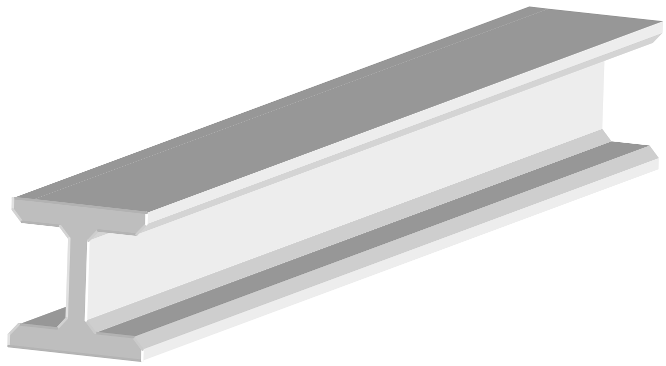
Chapter 4. Beam Elements

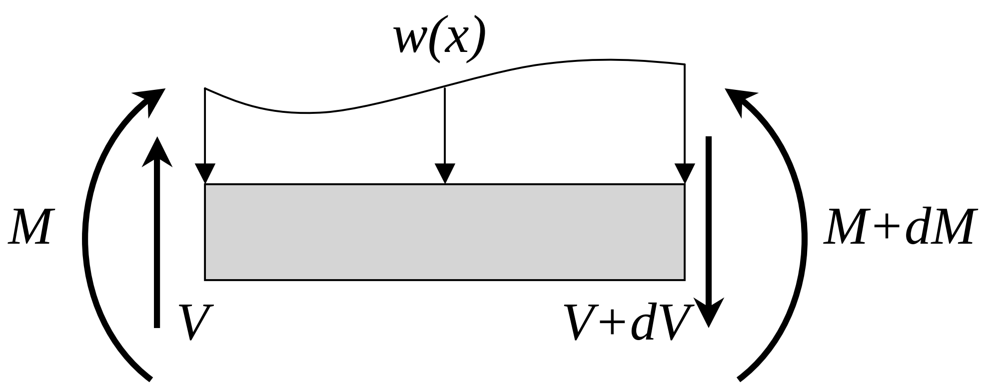


These have stiffnesses in two directions

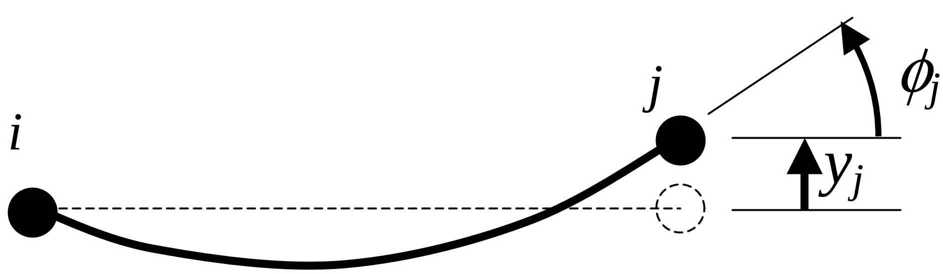
Axially (just like the truss member)

Transversely (bending moments, shears)

Simplified 1-D beam



Only subject to transverse loading (Ignore extension)



Each node has two DOF

Displacement in the transverse direction *y*

Rotation ϕ

Step 2. Select a displacement function

Let's assume that the transverse displacement is of the form:

$$\hat{v} = a_1\hat{x}^3 + a_2\hat{x}^2 + a_3\hat{x} + a_4$$

This is reasonable because we have 4 dof per element (2 per node)

Now solve for the a's

$$\hat{v}(0) = \hat{d}_{y1} = a_4$$

$$\frac{d\hat{v}(0)}{d\hat{x}} = \hat{\phi}_1 = a_3$$

$$\hat{v}(L) = \hat{d}_{y2} = a_1L^3 + a_2L^2 + a_3L + a_4$$

$$\frac{d\hat{v}(L)}{d\hat{x}} = \hat{\phi}_2 = 3a_1L^2 + 2a_2L + a_3$$

Solving for the a's gives:

$$\hat{v} = \left[\frac{2}{L^3}(\hat{d}_{y1} - \hat{d}_{y2}) + \frac{1}{L^2}(\hat{\phi}_1 + \hat{\phi}_2)\right]\hat{x}^3 +$$

$$\left[\frac{-3}{L^2}(\hat{d}_{y1} - \hat{d}_{y2}) - \frac{1}{L}(\hat{\phi}_1 + \hat{\phi}_2)\right]\hat{x}^2 + \hat{\phi}\hat{x} + \hat{d}_{y1}$$

Which we can write in matrix form as:

$$\hat{v} = \mathbf{N}\hat{\mathbf{d}} = \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix} \begin{Bmatrix} \hat{d}_{y1} \\ \hat{\phi}_1 \\ \hat{d}_{y2} \\ \hat{\phi}_2 \end{Bmatrix}$$

The shape functions are:

$$N_1 = \frac{1}{L^3}(2\hat{x}^3 - 3\hat{x}^2L + L^3) \qquad N_2 = \frac{1}{L^3}(\hat{x}^3L - 2\hat{x}^2L^2 + \hat{x}L^3)$$

$$N_3 = \frac{1}{L^3}(-2\hat{x}^3 + 3\hat{x}^2L) \qquad N_4 = \frac{1}{L^3}(\hat{x}^3L - \hat{x}^2L^2)$$

Step 3. Define strain/displacement and stress/strain relations

Assume the following relation:

$$\varepsilon_x(\hat{x}, \hat{y}) = -\hat{y}\frac{d^2\hat{v}}{d\hat{x}^2}$$

Then

$$\varepsilon_x = -\hat{y}\begin{bmatrix} \frac{12\hat{x}-6L}{L^3} & \frac{6\hat{x}L-4L^2}{L^3} & \frac{-12\hat{x}+6L}{L^3} & \frac{6\hat{x}L-2L^2}{L^3} \end{bmatrix}\begin{Bmatrix} \hat{d}_{y1} \\ \hat{\phi}_1 \\ \hat{d}_{y2} \\ \hat{\phi}_2 \end{Bmatrix} = -\hat{y}\,\mathbf{B}\,\hat{\mathbf{d}}$$

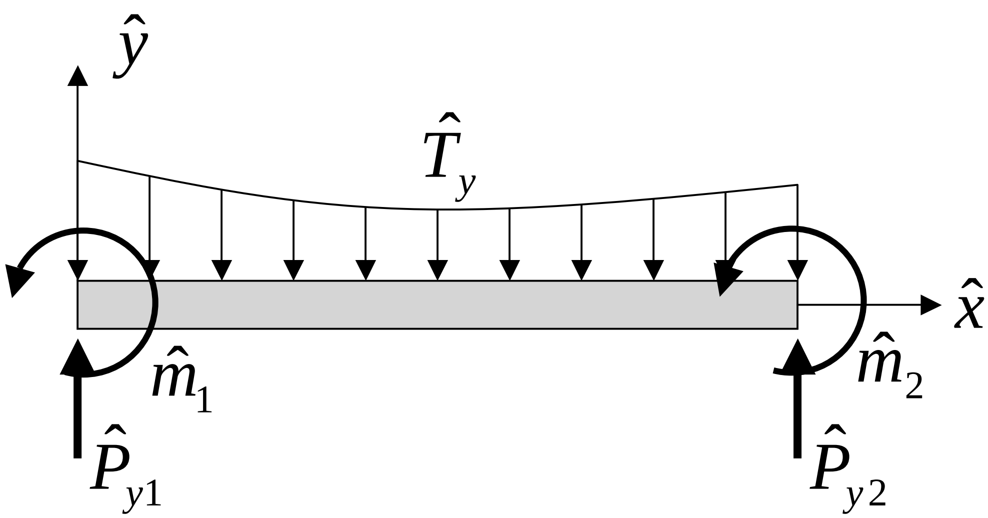
The stress/strain relation is expressed as:

$$\sigma_{\mathbf{x}} = \mathbf{D}\boldsymbol{\varepsilon}_{\mathbf{x}} \qquad \text{where} \qquad \mathbf{D} = E$$

Thus: $\sigma_x = -\hat{y}\mathbf{D}\mathbf{B}\hat{\mathbf{d}}$

Step 4. Derive the element stiffness matrix and equations

Potential Energy Method Derivation



The total PE

$$\pi_p = U + \Omega$$

Internal Strain Energy

$$U = \iiint_V \frac{1}{2}\sigma\varepsilon\, dV$$

Potential energy of the external forces

$$\Omega = -\iint_S \hat{T}_y \hat{v} ds - \hat{P}_{y1}\hat{d}_{y1} - \hat{P}_{y2}\hat{d}_{y2} - \hat{m}_1\hat{\phi}_1 - \hat{m}_2\hat{\phi}_2$$

$$\pi_p = \int_L \iint_A \frac{1}{2}\sigma_x \varepsilon_x dA dx - \int_L \hat{T}_y \hat{v} b dx - \sum_{i=1}^{2}\left(\hat{P}_{yi}\hat{d}_{yi} + \hat{m}_i\hat{\phi}_i\right)$$

Which we can rewrite in matrix form as:

$$\pi_p = \int_L \iint_A \frac{1}{2}\boldsymbol{\sigma}_{\mathbf{x}}{}^T \boldsymbol{\varepsilon}_{\mathbf{x}} dA dx - \int_L b\hat{T}_y \hat{\mathbf{v}}^T\, dx - \hat{\mathbf{d}}^T \hat{\mathbf{P}}$$

Making the appropriate substitutions (rewrite in terms of **d**)

$$\pi_p = \int_L \iint_A \frac{1}{2}\hat{y}^2\, \hat{\mathbf{d}}^T \mathbf{B}^T \mathbf{D} \mathbf{B} \hat{\mathbf{d}}\; dA d\hat{x} - \int_L b\hat{T}_y \hat{\mathbf{d}}^T \mathbf{N}^T\, d\hat{x} - \hat{\mathbf{d}}^T \hat{\mathbf{P}}$$

$$= \int_L \frac{EI}{2}\hat{\mathbf{d}}^T \mathbf{B}^T \mathbf{B} \hat{\mathbf{d}}\, d\hat{x} - \int_L w \hat{\mathbf{d}}^T \mathbf{N}^T\, d\hat{x} - \hat{\mathbf{d}}^T \hat{\mathbf{P}}$$

Where we have used the definition of moment of inertia

$$I = \iint_A \hat{y}^2\; dA$$

We want to find the minimum potential energy

So we differentiate w.r.t. **d** and set = 0

$$\int_L EI\, \mathbf{B}^T \mathbf{B}\; d\hat{x}\; \hat{\mathbf{d}} - \int_L w \mathbf{N}^T\; d\hat{x} - \hat{\mathbf{P}} = 0$$

And Voila!! $$\hat{\mathbf{K}}\;\hat{\mathbf{d}} - \hat{\mathbf{f}} = 0$$

$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ & 4L^2 & -6L & 2L^2 \\ & & 12 & -6L \\ sym & & & 4L^2 \end{bmatrix}\begin{Bmatrix} \hat{d}_{y1} \\ \hat{\phi}_1 \\ \hat{d}_{y2} \\ \hat{\phi}_2 \end{Bmatrix} = \begin{Bmatrix} \hat{f}_{y1} \\ \hat{f}_{\phi 1} \\ \hat{f}_{y2} \\ \hat{f}_{\phi 2} \end{Bmatrix}$$

What about extension along the length?

2-D Beams

Recall from truss element

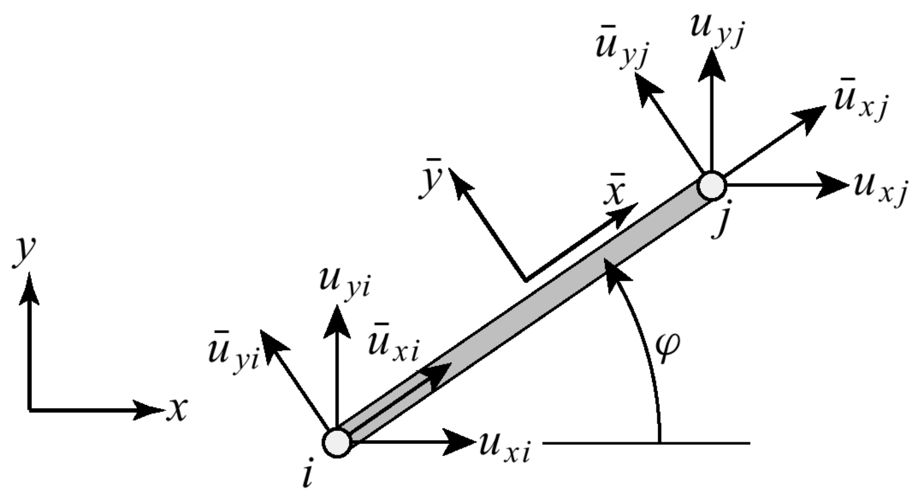
$$\begin{Bmatrix} \hat{f}_{x1} \\ \hat{f}_{x2} \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \hat{d}_{x1} \\ \hat{d}_{x2} \end{Bmatrix}$$

Then we simply insert the additional terms

$$\begin{bmatrix} \frac{AE}{L} & 0 & 0 & -\frac{AE}{L} & 0 & 0 \\ & 12\frac{EI}{L^3} & 6\frac{EI}{L^2} & 0 & -12\frac{EI}{L^3} & 6\frac{EI}{L^2} \\ & & 4\frac{EI}{L} & 0 & -6\frac{EI}{L^2} & 2\frac{EI}{L} \\ & & & \frac{AE}{L} & 0 & 0 \\ & & & & 12\frac{EI}{L^3} & -6\frac{EI}{L^2} \\ sym & & & & & 4\frac{EI}{L} \end{bmatrix}\begin{Bmatrix} \hat{d}_{x1} \\ \hat{d}_{y1} \\ \hat{\phi}_1 \\ \hat{d}_{x2} \\ \hat{d}_{y2} \\ \hat{\phi}_2 \end{Bmatrix} = \begin{Bmatrix} \hat{f}_{x1} \\ \hat{f}_{y1} \\ \hat{f}_{\phi 1} \\ \hat{f}_{x2} \\ \hat{f}_{y2} \\ \hat{f}_{\phi 2} \end{Bmatrix}$$

But we still have a beam that is lying on the x axis
Let's rotate this in space

Local to Global Transformation



$$\begin{Bmatrix} \hat{d}_{x1} \\ \hat{d}_{y1} \\ \hat{\phi}_1 \\ \hat{d}_{x2} \\ \hat{d}_{y2} \\ \hat{\phi}_2 \end{Bmatrix} = \begin{bmatrix} c & s & 0 & 0 & 0 & 0 \\ -s & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & s & 0 \\ 0 & 0 & 0 & -s & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} d_{x1} \\ d_{y1} \\ \phi_1 \\ d_{x2} \\ d_{y2} \\ \phi_2 \end{Bmatrix}$$

And following the same procedure as before

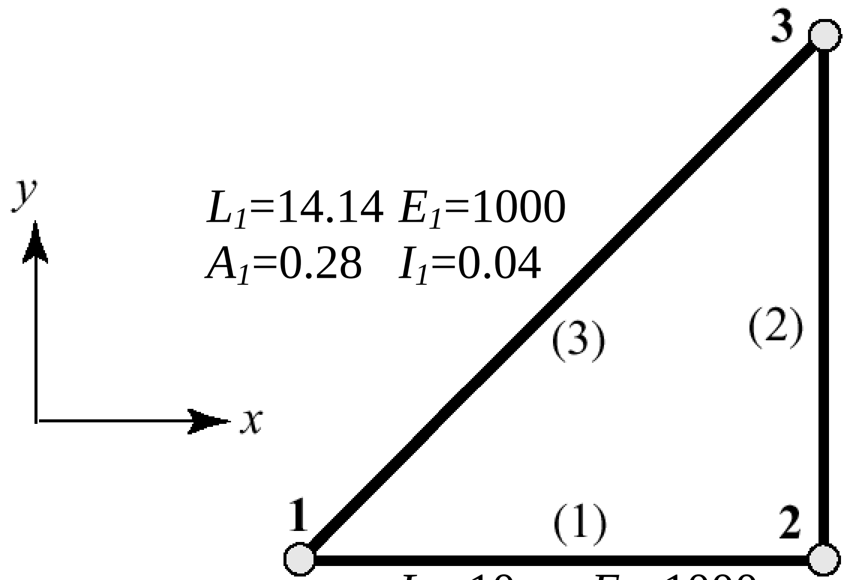
$$\hat{\mathbf{d}} = \mathbf{T}\mathbf{d}$$

$$\mathbf{f} = \mathbf{T}^T\hat{\mathbf{f}}$$

$$\mathbf{K} = \mathbf{T}^T\hat{\mathbf{K}}\mathbf{T}$$

Plane Frame Example

Let's use the same example as before, except we are going to weld all the joints together rather than pin them



The elemental stiffness matrices are for

Element 1

$$\begin{Bmatrix} f_{x1}^{(1)} \\ f_{y1}^{(1)} \\ f_{\phi 2}^{(1)} \\ f_{x2}^{(1)} \\ f_{y2}^{(1)} \\ f_{\phi 2}^{(1)} \end{Bmatrix} = \begin{bmatrix} 10 & 0 & 0 & -10 & 0 & 0 \\ & 0.06 & 0.3 & 0 & -0.06 & 0.3 \\ & & 2 & 0 & -0.3 & 1 \\ & & & 10 & 0 & 0 \\ & & & & 0.06 & -0.3 \\ sym & & & & & 2 \end{bmatrix} \begin{Bmatrix} d_{x1}^{(1)} \\ d_{y1}^{(1)} \\ \phi_1^{(1)} \\ u_{x2}^{(1)} \\ u_{y2}^{(1)} \\ \phi_2^{(1)} \end{Bmatrix}$$

Element 2

$$\begin{Bmatrix} f_{x2}^{(2)} \\ f_{y2}^{(2)} \\ f_{\phi 2}^{(2)} \\ f_{x3}^{(2)} \\ f_{y3}^{(2)} \\ f_{\phi 3}^{(2)} \end{Bmatrix} = \begin{bmatrix} 0.015 & 0 & -0.075 & -0.015 & 0 & -0.075 \\ & 5 & 0 & 0 & -5 & 0 \\ & & 0.5 & 0.075 & 0 & 0.25 \\ & & & 0.15 & 0 & 0.075 \\ & & & & 5 & 0 \\ sym & & & & & 0.5 \end{bmatrix} \begin{Bmatrix} d_{x2}^{(2)} \\ d_{y2}^{(2)} \\ \phi_2^{(2)} \\ u_{x3}^{(2)} \\ u_{y3}^{(2)} \\ \phi_3^{(2)} \end{Bmatrix}$$

Element 3

$$\begin{Bmatrix} f_{x1}^{(3)} \\ f_{y1}^{(3)} \\ f_{\phi 1}^{(3)} \\ f_{x3}^{(3)} \\ f_{y3}^{(3)} \\ f_{\phi 3}^{(3)} \end{Bmatrix} = \begin{bmatrix} 10.08 & 9.91 & -0.84 & -10.08 & -9.91 & -0.85 \\ & 10.08 & 0.84 & -9.91 & -10.08 & 0.85 \\ & & 11.31 & 0.84 & -0.84 & 5.65 \\ & & & 10.08 & 9.91 & 0.85 \\ & & & & 10.08 & -0.85 \\ sym & & & & & 11.31 \end{bmatrix} \begin{Bmatrix} d_{x1}^{(3)} \\ d_{y1}^{(3)} \\ \phi_1^{(3)} \\ u_{x3}^{(3)} \\ u_{y3}^{(3)} \\ \phi_3^{(3)} \end{Bmatrix}$$

Step 5: Assembly

Rules:

1. Compatibility

    The joint displacement of all the members meeting at a joint ***must be the same***

2. Equilibrium

    The sum of all the forces exerted by all the members that meet at a joint ***must balance*** the external forces acting on that joint

These are applied by expanding the element stiffness matrices by adding the missing rows and columns for each matrix

Reconnecting members

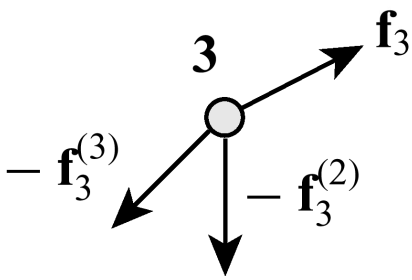
1. Enforce compatibility rule

$$\begin{Bmatrix} d_{x1}^{(1)} \\ d_{y1}^{(1)} \\ \phi_1^{(1)} \\ d_{x2}^{(1)} \\ d_{y2}^{(1)} \\ \phi_2^{(1)} \\ d_{x3}^{(1)} \\ d_{y3}^{(1)} \\ \phi_3^{(1)} \end{Bmatrix} = \begin{Bmatrix} d_{x1}^{(2)} \\ d_{y1}^{(2)} \\ \phi_1^{(2)} \\ d_{x2}^{(2)} \\ d_{y2}^{(2)} \\ \phi_2^{(2)} \\ d_{x3}^{(2)} \\ d_{y3}^{(2)} \\ \phi_3^{(2)} \end{Bmatrix} = \begin{Bmatrix} d_{x1}^{(3)} \\ d_{y1}^{(3)} \\ \phi_1^{(3)} \\ d_{x2}^{(3)} \\ d_{y2}^{(3)} \\ \phi_2^{(3)} \\ d_{x3}^{(3)} \\ d_{y3}^{(3)} \\ \phi_3^{(3)} \end{Bmatrix} = \begin{Bmatrix} d_{x1} \\ d_{y1} \\ \phi_1 \\ d_{x2} \\ d_{y2} \\ \phi_2 \\ d_{x3} \\ d_{y3} \\ \phi_3 \end{Bmatrix}$$

Drop the element index from the nodal displacements

2. Enforce equilibrium rule

sum of the forces at each node



$$f_{x1}^{(1)} + f_{x1}^{(2)} + f_{x1}^{(3)} = f_{x1}$$
$$\vdots \quad \vdots \quad \vdots$$

Form master (global) stiffness equations

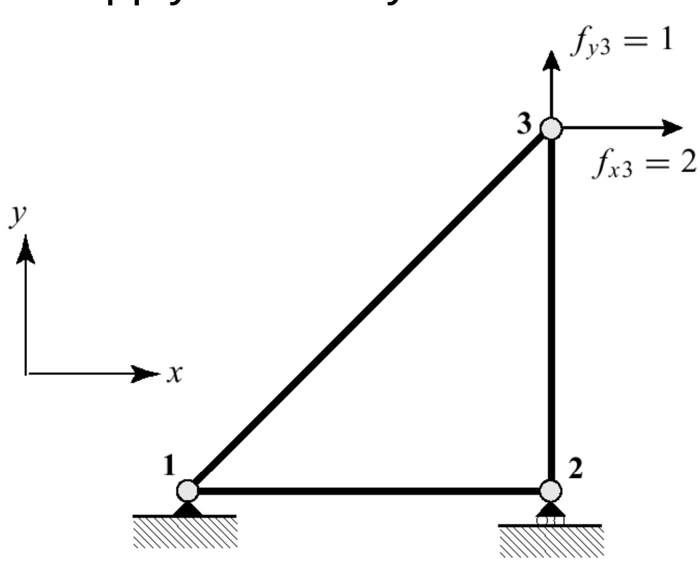
$$\mathbf{f} = \mathbf{f}^{(1)} + \mathbf{f}^{(2)} + \mathbf{f}^{(3)} = (\mathbf{K}^{(1)} + \mathbf{K}^{(2)} + \mathbf{K}^{(2)})\mathbf{d} = \mathbf{Kd}$$

Thus,

$$\begin{bmatrix} 20.1 & 9.9 & .84 & -10 & 0 & 0 & -10.1 & -9.9 & -.84 \\ & 10.1 & -.54 & 0 & -.06 & .3 & -9.9 & -10.1 & .84 \\ & & 13.3 & 0 & -.3 & 1 & .84 & -.84 & 5.65 \\ & & & 10.0 & 0 & -.075 & -.015 & 0 & -.075 \\ & & & & 5.06 & -.3 & 0 & -5 & 0 \\ & & & & & 2.5 & .075 & 0 & .25 \\ & & & & & & 10.1 & 9.9 & -.77 \\ & & & & & & & 15.1 & .84 \\ sym & & & & & & & & 11.8 \end{bmatrix} \begin{Bmatrix} d_{x1} \\ d_{y1} \\ \phi_1 \\ d_{x2} \\ d_{y2} \\ \phi_2 \\ d_{x3} \\ d_{y3} \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} f_{x1} \\ f_{y1} \\ f_{\phi_1} \\ f_{x2} \\ f_{y2} \\ f_{\phi_2} \\ f_{x3} \\ f_{y3} \\ f_{\phi_3} \end{Bmatrix}$$

Step 6:

Apply Boundary Conditions



Displacement Boundary Conditions:

$$u_{x1} = u_{y1} = u_{y2} = 0$$

Force Boundary Conditions

$$f_{y3} = 1 \qquad f_{x3} = 2$$

Applying Displacement Boundary Conditions:

$$u_{x1} = u_{y1} = u_{y2} = 0$$

Remove the rows and columns associated with zero displacements

$$\mathbf{Ku} = \mathbf{f}$$

Solve this using Gauss elimination
(or simply invert **K** using Mathematica)

$$\mathbf{K}^{-1}\mathbf{f} = \mathbf{u}$$

Step 7: Solution

$$\mathbf{d} = \begin{Bmatrix} d_{x1} \\ d_{y1} \\ \phi_1 \\ d_{x2} \\ d_{y2} \\ \phi_2 \\ d_{x3} \\ d_{y3} \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -0.078 \\ 0.0013 \\ 0 \\ 0.010 \\ 0.44 \\ -0.23 \\ 0.082 \end{Bmatrix}$$