Chapter 2. Review of Matrix Algebra

Matrices and Indicial Notation

$$\mathbf{a} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{ij}$$

← i is the row
j is the column

Summation Convention

2 repeated indices implies summation (Einstein's notation)
(3 repeated indices means nothing)

Ex: $$a_{ii} = \sum_{i=1}^{3} a_{ii} \qquad\qquad a_{ij}b_{jk} = \sum_{j=1}^{3} a_{ij}b_{jk}$$

Free indices

$$\hat{x}_i = a_{im}x_m$$

this implies three equations (i is called a free index)

$$\hat{x}_1 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3$$

$$\hat{x}_2 = a_{21}x_1 + a_{22}x_2 + a_{23}x_3$$

$$\hat{x}_3 = a_{31}x_1 + a_{32}x_2 + a_{33}x_3$$

Symmetric Matrices

$$a_{ij} = a_{ji}$$

Antisymetric Matrices

$$a_{ij} = -a_{ij}, \qquad i \neq j$$

Identity Matrix

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

or we can use Kronecker's delta

$$\delta_{ij} = 1 \qquad for \;\; i = j$$
$$= 0 \qquad for \;\; i \neq j$$

Matrix Multiplication

$$\mathbf{c} = \mathbf{ab} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \sum_{j=1}^{3} a_{ij} b_{jk} = a_{ij} b_{jk} = c_{ik}$$

$$= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} & a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} & a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33} \end{bmatrix}$$

$$\mathbf{c} = \mathbf{aI} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} = \sum_{j=1}^{3} a_{ij} \delta_{jk} = a_{ij} \delta_{jk} = c_{ik}$$

Transpose of a Matrix

$$\mathbf{a}^T = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

Swap rows and columns

$$x^T = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}^T = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}$$

Some interesting things

If a is symmetric

$$\mathbf{a}^T = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}^T = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} = \mathbf{a}$$

If a is antisymmetric

$$\frac{\mathbf{a}+\mathbf{a}^T}{2} = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} = \mathbf{a}I$$

Orthogonal Matrix

Each set of columns form an orthogonal set of unit vectors

Given:
$$\mathbf{a} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$\mathbf{a}$ is an orthogonal matrix if:

$$\begin{Bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{Bmatrix} \text{ Is orthogonal to } \begin{Bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{Bmatrix} \text{ Is orthogonal to } \begin{Bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{Bmatrix}$$

Example:

$$\mathbf{a} = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

$\mathbf{a}$ is an orthogonal matrix since

$$\begin{Bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{Bmatrix} \bullet \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} = 0 \qquad \begin{Bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{Bmatrix} \bullet \begin{Bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{Bmatrix} = 0 \qquad \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} \bullet \begin{Bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{Bmatrix} = 0$$

An orthogonal matrix also has the property

$$\mathbf{a}^{-1} = \mathbf{a}^T$$

Determinant of a square matrix

The determinant of a square matrix is a scalar quantity which summarizes the tensorial property in the form of a multilinear functional

Determinant of matrix products

$$\det(\mathbf{a}\cdot\mathbf{b}) = \det\mathbf{a}\det\mathbf{b}$$

Eigenvalues and Eigenvectors (in relation to stress tensors)

There exists a nonzero vector $\mathbf{x}$ such that the linear transformation $\boldsymbol{\sigma}\cdot\mathbf{x}$ is a multiple of $\mathbf{x}$

$$\boldsymbol{\sigma}\cdot\mathbf{x} = \lambda\mathbf{x}$$

where the eigenvalues λ_i define the three principle values of stress and the eigenvectors $\mathbf{x}_i$ span the triad of the principle directions.

This is equivalent to stating:

$$(\boldsymbol{\sigma} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$$

which for a nontrivial solution to exist:

$$\det(\boldsymbol{\sigma} - \lambda\mathbf{I}) = \mathbf{0}$$

which gives the characteristic polynomial

$$p(\lambda) = \det(\boldsymbol{\sigma} - \lambda\mathbf{I})$$

Note all aigenvalues are real as long as $\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$ is symmetric, which is the case for nonpolar materials because of the conjugate shear stresses $\sigma_{ij} = \sigma_{ji}$.