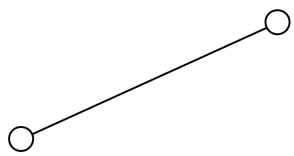
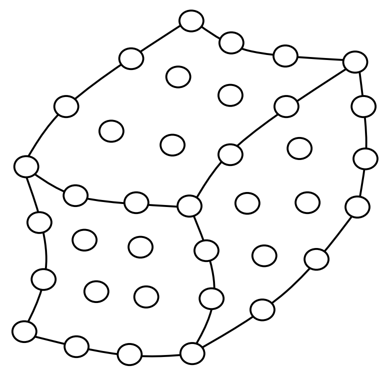
Chapter 3. FEA Preliminaries

The DSM is used by all major commercial finite element codes
Works the same for every element



Truss element
2 nodes, 4 DOFs



Tricubic brick element
64 nodes, 192 DOFs

Therefore, we will use the truss element to teach the DSM

Direct Stiffness Method (DSM) steps

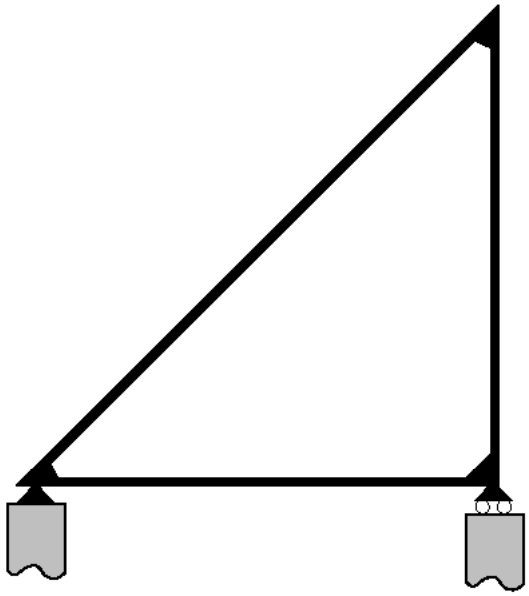
Breakdown
- Disconnection
- Localization
- Member (Element) Formation

Assembly & Solution
- Globalization
- Merge
- Application of BCs
- Solution
- Recovery of Derived Quantities

What we are going to solve….

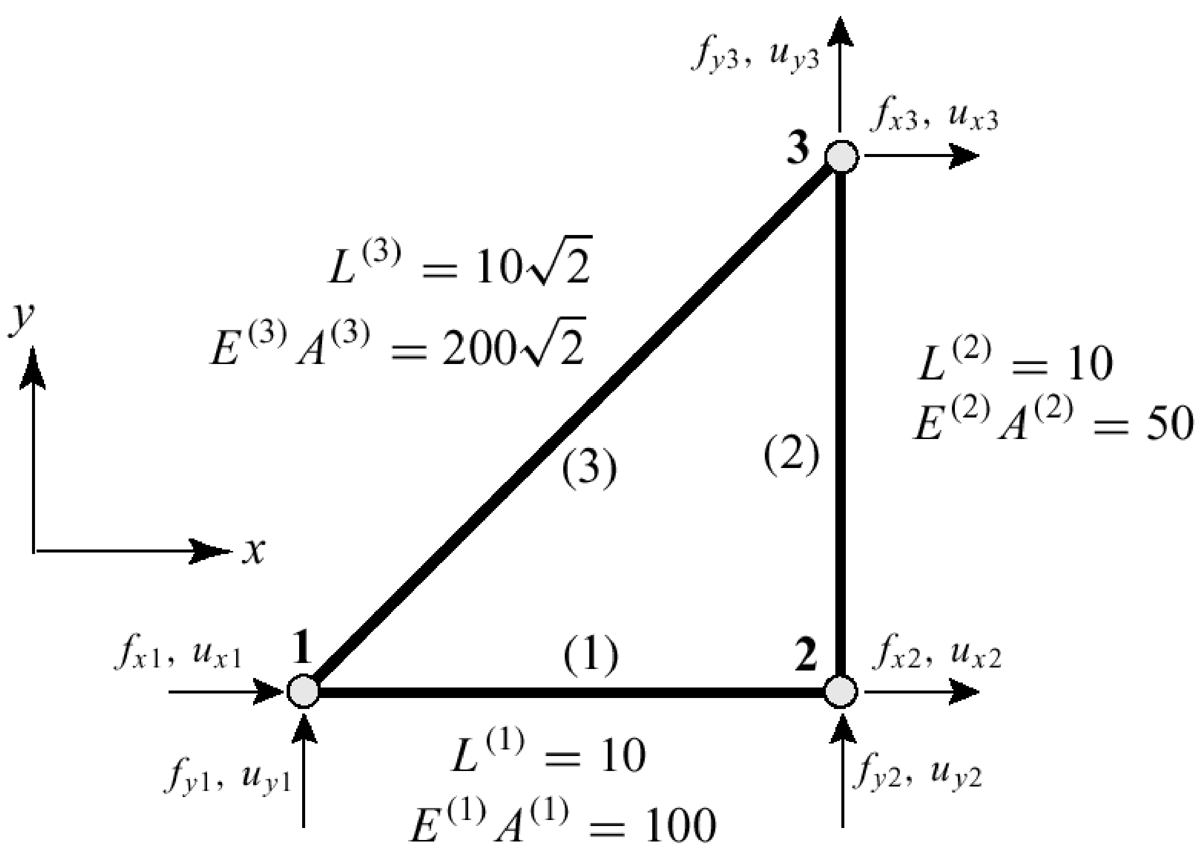
The Example Truss: Physical Model

(loads not shown)



The Example Truss: FEM Model

(nodes, elements and DOFs)



Master (Global) Stiffness Equations

Nodal forces

$$\mathbf{f} = \begin{Bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \end{Bmatrix}$$

Nodal displacements

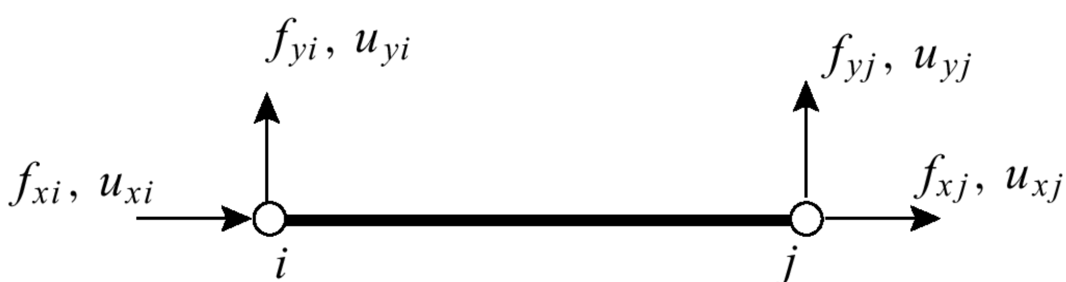
$$\mathbf{u} = \begin{Bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{Bmatrix}$$

Linear structure $\mathbf{f} = \mathbf{K}\mathbf{u}$

$$\begin{Bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \end{Bmatrix} = \begin{bmatrix} K_{x1x1} & K_{x1y1} & K_{x1x2} & K_{x1y2} & K_{x1x3} & K_{x1y3} \\ & K_{y1y1} & K_{y1x2} & K_{y1y2} & K_{y1x3} & K_{y1y3} \\ & & K_{x2x2} & K_{x2y2} & K_{x2x3} & K_{x2y3} \\ & & & K_{y2y2} & K_{y2x3} & K_{y2y3} \\ & sym & & & K_{x3x3} & K_{x3y3} \\ & & & & & K_{y3y3} \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{Bmatrix}$$

Global stiffness matrix

Elemental (Local) Stiffness Equations



Nodal forces

$$\bar{\mathbf{f}} = \begin{Bmatrix} \bar{f}_{xi} \\ \bar{f}_{yi} \\ \bar{f}_{xj} \\ \bar{f}_{yj} \end{Bmatrix}$$

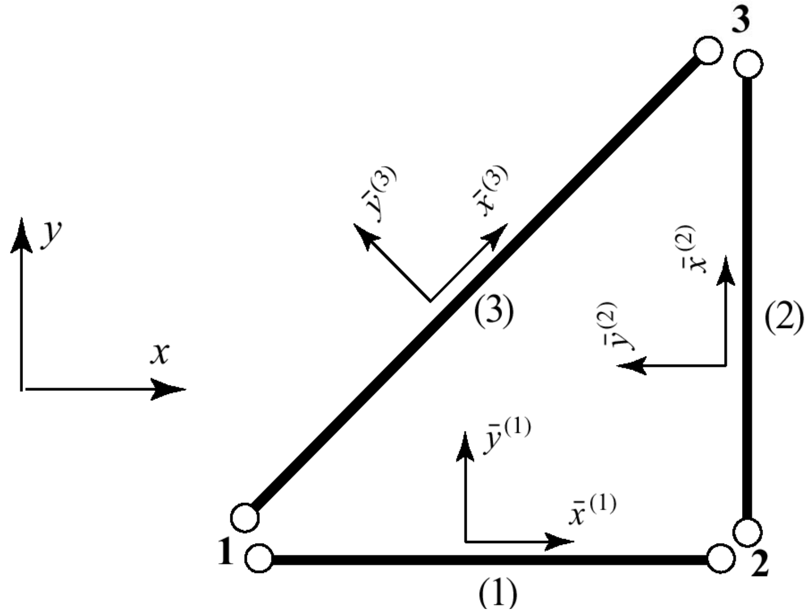
Nodal displacements

$$\bar{\mathbf{u}} = \begin{Bmatrix} \bar{u}_{xi} \\ \bar{u}_{yi} \\ \bar{u}_{xj} \\ \bar{u}_{yj} \end{Bmatrix}$$

Linear structure $\bar{\mathbf{f}} = \bar{\mathbf{K}}\bar{\mathbf{u}}$

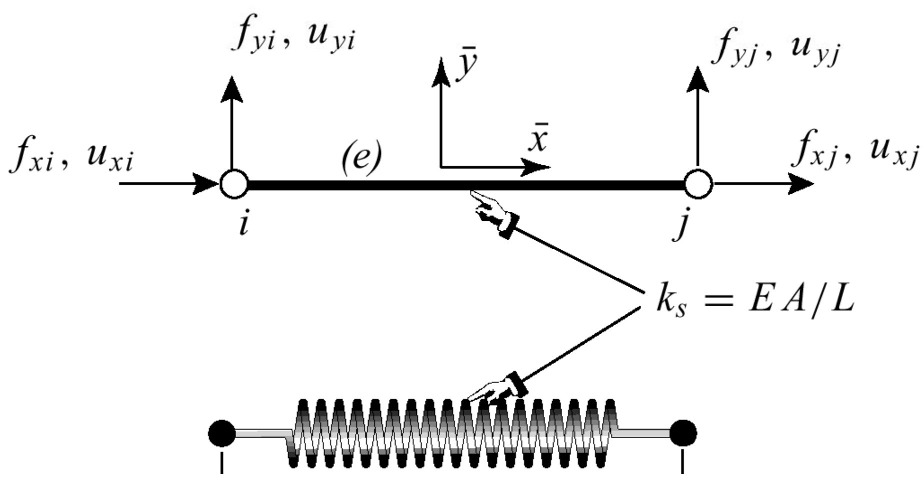
$$\begin{Bmatrix} \bar{f}_{xi} \\ \bar{f}_{yi} \\ \bar{f}_{xj} \\ \bar{f}_{yj} \end{Bmatrix} = \begin{bmatrix} \bar{K}_{xixi} & \bar{K}_{xiyi} & \bar{K}_{xixj} & \bar{K}_{xiyj} \\ & \bar{K}_{yiyi} & \bar{K}_{yixj} & \bar{K}_{yiyj} \\ & & \bar{K}_{xjxj} & \bar{K}_{xjyj} \\ sym & & & \bar{K}_{yjyj} \end{bmatrix} \begin{Bmatrix} \bar{u}_{xi} \\ \bar{u}_{yi} \\ \bar{u}_{xj} \\ \bar{u}_{yj} \end{Bmatrix}$$

Step 1&2: Disconnection & Localization



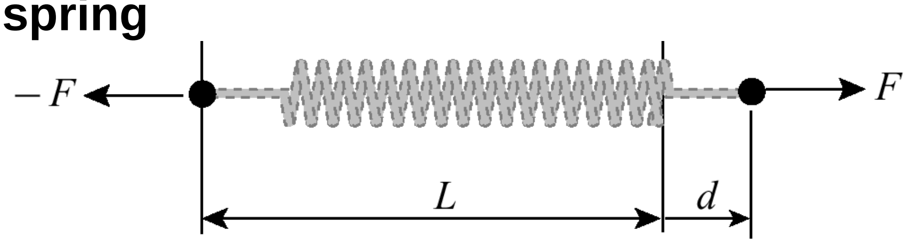
Step 3: Element Formulation

The 2-node Truss (Bar) Element



Truss-(Spring) Element Formulation (MofM Approach)

Consider a spring



$F = k_s d$ $\qquad F = f_2 = -f_1$ $\qquad d = u_2 - u_1$

We have two equations with two unknowns u_1 and u_2

$$f_1 = k_s(u_1 - u_2)$$

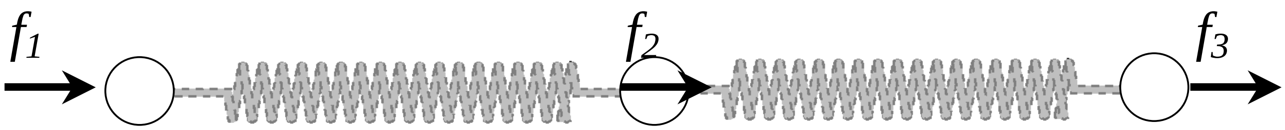
$$f_2 = k_s(u_2 - u_1)$$

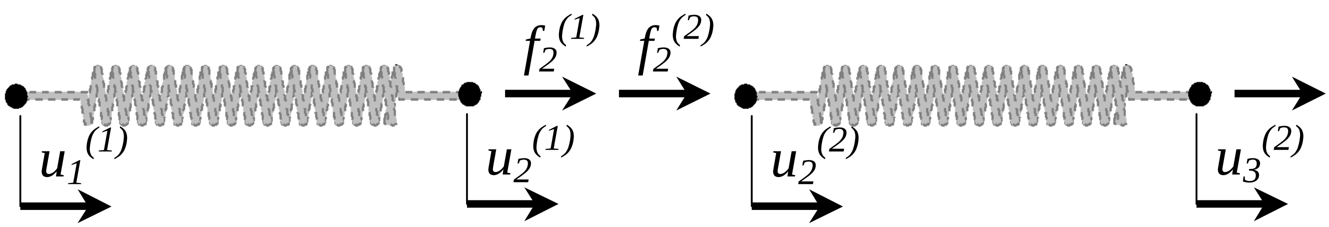
We write this in matrix form as:

$$\begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} = \begin{bmatrix} k_s & -k_s \\ -k_s & k_s \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

Force vector — Stiffness matrix — Displacement vector

Consider two springs in series





For element 1 we have

$$\begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} \end{Bmatrix} = \begin{bmatrix} k_{s1} & -k_{s1} \\ -k_{s1} & k_{s1} \end{bmatrix} \begin{Bmatrix} u_1^{(1)} \\ u_2^{(1)} \end{Bmatrix}$$

For element 2 we have

$$\begin{Bmatrix} f_2^{(2)} \\ f_3^{(2)} \end{Bmatrix} = \begin{bmatrix} k_{s2} & -k_{s2} \\ -k_{s2} & k_{s2} \end{bmatrix} \begin{Bmatrix} u_2^{(2)} \\ u_3^{(2)} \end{Bmatrix}$$

Now join the two elements

equilibrium requires that $f_2^{(1)} + f_2^{(2)} = f_2$

compatibility requires that $u_2^{(1)} = u_2^{(2)} = u_2$

We really have the following 3 equations with 3 unknowns

$$f_1 = k_{s1}(u_1 - u_2)$$

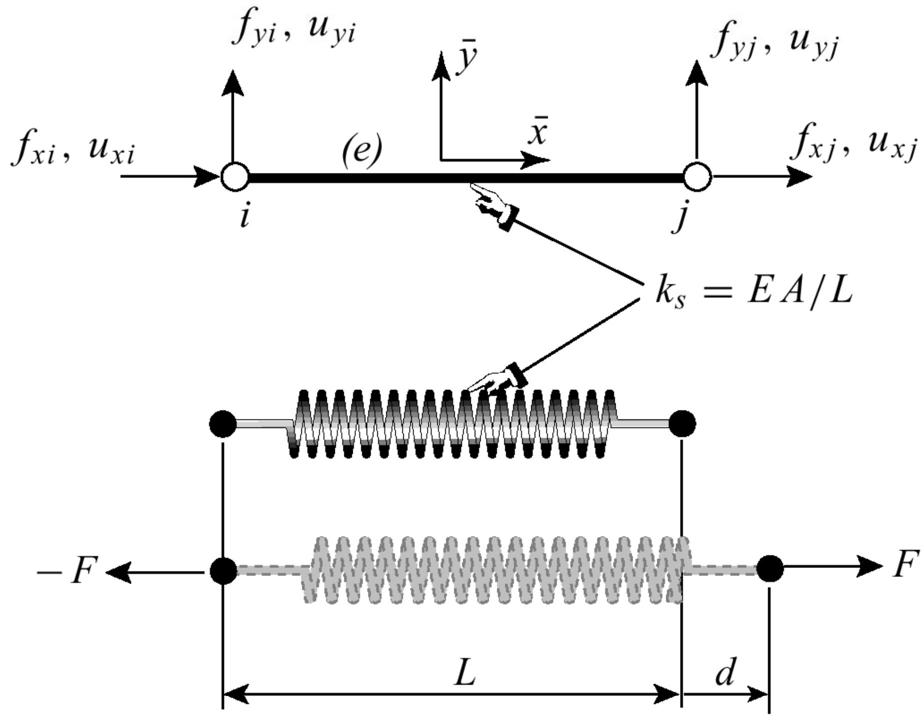
$$f_2 = k_{s1}(u_2 - u_1) + k_{s2}(u_2 - u_3)$$

$$f_3 = k_{s2}(u_3 - u_2)$$

In matrix from:

$$\begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix} = \begin{bmatrix} k_{s1} & -k_{s1} & 0 \\ -k_{s1} & k_{s1}+k_{s2} & -k_{s2} \\ 0 & -k_{s2} & k_{s2} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

Extension to truss element



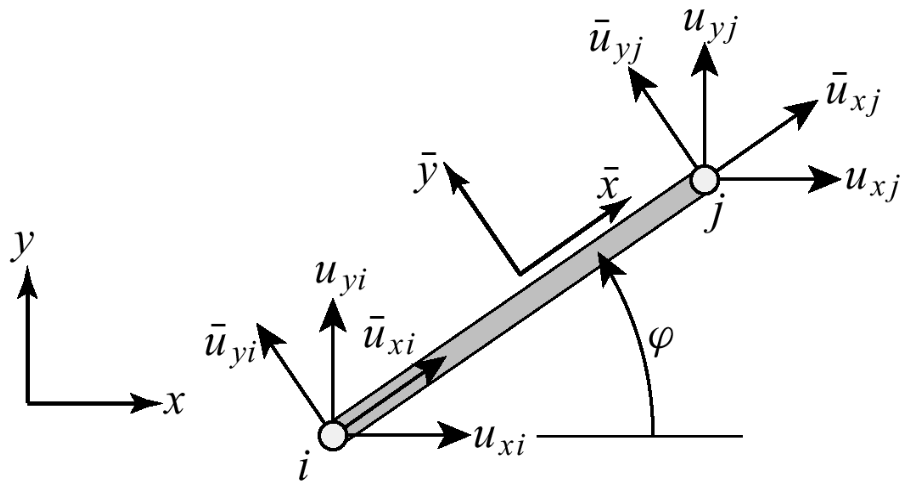
$$F = k_s d = \frac{EA}{L} d, \qquad F = \bar{f}_{xj} = -\bar{f}_{xi}, \qquad d = \bar{u}_{xj} - \bar{u}_{xi}$$

$$\begin{Bmatrix} \bar{f}_{xi} \\ \bar{f}_{yi} \\ \bar{f}_{xj} \\ \bar{f}_{yj} \end{Bmatrix} = \frac{EA}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \bar{u}_{xi} \\ \bar{u}_{yi} \\ \bar{u}_{xj} \\ \bar{u}_{yj} \end{Bmatrix} \qquad \bar{\mathbf{f}} = \bar{\mathbf{K}}\bar{\mathbf{u}}$$

Elemental stiffness matrix
in local coordinates

Step 4: Globalization

Displacement transformation



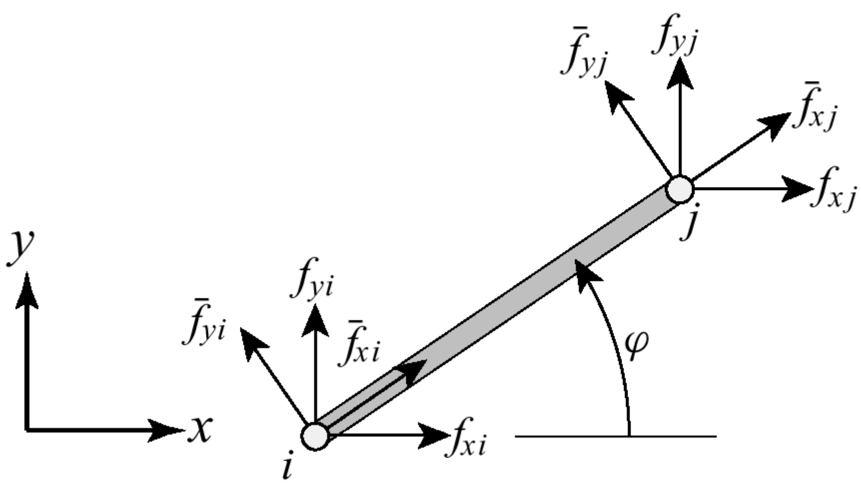
Local

$$\begin{Bmatrix} \bar{u}_{xi} \\ \bar{u}_{yi} \\ \bar{u}_{xj} \\ \bar{u}_{yj} \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} u_{xi} \\ u_{yi} \\ u_{xj} \\ u_{yj} \end{Bmatrix}$$

Global

$$\bar{\mathbf{u}} = \mathbf{T}\mathbf{u}$$

Force transformation



Global

$$\begin{Bmatrix} f_{xi} \\ f_{yi} \\ f_{xj} \\ f_{yj} \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} \bar{f}_{xi} \\ \bar{f}_{yi} \\ \bar{f}_{xj} \\ \bar{f}_{yj} \end{Bmatrix}$$

Local

$$\mathbf{f} = \mathbf{T}^T \vec{\mathbf{f}}$$

Element stiffness matrix transformation

$$\bar{\mathbf{f}}_e = \bar{\mathbf{K}}_e \bar{\mathbf{u}}_e$$

$$\bar{\mathbf{u}}_e = \mathbf{T}\mathbf{u}_e \qquad\qquad \mathbf{f}_e = \mathbf{T}^T \bar{\mathbf{f}}_e$$

$$\mathbf{K}_e = \mathbf{T}^T \bar{\mathbf{K}}_e \mathbf{T}$$

$$\mathbf{K}_e = \frac{EA}{L}\begin{bmatrix} c^2 & sc & -c^2 & -sc \\ sc & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -sc & -s^2 & sc & s^2 \end{bmatrix}$$

Recall the truss problem that we are trying to solve

The elemental stiffness matrices are for

Element 1

$$\begin{Bmatrix} f_{x1}^{(1)} \\ f_{y1}^{(1)} \\ f_{x2}^{(1)} \\ f_{y2}^{(1)} \end{Bmatrix} = 10\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} u_{x1}^{(1)} \\ u_{y1}^{(1)} \\ u_{x2}^{(1)} \\ u_{y2}^{(1)} \end{Bmatrix}$$

Element 2

$$\begin{Bmatrix} f_{x2}^{(2)} \\ f_{y2}^{(2)} \\ f_{x3}^{(2)} \\ f_{y3}^{(2)} \end{Bmatrix} = 5\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}\begin{Bmatrix} u_{x2}^{(2)} \\ u_{y2}^{(2)} \\ u_{x3}^{(2)} \\ u_{y3}^{(2)} \end{Bmatrix}$$

Element 3

$$\begin{Bmatrix} f_{x1}^{(3)} \\ f_{y1}^{(3)} \\ f_{x3}^{(3)} \\ f_{y3}^{(3)} \end{Bmatrix} = 20\begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \end{bmatrix}\begin{Bmatrix} u_{x1}^{(3)} \\ u_{y1}^{(3)} \\ u_{x3}^{(3)} \\ u_{y3}^{(3)} \end{Bmatrix}$$

Step 5: Assembly

1. Compatibility

   The joint displacement of all the members meeting at a joint ***must be the same***

2. Equilibrium

   The sum of all the forces exerted by all the members that meet at a joint ***must balance*** the external forces acting on that joint

Expanded element stiffness matrices

Element 1

$$\begin{Bmatrix} f_{x1}^{(1)} \\ f_{y1}^{(1)} \\ f_{x2}^{(1)} \\ f_{y2}^{(1)} \\ f_{x3}^{(1)} \\ f_{y3}^{(1)} \end{Bmatrix} = 10 \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_{x1}^{(1)} \\ u_{y1}^{(1)} \\ u_{x2}^{(1)} \\ u_{y2}^{(1)} \\ u_{x3}^{(1)} \\ u_{y3}^{(1)} \end{Bmatrix}$$

Element 2

$$\begin{Bmatrix} f_{x1}^{(2)} \\ f_{y1}^{(2)} \\ f_{x2}^{(2)} \\ f_{y2}^{(2)} \\ f_{x3}^{(2)} \\ f_{y3}^{(2)} \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 5 \end{bmatrix} \begin{Bmatrix} u_{x1}^{(2)} \\ u_{y1}^{(2)} \\ u_{x2}^{(2)} \\ u_{y2}^{(2)} \\ u_{x3}^{(2)} \\ u_{y3}^{(2)} \end{Bmatrix}$$

Element 3

$$\begin{Bmatrix} f_{x1}^{(3)} \\ f_{y1}^{(3)} \\ f_{x2}^{(3)} \\ f_{y2}^{(3)} \\ f_{x3}^{(3)} \\ f_{y3}^{(3)} \end{Bmatrix} = \begin{bmatrix} 10 & 10 & 0 & 0 & -10 & -10 \\ 10 & 10 & 0 & 0 & -10 & -10 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -10 & -10 & 0 & 0 & 10 & 10 \\ -10 & -10 & 0 & 0 & 10 & 10 \end{bmatrix} \begin{Bmatrix} u_{x1}^{(3)} \\ u_{y1}^{(3)} \\ u_{x2}^{(3)} \\ u_{y2}^{(3)} \\ u_{x3}^{(3)} \\ u_{y3}^{(3)} \end{Bmatrix}$$

Reconnecting members

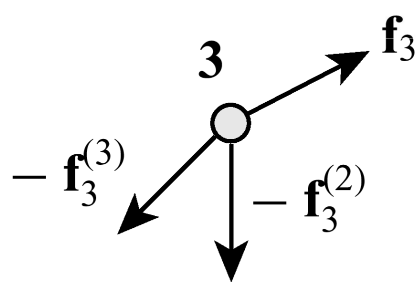
1. Enforce compatibility rule

$$\begin{Bmatrix} u_{x1}^{(1)} \\ u_{y1}^{(1)} \\ u_{x2}^{(1)} \\ u_{y2}^{(1)} \\ u_{x3}^{(1)} \\ u_{y3}^{(1)} \end{Bmatrix} = \begin{Bmatrix} u_{x1}^{(2)} \\ u_{y1}^{(2)} \\ u_{x2}^{(2)} \\ u_{y2}^{(2)} \\ u_{x3}^{(2)} \\ u_{y3}^{(2)} \end{Bmatrix} = \begin{Bmatrix} u_{x1}^{(3)} \\ u_{y1}^{(3)} \\ u_{x2}^{(3)} \\ u_{y2}^{(3)} \\ u_{x3}^{(3)} \\ u_{y3}^{(3)} \end{Bmatrix} = \begin{Bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{Bmatrix}$$

Drop the element index from the nodal displacements

2. Enforce equilibrium rule
   (sum of the forces at each node)

$$\begin{Bmatrix} f_{x1}^{(1)} \\ f_{y1}^{(1)} \\ f_{x2}^{(1)} \\ f_{y2}^{(1)} \\ f_{x3}^{(1)} \\ f_{y3}^{(1)} \end{Bmatrix} + \begin{Bmatrix} f_{x1}^{(2)} \\ f_{y1}^{(2)} \\ f_{x2}^{(2)} \\ f_{y2}^{(2)} \\ f_{x3}^{(2)} \\ f_{y3}^{(2)} \end{Bmatrix} + \begin{Bmatrix} f_{x1}^{(3)} \\ f_{y1}^{(3)} \\ f_{x2}^{(3)} \\ f_{y2}^{(3)} \\ f_{x3}^{(3)} \\ f_{y3}^{(3)} \end{Bmatrix} = \begin{Bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \end{Bmatrix}$$



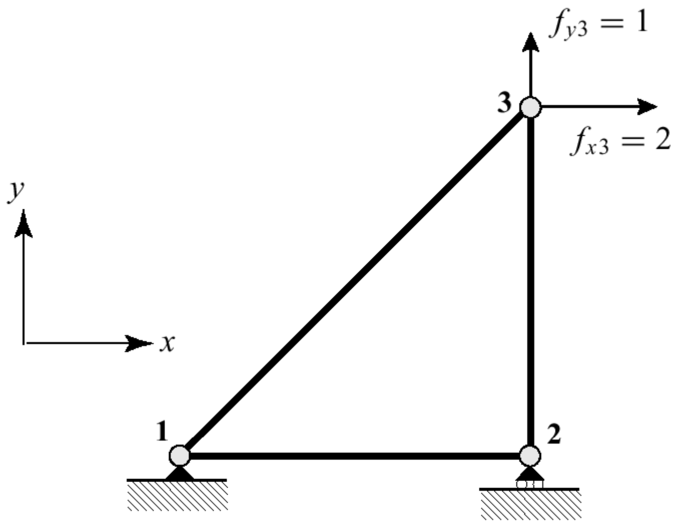
Form master (global) stiffness equations

$$\mathbf{f} = \mathbf{f}^{(1)} + \mathbf{f}^{(2)} + \mathbf{f}^{(3)} = (\mathbf{K}^{(1)} + \mathbf{K}^{(2)} + \mathbf{K}^{(2)})\mathbf{u} = \mathbf{K}\mathbf{u}$$

Thus,

$$\begin{Bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \end{Bmatrix} = \begin{bmatrix} 20 & 10 & -10 & 0 & -10 & -10 \\ 10 & 10 & 0 & 0 & -10 & -10 \\ -10 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & -5 \\ -10 & -10 & 0 & 0 & 10 & 10 \\ -10 & -10 & 0 & -5 & 10 & 15 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{Bmatrix}$$

Step 6: Apply Boundary Conditions



Applying Displacement Boundary Conditions:

$$u_{x1} = u_{y1} = u_{y2} = 0$$

Remove the rows and columns associated with zero displacements

$$\begin{Bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \end{Bmatrix} = \begin{bmatrix} 20 & 10 & -10 & 0 & -10 & -10 \\ 10 & 10 & 0 & 0 & -10 & -10 \\ -10 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & -5 \\ -10 & -10 & 0 & 0 & 10 & 10 \\ -10 & -10 & 0 & -5 & 10 & 15 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_{x2} \\ 0 \\ u_{x3} \\ u_{y3} \end{Bmatrix}$$

Reduced master (global) stiffness equations and include force boundary conditions

$$\begin{Bmatrix} f_{x2} \\ f_{x3} \\ f_{y3} \end{Bmatrix} = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 10 \\ 0 & 10 & 15 \end{bmatrix} \begin{Bmatrix} u_{x2} \\ u_{x3} \\ u_{y3} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 2 \\ 1 \end{Bmatrix}$$

$$\hat{\mathbf{K}}\hat{\mathbf{u}} = \hat{\mathbf{f}}$$

Solve this using Gauss elimination

(or simply invert **K** using Mathematica $\hat{\mathbf{K}}^{-1}\hat{\mathbf{f}} = \hat{\mathbf{u}}$)

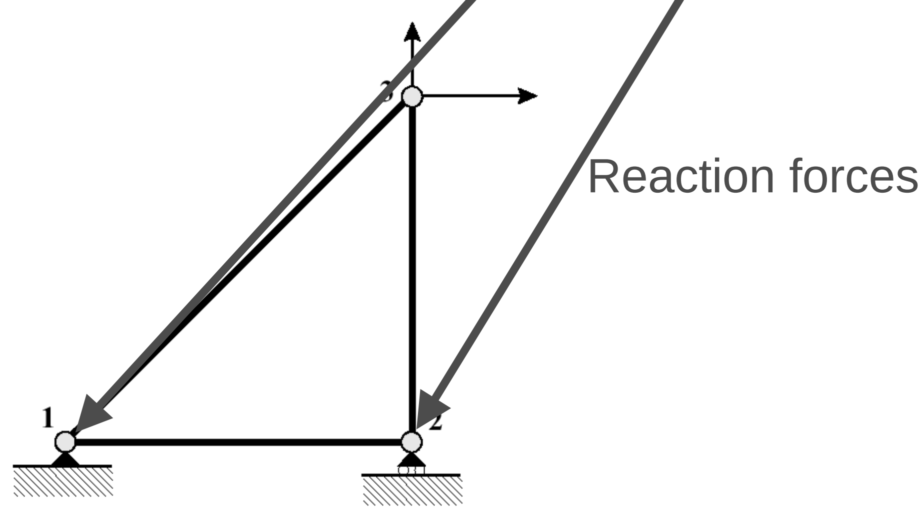
Step 7: Solution

$$\begin{Bmatrix} u_{x2} \\ u_{x3} \\ u_{y3} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0.4 \\ -0.2 \end{Bmatrix} \qquad \mathbf{u} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.4 \\ -0.2 \end{Bmatrix}$$

Step 8: Post-processing

Compute nodal forces and reactions

$$\mathbf{Ku} = \mathbf{f} = \begin{bmatrix} 20 & 10 & -10 & 0 & -10 & -10 \\ 10 & 10 & 0 & 0 & -10 & -10 \\ -10 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & -5 \\ -10 & -10 & 0 & 0 & 10 & 10 \\ -10 & -10 & 0 & -5 & 10 & 15 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.4 \\ -0.2 \end{Bmatrix} = \begin{Bmatrix} -2 \\ -2 \\ 0 \\ 1 \\ 2 \\ 1 \end{Bmatrix}$$



Compute member internal forces (axial forces in trusses)

1. Extract the $\mathbf{u}^{(e)}$ from $\mathbf{u}$
2. Transform to local (element) displacements

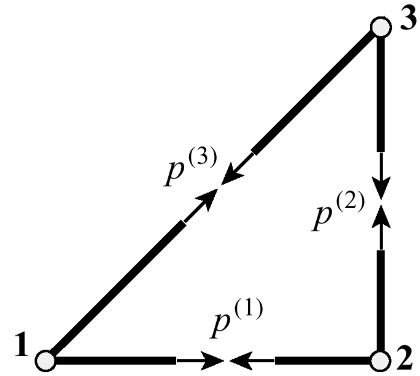
$$\overline{\mathbf{u}}^{(e)} = \mathbf{T}\mathbf{u}^{(e)}$$

3. Compute elongation

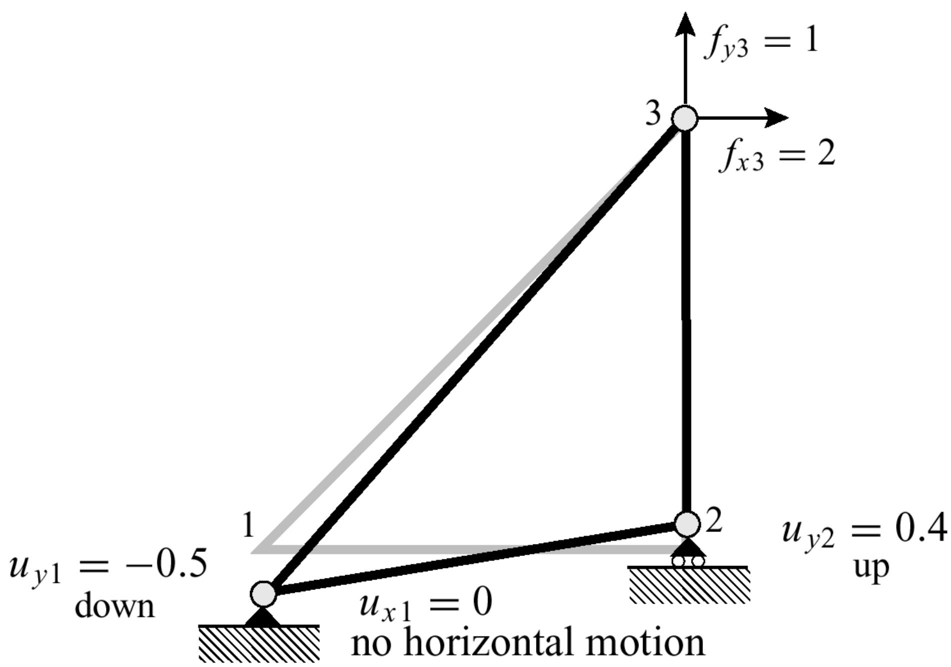
$$d^{(e)} = \overline{\mathbf{u}}_{xj}^{(e)} - \overline{\mathbf{u}}_{xi}^{(e)}$$

4. Compute axial force

$$p^{(e)} = \frac{E^{(e)} A^{(e)}}{L^{(e)}} d^{(e)}$$



Prescribed initial displacements



Recall we had

$$\begin{bmatrix} 20 & 10 & -10 & 0 & -10 & -10 \\ 10 & 10 & 0 & 0 & -10 & -10 \\ -10 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & -5 \\ -10 & -10 & 0 & 0 & 10 & 10 \\ -10 & -10 & 0 & -5 & 10 & 15 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{Bmatrix} = \begin{Bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \end{Bmatrix}$$

and now we also have

$$u_{x1} = 0, \quad u_{y1} = -0.5 \quad u_{y2} = 0.4$$

Put the boundary conditions in:

$$\begin{bmatrix} 20 & 10 & -10 & 0 & -10 & -10 \\ 10 & 10 & 0 & 0 & -10 & -10 \\ -10 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & -5 \\ -10 & -10 & 0 & 0 & 10 & 10 \\ -10 & -10 & 0 & -5 & 10 & 15 \end{bmatrix} \begin{Bmatrix} 0 \\ -0.5 \\ u_{x2} \\ 0.4 \\ u_{x3} \\ u_{y3} \end{Bmatrix} = \begin{Bmatrix} f_{x1} \\ f_{y1} \\ 0 \\ f_{y2} \\ 2 \\ 1 \end{Bmatrix}$$

and remove rows 1,2,and 4 (keep the columns for now)

$$\begin{bmatrix} -10 & 0 & 10 & 0 & 0 & 0 \\ -10 & -10 & 0 & 0 & 10 & 10 \\ -10 & -10 & 0 & -5 & 10 & 15 \end{bmatrix} \begin{Bmatrix} 0 \\ -0.5 \\ u_{x2} \\ 0.4 \\ u_{x3} \\ u_{y3} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 2 \\ 1 \end{Bmatrix}$$

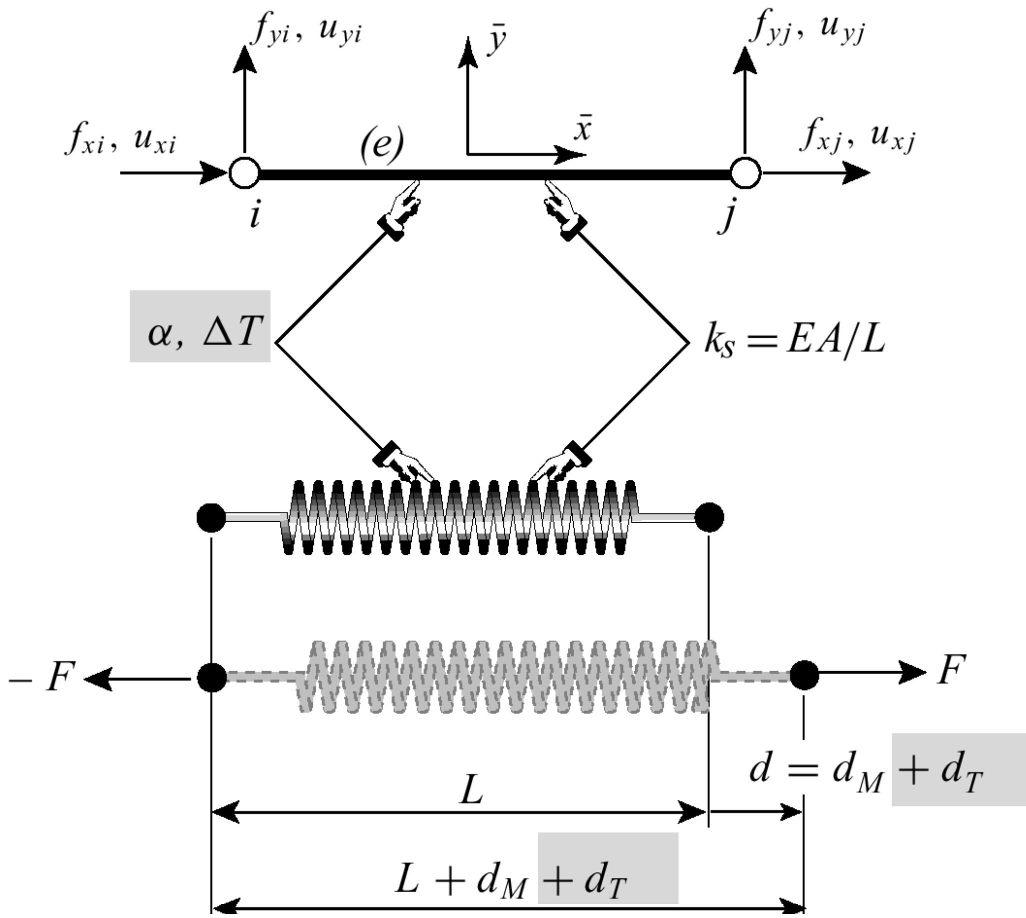
Use the effect of the known displacements to change the force vector and allow us to delete the extra columns

$$\begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 10 \\ 0 & 10 & 15 \end{bmatrix} \begin{Bmatrix} u_{x2} \\ u_{x3} \\ u_{y3} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 2-5 \\ 1-3 \end{Bmatrix}$$

Solution:

$$\begin{Bmatrix} u_{x2} \\ u_{x3} \\ u_{y3} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -0.5 \\ .2 \end{Bmatrix} \qquad \mathbf{u} = \begin{Bmatrix} 0 \\ -.05 \\ 0 \\ 0.4 \\ -0.5 \\ 0.2 \end{Bmatrix}$$

Thermal effects



$$e = d / L \qquad d = \bar{u}_{xj} - \bar{u}_{xi}$$

$$d = \frac{\bar{u}_{xj} - \bar{u}_{xi}}{L} = \frac{\sigma}{E} + \alpha\Delta T$$

$$\frac{EA}{L}\left(\bar{u}_{xj} - \bar{u}_{xi}\right) = A\sigma + EA\alpha\Delta T = p^m + p^t = F$$

$$F = \frac{EA}{L}\begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} \bar{u}_{xi} \\ \bar{u}_{yi} \\ \bar{u}_{xj} \\ \bar{u}_{yj} \end{bmatrix}$$

$$\begin{Bmatrix} \bar{f}_{xi} \\ \bar{f}_{yi} \\ \bar{f}_{xj} \\ \bar{f}_{yj} \end{Bmatrix} = \begin{Bmatrix} \bar{f}_{xi}{}^{m} \\ \bar{f}_{yi}{}^{m} \\ \bar{f}_{xj}{}^{m} \\ \bar{f}_{yj}{}^{m} \end{Bmatrix} + \begin{Bmatrix} \bar{f}_{xi}{}^{t} \\ \bar{f}_{yi}{}^{t} \\ \bar{f}_{xj}{}^{t} \\ \bar{f}_{yj}{}^{t} \end{Bmatrix} == \begin{Bmatrix} \bar{f}_{xi}{}^{m} \\ \bar{f}_{yi}{}^{m} \\ \bar{f}_{xj}{}^{m} \\ \bar{f}_{yj}{}^{m} \end{Bmatrix} + EA\alpha\Delta T \begin{Bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{Bmatrix} =$$

$$= \frac{EA}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} \bar{u}_{xi} \\ \bar{u}_{yi} \\ \bar{u}_{xj} \\ \bar{u}_{yj} \end{Bmatrix}$$

Matrix Form of elemental stiffness equations

$$\overline{\mathbf{K}}\overline{\mathbf{u}} - \overline{\mathbf{f}}^{t} = \overline{\mathbf{f}}^{m}$$

$$\overline{\mathbf{K}}\overline{\mathbf{u}} = \overline{\mathbf{f}}^{m} + \overline{\mathbf{f}}^{t} = \overline{\mathbf{f}}$$

Effective force vector

The same approach is used to treat other initial forces
(thermal, prestress, moisture, lack of fit, residual stress)