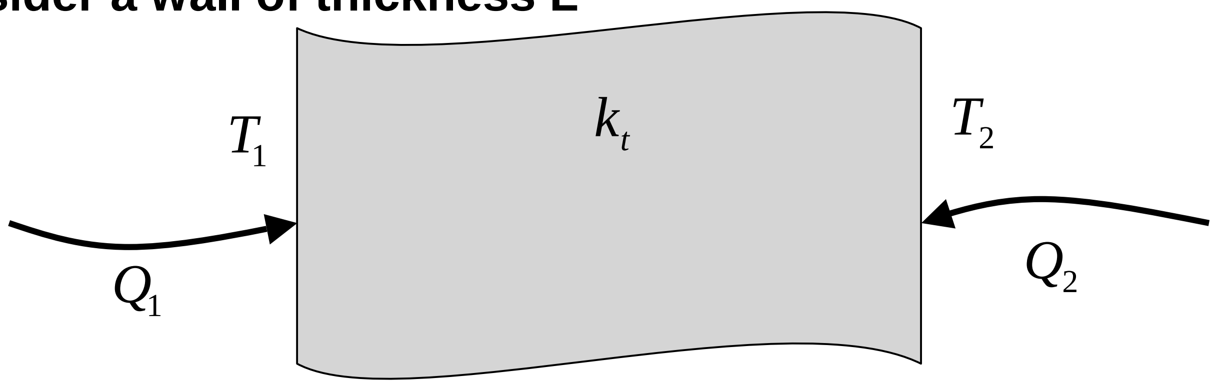
Chapter 4.5 Field Problems

Thermal problems (1dof – Temperature, T)
Electrostatic problems (1dof – Potential, V)
Antiplane problems (1dof – Out of plane displacement, w)

1-D Field Problems

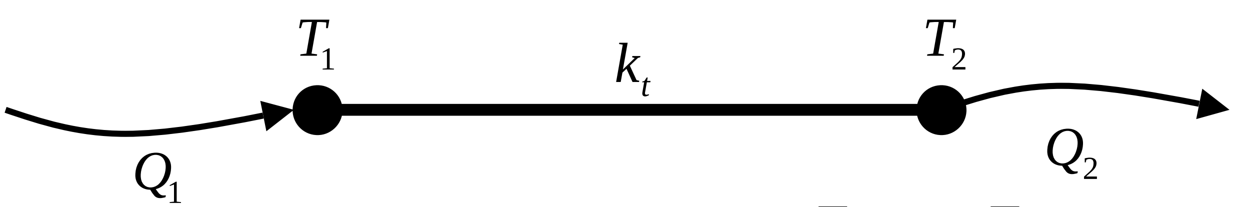
(1-D heat conduction)

Consider a wall of thickness L



$$q = -k_t \, A \frac{dT}{dx} \qquad q = Q_1 = -Q_2 \qquad \frac{dT}{dx} = \frac{T_2 - T_1}{L}$$

which we model as a single element



We have two equations with two unknowns T_1 and T_2

$$Q_1 = \frac{k_t \, A}{L}(T_1 - T_2)$$

$$Q_2 = \frac{k_t \, A}{L}(T_2 - T_1)$$

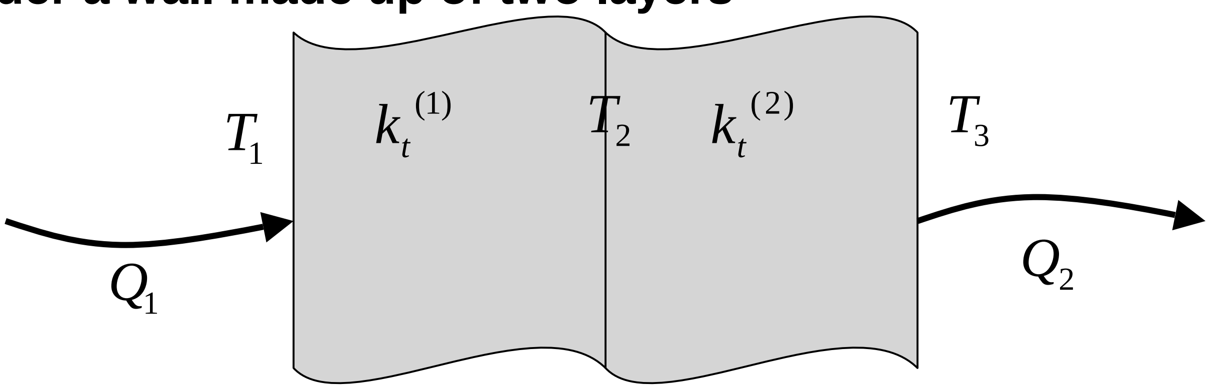
We write this in matrix form as:

$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \begin{bmatrix} \frac{k_t A}{L} & -\frac{k_t A}{L} \\ -\frac{k_t A}{L} & \frac{k_t A}{L} \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

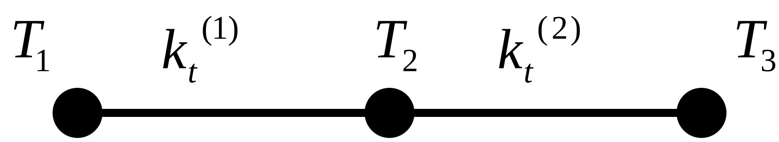
Flux vector — Conductance matrix — Temperature vector

This is the same thing that we had for the 1-D truss element

Consider a wall made up of two layers



We model this as two elements



For element 1 we have

$$\begin{Bmatrix} Q_1^{(1)} \\ Q_2^{(1)} \end{Bmatrix} = \begin{bmatrix} \frac{k_t A^{(1)}}{L} & -\frac{k_t A^{(1)}}{L} \\ -\frac{k_t A^{(1)}}{L} & \frac{k_t A^{(1)}}{L} \end{bmatrix} \begin{Bmatrix} T_1^{(1)} \\ T_2^{(1)} \end{Bmatrix}$$

For element 2 we have

$$\begin{Bmatrix} Q_2^{(2)} \\ Q_3^{(2)} \end{Bmatrix} = \begin{bmatrix} \frac{k_t A^{(2)}}{L} & -\frac{k_t A^{(2)}}{L} \\ -\frac{k_t A^{(2)}}{L} & \frac{k_t A^{(2)}}{L} \end{bmatrix} \begin{Bmatrix} T_2^{(2)} \\ T_3^{(2)} \end{Bmatrix}$$

Now join the two elements

equilibrium requires that $Q_2^{(1)} + Q_2^{(2)} = Q_2$

compatibility requires that $T_2^{(1)} = T_2^{(2)} = T_2$

We really have the following 3 equations with 3 unknowns
In matrix from:

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{Bmatrix} = \begin{bmatrix} \frac{k_{t1}A_1}{L_1} & -\frac{k_{t1}A_1}{L_1} & 0 \\ -\frac{k_{t1}A_1}{L_1} & \frac{k_{t1}A_1}{L_1} + \frac{k_{t2}A_2}{L_2} & -\frac{k_{t2}A_2}{L_2} \\ 0 & -\frac{k_{t2}A_2}{L_2} & \frac{k_{t2}A_2}{L_2} \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix}$$

Or

$$\mathbf{q} = \mathbf{K}\mathbf{t}$$

Nt. We don't have to rotate this –*T* has no direction

2-D Field Problems

(General case)

The field variable is ϕ

The steady state problem is given by

$$\frac{\partial}{\partial x}\left(k_{xx}\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial x}\left(k_{yy}\frac{\partial T}{\partial y}\right) + Q = 0$$

Which, when k = constant, reduces to the well known Poisson's equation as:

$$\nabla^2\phi = \frac{Q}{k}$$

Which, when Q = 0, reduces to the well known Laplace equation as:

$$\nabla^2\phi = 0$$

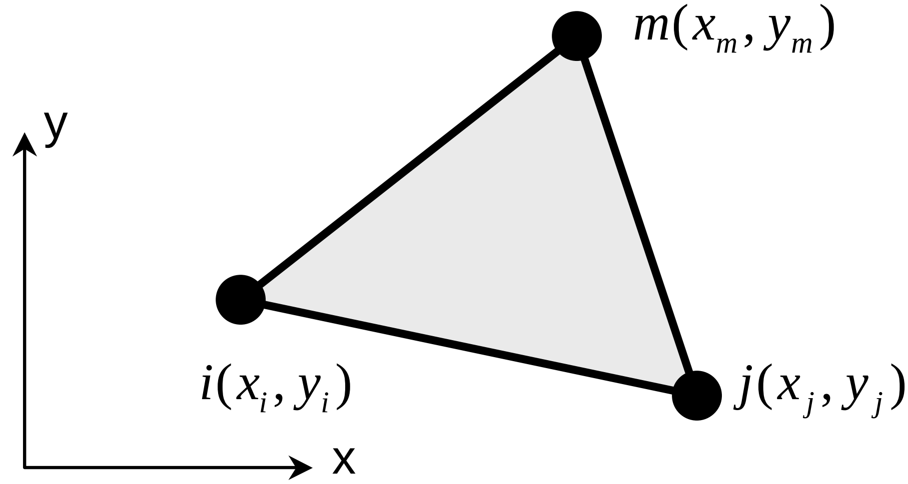
For now let's consider the more general case

$$\frac{\partial}{\partial x}\left(k_{xx}\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial x}\left(k_{yy}\frac{\partial T}{\partial y}\right) + Q = 0$$

2-D heat transfer finite element derivation
Again we are going to go through the same steps as before

Step 1 Select element type

Consider the three nodded triangular element



Nt. Node numbering is clockwise

Here each node has only 1 dof (Temperature)

Step 2 Select a temperature function

Since we have 3 nodes and 1 dof per node, let

$$T(x,y) = a_1 + a_2 x + a_3 y$$

which upon solving for the a's can be written as:

$$T(x,y) = \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} \begin{Bmatrix} T_i \\ T_j \\ T_m \end{Bmatrix} = \mathbf{Nt}$$

where the shape functions are solved as follows:

We really have 3 equations here

In matrix form we have for u_x

$$\begin{Bmatrix} T_i \\ T_j \\ T_m \end{Bmatrix} = \begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_m & y_m \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

Which can be solved as:

$$\mathbf{a} = \mathbf{x}^{-1}\mathbf{t}$$

To get the inverse of **x** we use the method of cofactors

$$\mathbf{x}^{-1} = \frac{1}{2A} adj[\mathbf{x}]$$

where

$$2A = \det\begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_m & y_m \end{bmatrix}$$

A is the area of the triangle and

$$adj[\mathbf{x}] = (cof[\mathbf{x}])^T$$

Step 3 Define the temperature gradient/temperature and heat flux/temperature gradient relations

$$\mathbf{g} = \begin{Bmatrix} \dfrac{\partial T}{\partial x} \\ \dfrac{\partial T}{\partial y} \end{Bmatrix} = \mathbf{Bt}$$

where $\mathbf{g}$ is the temperature gradient matrix (analogous to strain) and

$$\mathbf{B} = \frac{1}{2A}\begin{bmatrix} \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix}$$

The heat flux/temperature gradient relation is:

$$\mathbf{q} = \begin{Bmatrix} q_x \\ q_y \end{Bmatrix} = \mathbf{Dg} = \begin{bmatrix} k_{xx} & 0 \\ 0 & k_{yy} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial T}{\partial x} \\ \dfrac{\partial T}{\partial y} \end{Bmatrix}$$

Step 4 Derive the element conduction matrix and equations

Here we are going to use the variational method (analogous to the principle of minimum potential energy seen earlier)

Given the functional

$$\pi_h = U + \Omega_Q + \Omega_q + \Omega_h$$

Internal Energy

$$U = \iiint_V \frac{1}{2}\left[k_{xx}\left(\frac{\partial T}{\partial x}\right)^2 + k_{yy}\left(\frac{\partial T}{\partial y}\right)^2 \right] dV$$

Potential energy of the external forces

$$\Omega_Q = -\iiint_V QT dV$$

$$\Omega_q = -\iint_{Sq} q^* T \, ds_q$$

$$\Omega_h = \iint_{Sh} \frac{1}{2} h (T - T_\infty)^2 \, ds_h$$

Where Q is internal heat generation, q is heat flow and h is convection loss

(Nt, we can't specify both q and h on the same surface)

We can rewrite in matrix form as:

$$\pi_h = \iiint_V \frac{1}{2}\mathbf{g}^T\mathbf{D}\mathbf{g}\ dv - \iiint_V \mathbf{t}^T\mathbf{N}^T Q dV$$

$$-\iint_{Sq} \mathbf{t}^T\mathbf{N}^T q^*\, ds_q + \iint_{Sh} \frac{1}{2} h(\mathbf{t}^T\mathbf{N}^T - T_\infty)^2 ds_h$$

t is not a function of x and y so lets pull it out if the integrals to get

$$\pi_h = \frac{1}{2}\mathbf{t}^T \iiint_V \frac{1}{2}\mathbf{B}^T\mathbf{D}\mathbf{B}\ dv\ \mathbf{t} - \mathbf{t}^T \iiint_V \mathbf{N}^T Q dV$$

$$-\mathbf{t}^T \iint_{Sq} \mathbf{N}^T q^*\, ds_q$$

$$+\iint_{Sh} \frac{1}{2} h(\mathbf{t}^T\mathbf{N}^T\mathbf{N}\mathbf{t} - \mathbf{t}^T\mathbf{N}^T T_\infty + T_\infty{}^2) ds_h$$

We want to find the minimum energy (differentiate w.r.t. **t** and = 0)

$$\frac{\partial \pi_h}{\partial \mathbf{t}} = \iiint_V \mathbf{B}^T\mathbf{D}\mathbf{B}\ dv\ \mathbf{t} - \iiint_V \mathbf{N}^T Q dV$$

$$-\iint_{Sq} \mathbf{N}^T q^*\, ds_q$$

$$+\iint_{Sh} h(\mathbf{N}^T\mathbf{N}\mathbf{t}) ds_h + \iint_{Sh} \frac{1}{2} h(\mathbf{N}^T T_\infty) ds_h$$

And rewriting

$$\left[\iiint_V \mathbf{B}^T\mathbf{D}\mathbf{B}\, dv + \iint_{Sh} h(\mathbf{N}^T\mathbf{N})ds_h\right]\mathbf{t} =$$

$$+\iiint_V \mathbf{N}^T Q dV + \iint_{Sq} \mathbf{N}^T q^*\, ds_q - \iint_{Sh} \frac{1}{2} h(\mathbf{N}^T T_\infty)ds_h$$

And Voila!!

$$\mathbf{K}\ \mathbf{t} - \mathbf{f} = 0$$

We can simplify the conduction matrix since all terms in the $\mathbf{B}$ matrix are constant as

$$\mathbf{K} = \left[\iiint_V \mathbf{B}^T\mathbf{D}\mathbf{B}\, dv + \iint_{Sh} h(\mathbf{N}^T\mathbf{N})ds_h\right]$$

$$= tA\mathbf{B}^T\mathbf{D}\mathbf{B} + h\iint_{Sh}(\mathbf{N}^T\mathbf{N})ds_h$$

Step 5 Assembly

Use the same approach as shown before by applying compatibility and equilibrium

Step 6 Solve for the nodal temperatures…

Step 7 Postprocess for the temperature gradients and heat flux