative can be calculated if the unrelaxed modulus and anelastic coefficients are known. The unrelaxed modulus is directly related to the elastic speed of wave propagation, the anelastic coefficients have to be determined from $Q(\omega)$-law.

Using the anelastic coefficient, the elastic modulus and viscosity in the *l*-th MB are $M_U Y_l$ and $\frac{1}{\omega_l} M_U Y_l$, respectively, the relaxed modulus is

$$M_R = M_U \left(1 - \sum_{l=1}^{n} Y_l \right),$$

and viscoelastic modulus

$$M(\omega) = M_U \left[1 - \sum_{l=1}^{n} Y_l \frac{\omega_l}{\omega_l + i\omega} \right] . \tag{127}$$

(Note that Emmerich and Korn 1987, used slightly less numerically accurate $y_l = a_l \, \delta M / M_R$; $l = 1,...,n$.) The quality factor (26) is then

$$\frac{1}{Q(\omega)} = \frac{\sum_{l=1}^{n} Y_l \frac{\omega_l \, \omega}{\omega_l^2 + \omega^2}}{1 - \sum_{l=1}^{n} Y_l \frac{\omega_l^2}{\omega_l^2 + \omega^2}} . \tag{128}$$

From equation (128) we can get

$$Q^{-1}(\omega) = \sum_{l=1}^{n} \frac{\omega_l \, \omega + \omega_l^2 \, Q^{-1}(\omega)}{\omega_l^2 + \omega^2} Y_l . \tag{129}$$

Equation (129) can be used to numerically fit any $Q(\omega)$-law. Emmerich and Korn (1987) demonstrated that a sufficiently accurate approximation to nearly constant $Q(\omega)$ is obtained if the relaxation frequencies ω_l cover the frequency range under interest logarithmically equidistantly. If, for example, $Q(\omega)$ values are known at frequencies $\tilde{\omega}_k \, ; k = 1,...,2n-1$, with $\tilde{\omega}_1 = \omega_1$, $\tilde{\omega}_{2n-1} = \omega_n$, equation (129) can be solved for the anelastic coefficients using the least square method.

A more detailed discussion of the frequency range and its sampling by frequencies $\tilde{\omega}_k$ can be found in the paper by Graves and Day (2003; equations 13 and 14).

In practice, a phase velocity at certain reference frequency ω_r, instead of the elastic velocity corresponding to the unrelaxed modulus, is known from measurements. The phase velocity $c(\omega)$ is given by

$$\frac{1}{c(\omega)} = \mathrm{Re}\left[\left(\frac{M(\omega)}{\rho} \right)^{-1/2} \right] . \tag{130}$$

From equations (127) and (130) we get (Moczo et al. 1997) for the phase velocity $c(\omega_r)$

$$M_U = \rho c^2(\omega_r) \frac{R + \Theta_1}{2R^2} , \tag{131}$$

where

$$R = \left(\Theta_1^2 + \Theta_2^2 \right)^{1/2} ,$$
$$\Theta_1 = 1 - \sum_{l=1}^{n} Y_l \frac{1}{1 + \left(\omega_r / \omega_l \right)^2} \;\; , \quad \Theta_2 = \sum_{l=1}^{n} Y_l \frac{\omega_r / \omega_l}{1 + \left(\omega_r / \omega_l \right)^2} . \tag{132}$$

Thus, using equations (131) and (132), the unrelaxed modulus can be determined from the anelastic coefficients Y_l ; $l = 1,...,n$, and phase velocity $c(\omega_r)$.

Viscoelastic moduli and attenuations for Hooke, Stokes, Maxwell, Kelvin-Voigt, Zener and GMB-EK (=GZB) bodies - - Graphical examples

Here we graphically illustrate viscoelastic modulus M and attenuation $1/Q$ for Hooke, Stokes, Maxwell, Kelvin-Voigt, Zener and GMB-EK (=GZB) body. For each type of rheology we show a real part of modulus, imaginary part of modulus, absolute value of modulus, and attenuation as functions of frequency. Graphs clearly illustrate different viscoelastic properties of the basic type of rheological models. Figure legends explain how the graphs were obtained. Hooke and Stokes rheologies are illustrated in Fig. 10, Maxwell and Kelvin-Voigt in Fig. 11, and Zener together with GMB-EK (=GZB) for three relaxation mechanisms in Fig. 12.

Fig. 10. Viscoelastic modulus M and attenuation $1/Q$ as functions of frequency f. The horizontal axis (frequency) is logarithmic, the vertical axis decadic. Modulus for Hooke body is calculated using eq. (2), modulus for Stokes body using eq. (5). Frequency dependence of $1/Q$ is calculated using eq. (26). Note that $1/Q$ gives infinity for Stokes body.

Fig. 11. Viscoelastic modulus M and attenuation $1/Q$ as functions of frequency f. The horizontal axis (frequency) is logarithmic, the vertical axis decadic. Modulus for Maxwell body is calculated using eq. (50), modulus for Kelvin-Voigt body using eq. (68). Frequency dependence of $1/Q$ is calculated using eq. (26).

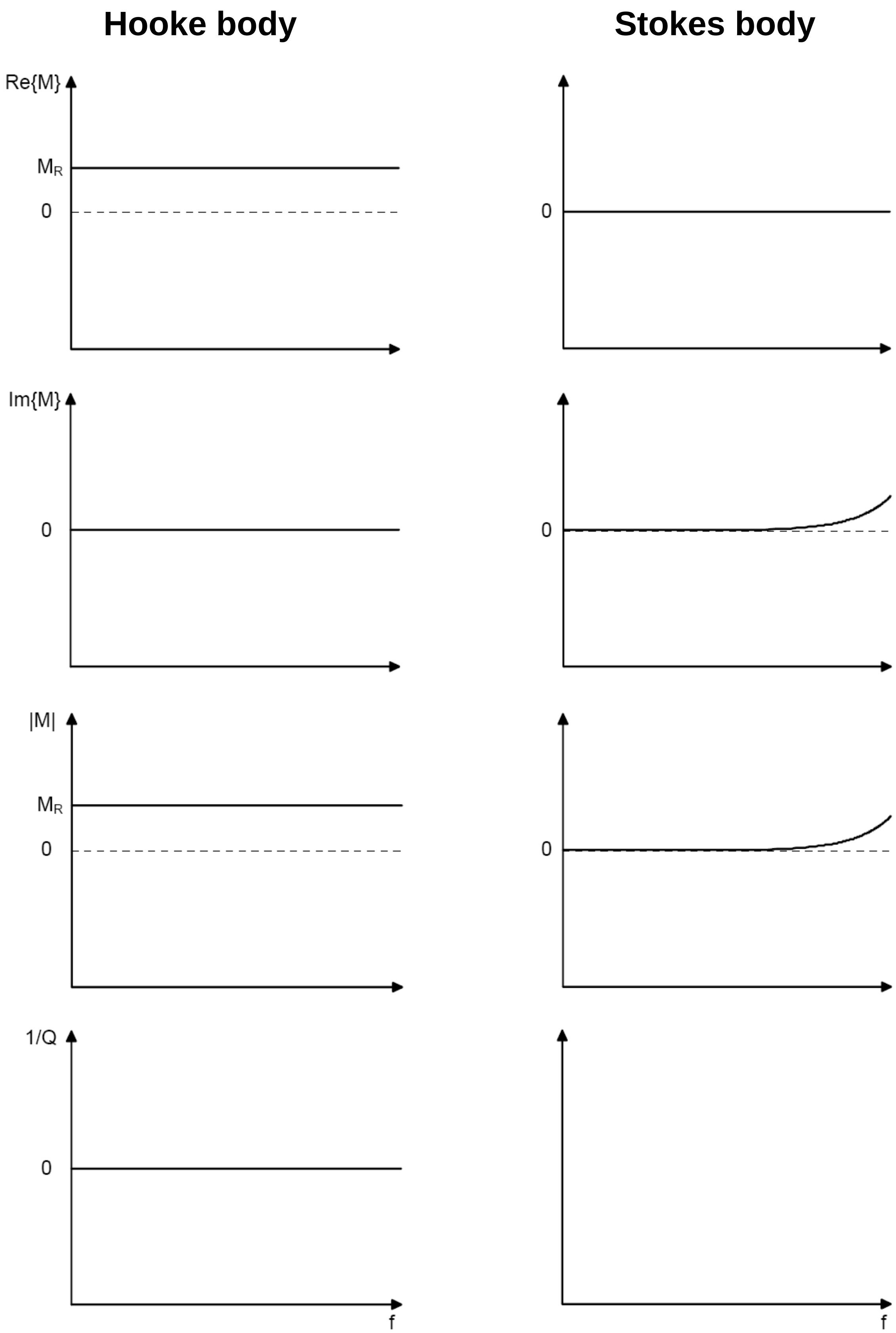


Fig. 10

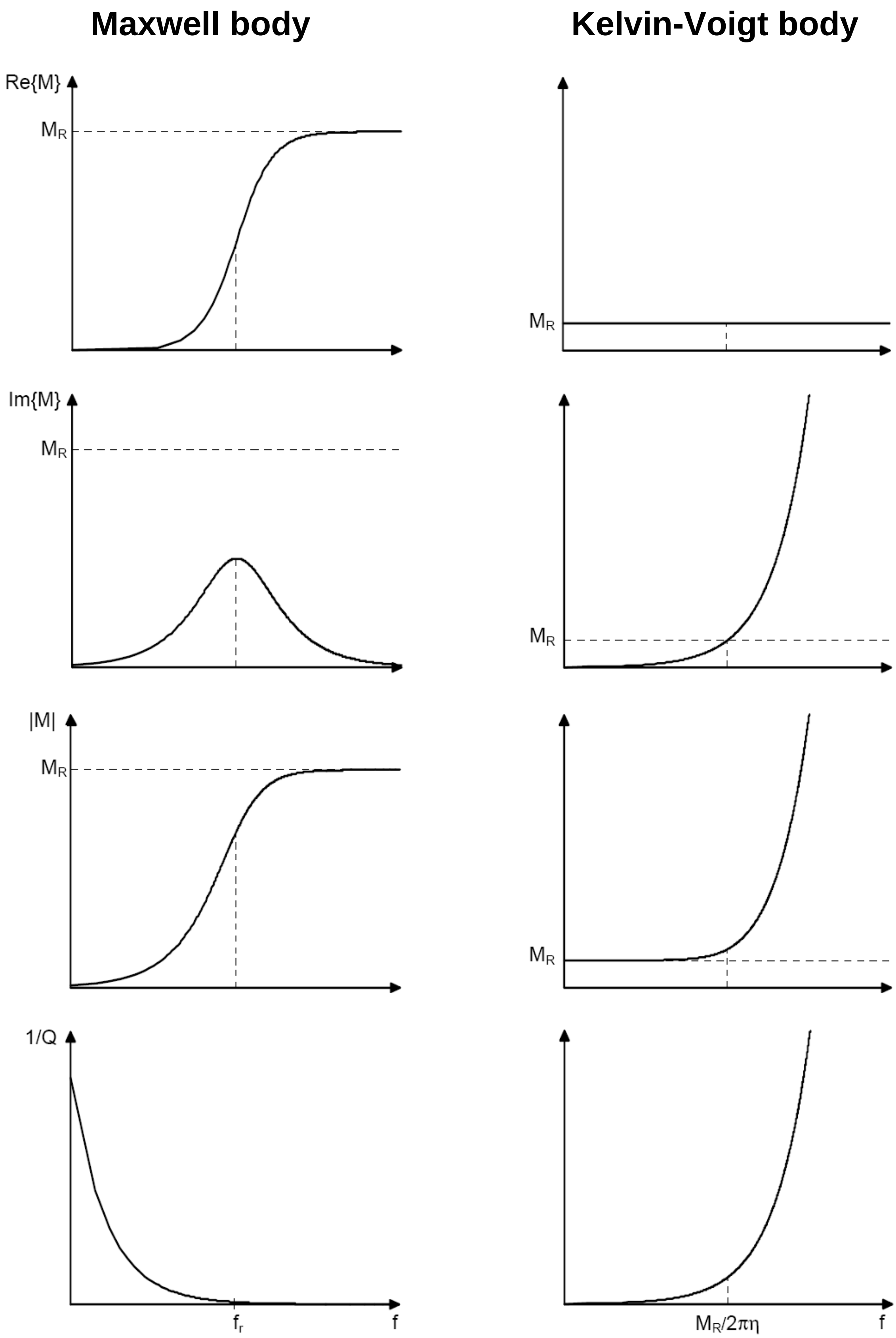

Maxwell body
Kelvin-Voigt body
Re{M}
M_R
Im{M}
M_R
|M|
M_R
1/Q
f_r
f
M_R
M_R
M_R
M_R/2πη
f

Fig. 11

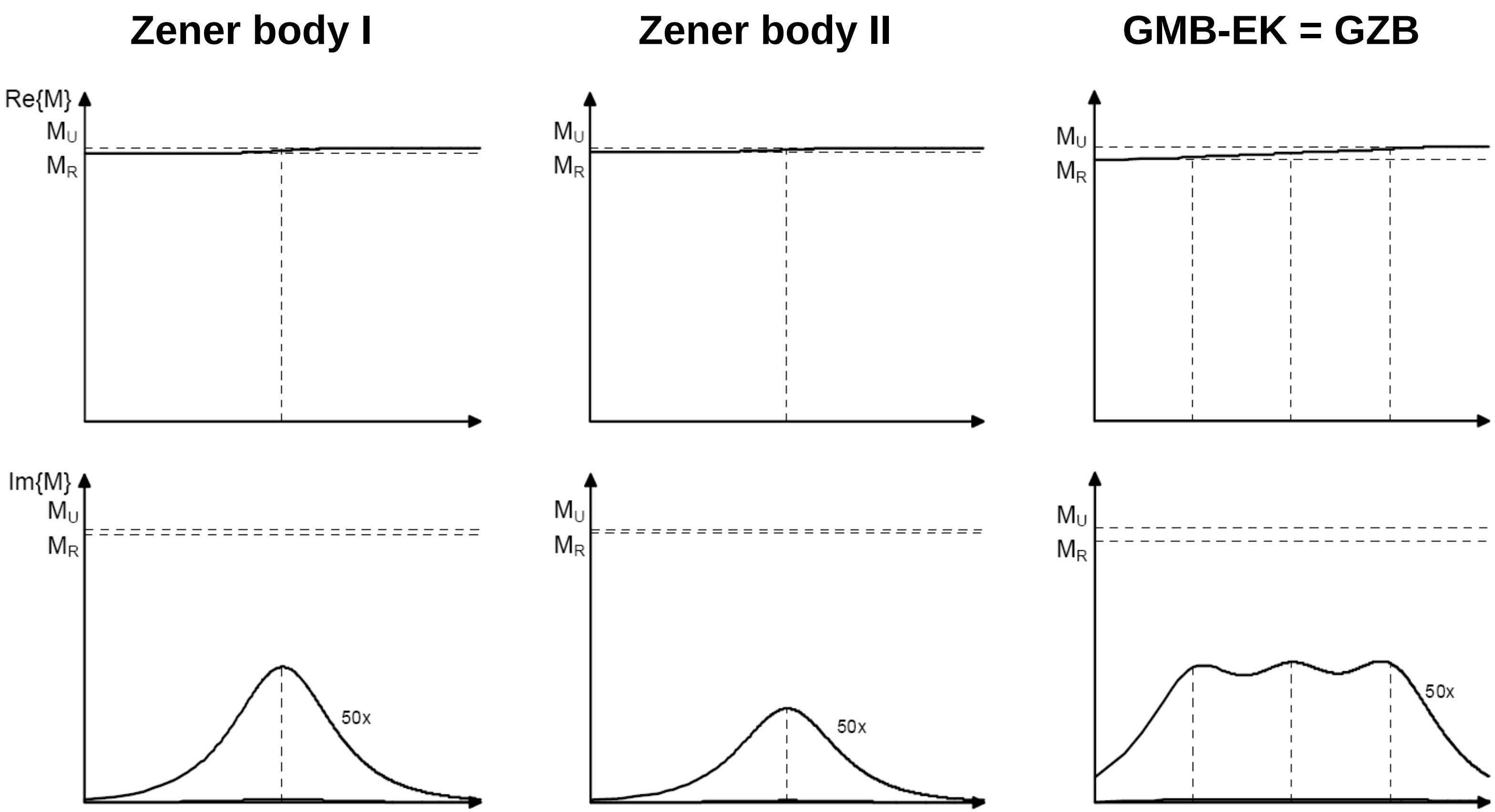
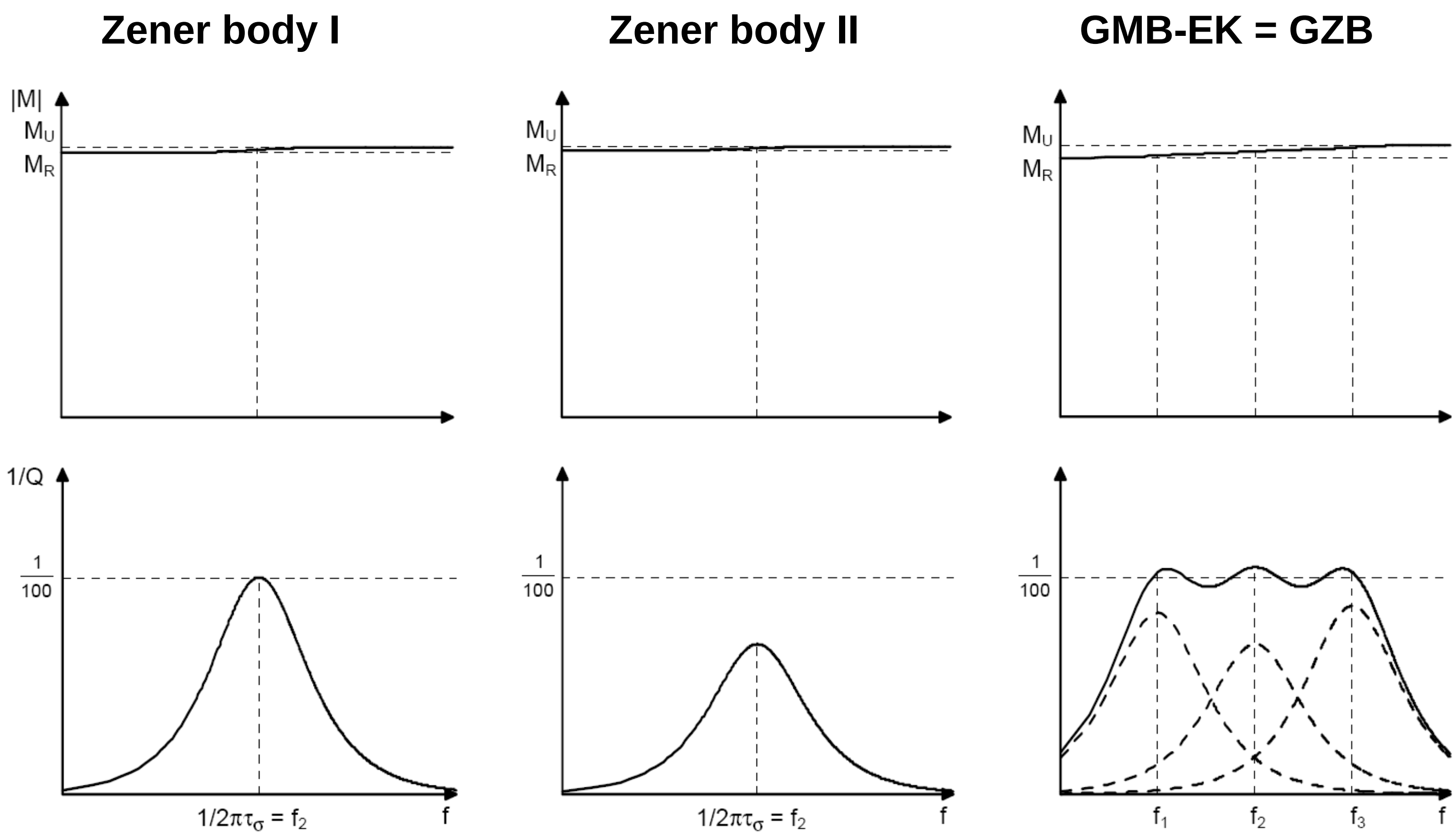


Fig. 12. Viscoelastic modulus M and attenuation $1/Q$ as functions of frequency f. The horizontal axis (frequency) is logarithmic, the vertical axis decadic. *Left column*: First, eq. (129) is solved for an anelastic coefficient Y of Zener body (GMB-EK with $n=1$ gives Zener body) for $Q=100$ at relaxation frequency $f_2=1\,\mathrm{Hz}$. Then viscoelastic modulus M and attenuation $1/Q$ as functions of frequency f are calculated using eqs. (127) and (128), respectively. (Continued on the next page.)



Right column: First, eq. (129) is solved (using the least square method) for anelastic coefficients Y_l $(l = 1,2,3)$ of GMB-EK with $n = 3$ and $Q = 100$ at relaxation frequencies $f_1 = 0.1\,\mathrm{Hz}$, $f_2 = 1\,\mathrm{Hz}$, and $f_3 = 10\,\mathrm{Hz}$. Then viscoelastic modulus M and attenuation $1/Q$ as functions of frequency f are calculated using eqs. (127) and (128), respectively. *Middle column*: Viscoelastic modulus M and attenuation $1/Q$ as functions of frequency f calculated only for anelastic coefficient Y_2 , that is, for a single relaxation mechanism at $f_2 = 1\,\mathrm{Hz}$ using eqs. (127) and (128), respectively. Anelastic coefficient Y_2 is the same as in the case of GMB-EK with $n = 3$, that is, 3 relaxation mechanisms.

Equations for the 1D Case – A Summary

We can now generalize 1D equations of motion for the **smoothly heterogeneous viscoelastic medium**. The considered formulations are:

displacement-stress formulation

$$\rho \ddot{d} = \sigma_{,x} + f \quad , \quad \sigma = M_U \cdot \varepsilon - \sum_{l=1}^{n} M_U Y_l^M \zeta_l \tag{133}$$

$$\dot{\zeta}_l + \omega_l \zeta_l = \omega_l \varepsilon \quad ; \quad l = 1,...,n \tag{134}$$

displacement-velocity-stress formulation

$$\rho \dot{\upsilon} = \sigma_{,x} + f \quad , \quad \upsilon = \dot{d} \quad , \quad \sigma = M_U \cdot \varepsilon - \sum_{l=1}^{n} M_U Y_l^M \zeta_l \tag{135}$$

$$\dot{\zeta}_l + \omega_l \zeta_l = \omega_l \varepsilon \quad ; \quad l = 1,...,n \tag{136}$$

velocity-stress formulation

$$\rho \dot{\upsilon} = \sigma_{,x} + f \quad , \quad \dot{\sigma} = M_U \cdot \dot{\varepsilon} - \sum_{l=1}^{n} M_U Y_l^M \xi_l \tag{137}$$

$$\dot{\xi}_l + \omega_l \xi_l = \omega_l \dot{\varepsilon} \quad ; \quad l = 1,...,n \tag{138}$$

displacement formulation

$$\rho \ddot{d} = \left(M_U \cdot \varepsilon \right)_{,x} - \sum_{l=1}^{n} \left(M_U Y_l^M \zeta_l \right)_{,x} + f \tag{139}$$

$$\dot{\zeta}_l + \omega_l \zeta_l = \omega_l \varepsilon \quad ; \quad l = 1,...,n \tag{140}$$

In equations (133) - (140), M and ε stand for

$\lambda + 2\mu$ and $d_{,x}$ in the case of the P wave

or

2μ and $\frac{1}{2} d_{,x}$ in the case of the S wave.

Note that in the above equations we used the upper index *M* for the anelastic functions to indicate that the anelastic function corresponds to modulus *M*.

References

Ben-Menahem, A. and S. J. Singh, 1981. Seismic waves and sources. *Springer*.

Blanch, J. O., J. O. A. Robertsson, and W. W. Symes, 1995. Modeling of a constant Q: Methodology and algorithm for an efficient and optimally inexpensive viscoelastic technique. *Geophysics* 60, 176-184.

Carcione, J. M., 2001. Wave fields in real media: wave propagation in anisotropic, anelastic and porous media. *Pergamon*.

Carcione, J. M., D. Kosloff, and R. Kosloff, 1988a. Wave propagation simulation in a linear viscoacoustic medium. *Geophys. J.* 93, 393-407.

Carcione, J. M., D. Kosloff, and R. Kosloff, 1988b. Wave propagation simulation in a linear viscoelastic medium. *Geophys. J.* 95, 597-611.

Dahlen, F. A., and J. Tromp, 1998. Theoretical Global Seismology. *Princeton University Press.*

Day, S. M. and J. B. Minster, 1984. Numerical simulation of wavefields using a Padé approximant method. *Geophys. J. Roy. astr. Soc.* 78, 105-118.

Emmerich, H. and M. Korn, 1987. Incorporation of attenuation into time-domain computations of seismic wave fields. *Geophysics* 52, 1252-1264.

Fäh, D., 1992. A hybrid technique for the estimation of strong ground motion in sedimentary basins. Diss. ETH Nr. 9767, *Swiss Federal Institute of Technology*, Zurich.

Graves, R. W. and S. M. Day, 2003. Stability and accuracy analysis of coarse-grain viscoelastic simulations. *Bull. Seism. Soc. Am.* 93, 283-300.

Kristek, J. and P. Moczo, 2003. Seismic wave propagation in viscoelastic media with material discontinuities – a 3D 4th-order staggered-grid finite-difference modeling. *Bull. Seism. Soc. Am.* 93, 2273-2280.

Liu, H.-P., D. L. Anderson, and H. Kanamori, 1976. Velocity dispersion due to anelasticity; implications for seismology and mantle composition. *Geophys. J. R. astr. Soc.* 47, 41-58.

Moczo, P. and P.-Y. Bard, 1993. Wave diffraction, amplification and differential motion near strong lateral discontinuities. *Bull. Seism. Soc. Am.* 83, 85-106.

Moczo, P. and J. Kristek, 2005. On the rheological models in the time-domain methods for seismic wave propagation. *Geophys. Res. Lett.* 32, L01306, doi: 10.1029/2004GL021598.

Moczo, P., J. Kristek, and L. Halada, 2004. The finite-difference method for seismologists. An introduction. *Comenius University*, Bratislava.

Ranalli, G., 1995. Rheology of the Earth. *Chapman & Hall*.

Robertsson, J. O. A., J. O. Blanch, and W. W. Symes, 1994. Viscoelastic finite-difference modeling. *Geophysics* 59, 1444-1456.

Xu, T. and G. A. McMechan, 1995. Composite memory variables for viscoelastic synthetic seismograms. *Geophys. J. Int.* 121, 634-639.