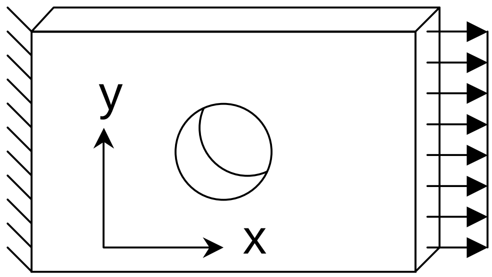
Chapter 5. Two-dimensional finite elements

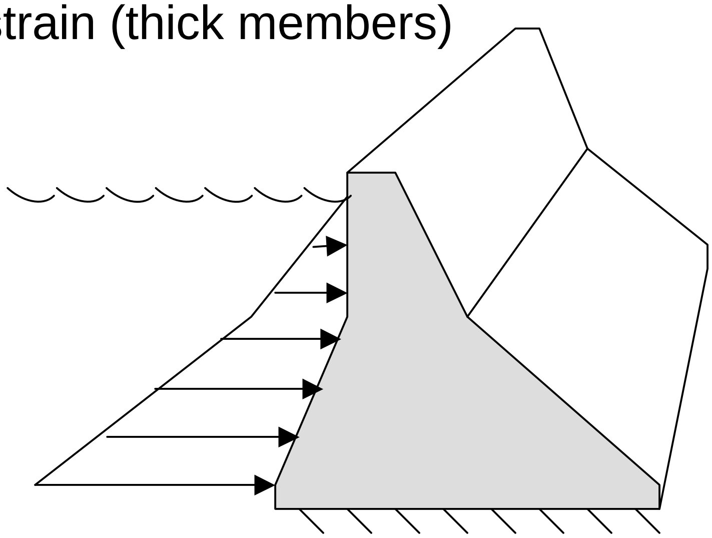
Plane stress (thin members)



Out-of-plane normal and shear stress are zero

$$\sigma_z = \sigma_{xz} = \sigma_{yz} = 0$$

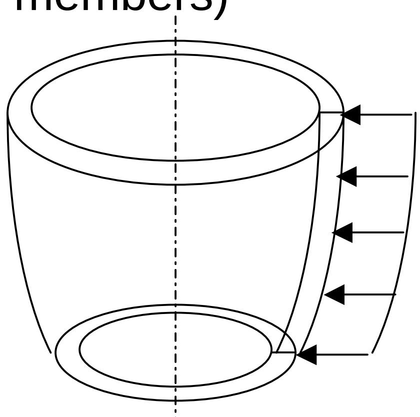
Plane strain (thick members)



Out-of-plane normal and shear strains are zero

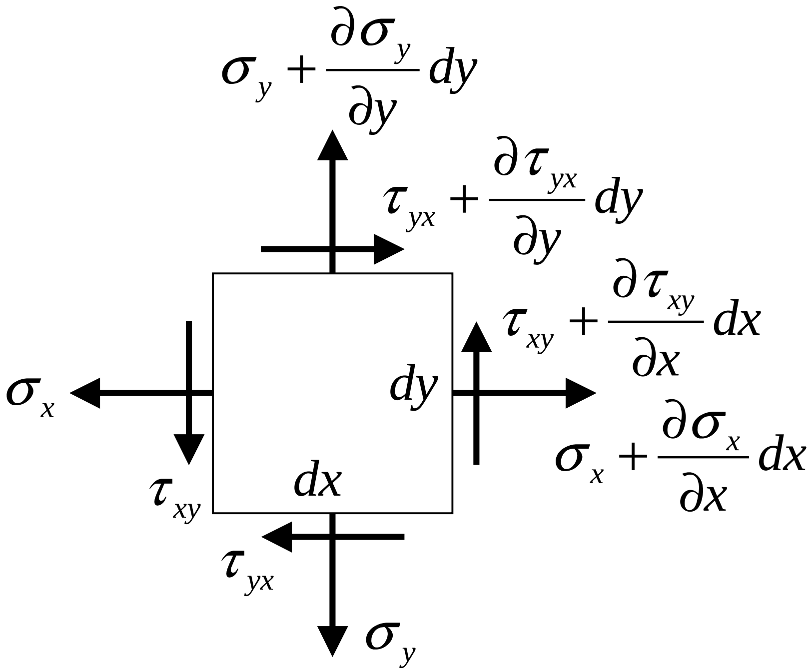
$$\varepsilon_z = \varepsilon_{xz} = \varepsilon_{yz} = 0$$

Axisymmetry (round members)



Things are a little more complicated here

Review: Two dimensional state of stress



In general, the stress/strain relation can be written in matrix form as

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon}$$

where

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} \quad \text{and} \quad \boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}$$

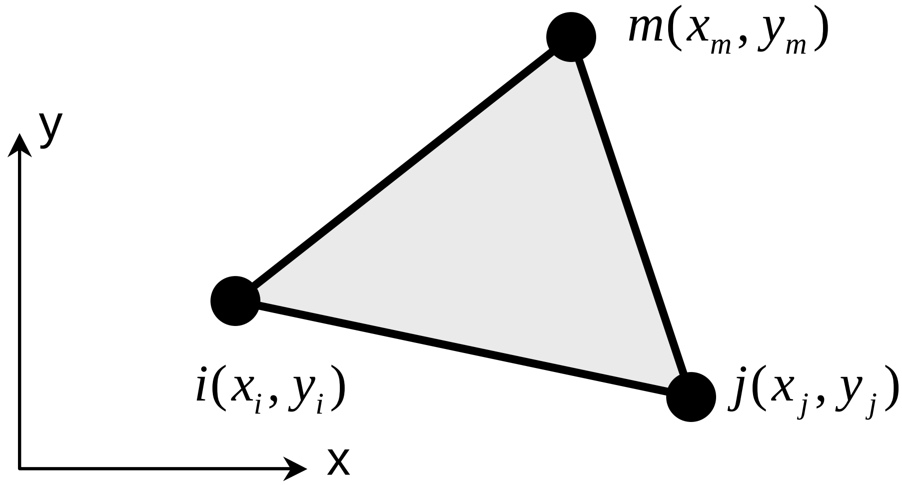
For plane strain

$$\mathbf{D} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

For plane stress

$$\mathbf{D} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

Element Formulation
Constant Strain Triangle (CST)



Nt. Node numbering is clockwise

Why is it called constant strain?

We assume that the displacements over the element are linear

This implies that the strain must be constant from

$$\varepsilon_x = \frac{\partial u_x}{\partial x}, \quad \varepsilon_y = \frac{\partial u_y}{\partial y}, \quad \text{and} \quad \gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}$$

Step 1: Choose the functional form of the displacement

The displacement vector for the element is:

$$\mathbf{u} = \begin{bmatrix} u_{xi} \\ u_{yi} \\ u_{xj} \\ u_{yj} \\ u_{xm} \\ u_{ym} \end{bmatrix}$$

we choose a linear displacement function as:

$$u_x(x, y) = a_1 + a_2 x \;\; + a_3 y$$

$$u_y(x, y) = a_4 + a_5 x \;\; + a_6 y$$

which has 6 a's since we have 3 nodes and 2dof per node

Now what are the a's?

We really have 6 equations here, 3 for u_x and three for u_y
In matrix form we have for u_x

$$\begin{Bmatrix} u_{xi} \\ u_{xj} \\ u_{xm} \end{Bmatrix} = \begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_m & y_m \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

Which can be solved as:

$$\mathbf{a}_x = \mathbf{x}^{-1}\mathbf{u}_x$$

Similarly in matrix form we have for u_y

$$\begin{Bmatrix} u_{yi} \\ u_{yj} \\ u_{ym} \end{Bmatrix} = \begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_m & y_m \end{bmatrix} \begin{Bmatrix} a_4 \\ a_5 \\ a_6 \end{Bmatrix}$$

Which can be solved as:

$$\mathbf{a}_y = \mathbf{x}^{-1}\mathbf{u}_y$$

So having solved this, we can write the element displacement as a function of nodal degrees of freedom as:

$$\begin{Bmatrix} u_x(x,y) \\ u_y(x,y) \end{Bmatrix} = \begin{bmatrix} N_i & 0 & N_j & 0 & N_m & 0 \\ 0 & N_i & 0 & N_j & 0 & N_m \end{bmatrix} \begin{Bmatrix} u_{xi} \\ u_{yi} \\ u_{xj} \\ u_{yj} \\ u_{xm} \\ u_{ym} \end{Bmatrix}$$

or

$$\mathbf{\Psi} = \mathbf{N}\mathbf{u}$$

Step 2 Strain/displacement and stress/strain relations

In 2-D the strain displacement relations are:

$$\varepsilon_x = \frac{\partial u_x}{\partial x}, \quad \varepsilon_y = \frac{\partial u_y}{\partial y}, \quad \text{and} \quad \gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}$$

or in matrix form as:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{\partial N_i}{\partial x} & 0 & \frac{\partial N_j}{\partial x} & 0 & \frac{\partial N_m}{\partial x} & 0 \\ 0 & \frac{\partial N_i}{\partial y} & 0 & \frac{\partial N_j}{\partial y} & 0 & \frac{\partial N_m}{\partial y} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} & \frac{\partial N_j}{\partial y} & \frac{\partial N_j}{\partial x} & \frac{\partial N_m}{\partial y} & \frac{\partial N_m}{\partial x} \end{bmatrix} \begin{Bmatrix} u_{xi} \\ u_{yi} \\ u_{xj} \\ u_{yj} \\ u_{xm} \\ u_{ym} \end{Bmatrix}$$

$$\boldsymbol{\varepsilon} = \mathbf{B}\mathbf{u}$$

In 2-D the stress/strain relations are:

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon} = \mathbf{D}\mathbf{B}\mathbf{u}$$

Where **D** depends on whether plane stress or plane strain conditions prevails

For plane strain

$$\mathbf{D} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

For plane stress

$$\mathbf{D} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

Step 3 Derive the element stiffness equations

Potential Energy Method Derivation

The total PE

$$\pi_p = U + \Omega$$

Internal Strain Energy

$$U = \iiint_V \frac{1}{2} \sigma^T \varepsilon \, dV$$

Potential energy of the external forces

Point loads acting on nodes

$$\Omega = P_{xi} u_{xi} - P_{yi} u_{yi} - P_{xj} u_{xj} - P_{yj} u_{yj} - P_{xm} u_{xm} - P_{ym} u_{ym}$$

$$-\iiint_V \boldsymbol{\Psi}^T \mathbf{X}_{body} \, dV \qquad -\iint_S \boldsymbol{\Psi}^T \mathbf{T}_{tract} \, dS$$

Body force X acting over volume

Tractions T_{tract} in x-y-plane acting over surface

$$\pi_p = \int_L \iint_A \frac{1}{2} \sigma_x \varepsilon_x dA dx - \sum_{n=i}^{m} \left(P_{ni} u_{ni} + P_{ni} u_{ni} \right)$$

$$-\iiint_V \boldsymbol{\Psi}^T \mathbf{X}_{body} \, dV - \iint_S \boldsymbol{\Psi}^T \mathbf{T}_{tract} \, dS$$

Which we can rewrite in matrix form as:

$$\pi_p = \int_L \iint_A \frac{1}{2} \boldsymbol{\sigma}_{\mathbf{x}}^T \boldsymbol{\varepsilon}_{\mathbf{x}} dA dx - \mathbf{u}^T \mathbf{P}$$

$$-\iiint_V \boldsymbol{\Psi}^T \mathbf{X}_{body} \, dV - \iint_S \boldsymbol{\Psi}^T \mathbf{T}_{tract} \, dS$$

Making the appropriate substitutions (rewrite in terms of **d**)

$$\pi_p = \int_z \iint_A \frac{1}{2} \mathbf{u}^T \mathbf{B}^T \mathbf{D} \mathbf{B} \mathbf{u} \; dA dz - \mathbf{u}^T \mathbf{P}$$

$$- \iiint_V \mathbf{u}^T \mathbf{N}^T \mathbf{X}_{body} \, dV - \iint_S \mathbf{u}^T \mathbf{N}^T \mathbf{T}_{tract} \, dS$$

$$= t \iint_A \frac{1}{2} \mathbf{u}^T \mathbf{B}^T \mathbf{D} \mathbf{B} \mathbf{u} \; dA - \mathbf{u}^T \mathbf{P}$$

$$- \iiint_V \mathbf{u}^T \mathbf{N}^T \mathbf{X}_{body} \, dV - \iint_S \mathbf{u}^T \mathbf{N}^T \mathbf{T}_{tract} \, dS$$

Thickness

We want to find the minimum potential energy

So we differentiate w.r.t. $\mathbf{u}$ and set = 0

$$t \iint_A \mathbf{B}^T \mathbf{D} \mathbf{B} \mathbf{u} \, dA - \mathbf{P} \quad - \iiint_V \mathbf{N}^T \mathbf{X}_{body} \, dV - \iint_S \mathbf{N}^T \mathbf{T}_{tract} \, dS = 0$$

And Voila!!

$$\mathbf{K} \; \mathbf{u} - \mathbf{f} = 0$$

In the absence of body forces and surface tractions, this is simply

$$t \iint_A \mathbf{B}^T \mathbf{D} \mathbf{B} \, dA \mathbf{u} - \mathbf{P} \quad = 0$$

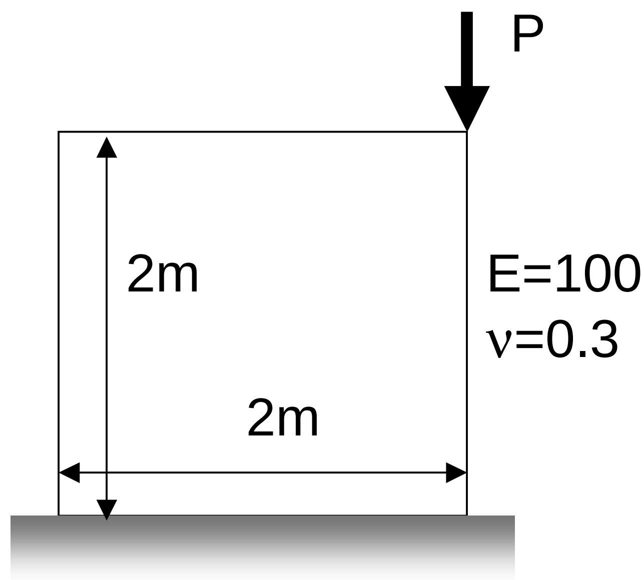
Written out explicitly, this is:

$$\begin{bmatrix} Lots & & & & & \\ & of & & & & \\ & & stuff & & & \\ & & & goes & & \\ & & & & in & \\ & & & & & here \end{bmatrix} \begin{Bmatrix} u_{xi} \\ u_{yi} \\ u_{xj} \\ u_{yj} \\ u_{xm} \\ u_{ym} \end{Bmatrix} = \begin{Bmatrix} f_{xi} \\ f_{yi} \\ f_{xj} \\ f_{yj} \\ f_{xm} \\ f_{ym} \end{Bmatrix}$$

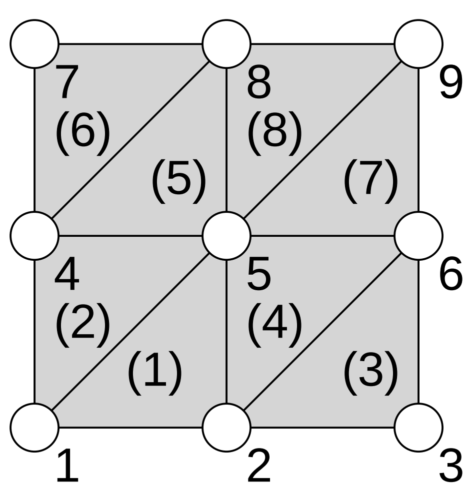
2D Example

Consider the following simple 2-D problem

Plane stress
thickness=0.1



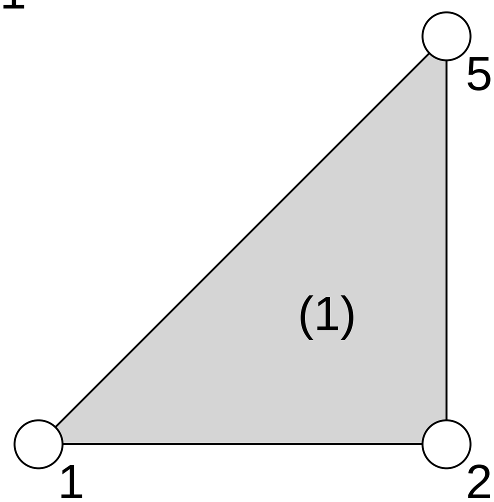
Start by breaking the surface up into elements and assigning node numbers and element numbers



Here we have 9 nodes (with 2 dof per node)

Thus we would expect a 18by18 stiffness matrix, an 18by1 displacement vector and an 18by1 force vector

Starting with Element 1



The elemental stiffness matrix is derived as follows:

The displacement function over this element can be written as:

$$\begin{Bmatrix} u_x(x,y) \\ u_y(x,y) \end{Bmatrix} = \mathbf{Nu} = \begin{bmatrix} 1-x & 0 & x-y & 0 & y & 0 \\ 0 & 1-x & 0 & x-y & 0 & y \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x5} \\ u_{y5} \end{Bmatrix}$$

$$\boldsymbol{\varepsilon} = \mathbf{Bu} = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & -1 & -1 & 1 & 1 & 0 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x5} \\ u_{y5} \end{Bmatrix}$$

$$\boldsymbol{\sigma} = \mathbf{DBu} = \begin{bmatrix} 109 & 33 & 0 \\ 33 & 109 & 0 \\ 0 & 0 & 38.5 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & -1 & -1 & 1 & 1 & 0 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x5} \\ u_{y5} \end{Bmatrix}$$

$$\mathbf{K}^{(1)} = A\,t\,\boldsymbol{\sigma}^T \boldsymbol{\varepsilon} = A\,t\,\mathbf{B}^T \mathbf{D}^T \mathbf{B} =$$

$$\begin{pmatrix} 0.275 & 0. & -0.275 & 0.0824 & 0. & -0.0824 \\ 0. & 0.0962 & 0.0962 & -0.0962 & -0.0962 & 0. \\ -0.275 & 0.0962 & 0.371 & -0.179 & -0.0962 & 0.0824 \\ 0.0824 & -0.0962 & -0.179 & 0.371 & 0.0962 & -0.275 \\ 0. & -0.0962 & -0.0962 & 0.0962 & 0.0962 & 0. \\ -0.0824 & 0. & 0.0824 & -0.275 & 0. & 0.275 \end{pmatrix}$$

The global stiffness matrix is:

$$\begin{pmatrix}
0.371 & 0. & -0.275 & 0.0824 & 0. & 0. & -0.0962 & 0.0962 & 0. & -0.179 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0.371 & 0.0962 & -0.0962 & 0. & 0. & 0.0824 & -0.275 & -0.179 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
-0.275 & 0.0962 & 0.742 & -0.179 & -0.275 & 0.0824 & 0. & 0. & -0.192 & 0.179 & 0. & -0.179 & 0. & 0. & 0. & 0. & 0. & 0. \\
0.0824 & -0.0962 & -0.179 & 0.742 & 0.0962 & -0.0962 & 0. & 0. & 0.179 & -0.549 & -0.179 & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & -0.275 & 0.0962 & 0.371 & -0.179 & 0. & 0. & 0. & 0. & -0.0962 & 0.0824 & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0.0824 & -0.0962 & -0.179 & 0.371 & 0. & 0. & 0. & 0. & 0.0962 & -0.275 & 0. & 0. & 0. & 0. & 0. & 0. \\
-0.0962 & 0.0824 & 0. & 0. & 0. & 0. & 0.742 & -0.179 & -0.549 & 0.179 & 0. & 0. & -0.0962 & 0.0962 & 0. & -0.179 & 0. & 0. \\
0.0962 & -0.275 & 0. & 0. & 0. & 0. & -0.179 & 0.742 & 0.179 & -0.192 & 0. & 0. & 0.0824 & -0.275 & -0.179 & 0. & 0. & 0. \\
0. & -0.179 & -0.192 & 0.179 & 0. & 0. & -0.549 & 0.179 & 1.48 & -0.357 & -0.549 & 0.179 & 0. & 0. & -0.192 & 0.179 & 0. & -0.179 \\
-0.179 & 0. & 0.179 & -0.549 & 0. & 0. & 0.179 & -0.192 & -0.357 & 1.48 & 0.179 & -0.192 & 0. & 0. & 0.179 & -0.549 & -0.179 & 0. \\
0. & 0. & 0. & -0.179 & -0.0962 & 0.0962 & 0. & 0. & -0.549 & 0.179 & 0.742 & -0.179 & 0. & 0. & 0. & 0. & -0.0962 & 0.0824 \\
0. & 0. & -0.179 & 0. & 0.0824 & -0.275 & 0. & 0. & 0.179 & -0.192 & -0.179 & 0.742 & 0. & 0. & 0. & 0. & 0.0962 & -0.275 \\
0. & 0. & 0. & 0. & 0. & 0. & -0.0962 & 0.0824 & 0. & 0. & 0. & 0. & 0.371 & -0.179 & -0.275 & 0.0962 & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0.0962 & -0.275 & 0. & 0. & 0. & 0. & -0.179 & 0.371 & 0.0824 & -0.0962 & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & -0.179 & -0.192 & 0.179 & 0. & 0. & -0.275 & 0.0824 & 0.742 & -0.179 & -0.275 & 0.0962 \\
0. & 0. & 0. & 0. & 0. & 0. & -0.179 & 0. & 0.179 & -0.549 & 0. & 0. & 0.0962 & -0.0962 & -0.179 & 0.742 & 0.0824 & -0.0962 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -0.179 & -0.0962 & 0.0962 & 0. & 0. & -0.275 & 0.0824 & 0.371 & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -0.179 & 0. & 0.0824 & -0.275 & 0. & 0. & 0.0962 & -0.0962 & 0. & 0.371
\end{pmatrix}$$

Next we delete the rows and columns from both the global stiffness matrix and the force vector corresponding to fixed (zero) displacements as:

$$
\begin{pmatrix}
0.74 & -0.27 & 0. & 0. & -0.19 & 0.18 & 0. & -0.18 & 0. & 0. & 0. & 0. & 0. & 0. \\
-0.27 & 0.37 & 0. & 0. & 0. & 0. & -0.096 & 0.082 & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0.74 & -0.18 & -0.55 & 0.18 & 0. & 0. & -0.096 & 0.096 & 0. & -0.18 & 0. & 0. \\
0. & 0. & -0.18 & 0.74 & 0.18 & -0.19 & 0. & 0. & 0.082 & -0.27 & -0.18 & 0. & 0. & 0. \\
-0.19 & 0. & -0.55 & 0.18 & 1.5 & -0.36 & -0.55 & 0.18 & 0. & 0. & -0.19 & 0.18 & 0. & -0.18 \\
0.18 & 0. & 0.18 & -0.19 & -0.36 & 1.5 & 0.18 & -0.19 & 0. & 0. & 0.18 & -0.55 & -0.18 & 0. \\
0. & -0.096 & 0. & 0. & -0.55 & 0.18 & 0.74 & -0.18 & 0. & 0. & 0. & 0. & -0.096 & 0.082 \\
-0.18 & 0.082 & 0. & 0. & 0.18 & -0.19 & -0.18 & 0.74 & 0. & 0. & 0. & 0. & 0.096 & -0.27 \\
0. & 0. & -0.096 & 0.082 & 0. & 0. & 0. & 0. & 0.37 & -0.18 & -0.27 & 0.096 & 0. & 0. \\
0. & 0. & 0.096 & -0.27 & 0. & 0. & 0. & 0. & -0.18 & 0.37 & 0.082 & -0.096 & 0. & 0. \\
0. & 0. & 0. & -0.18 & -0.19 & 0.18 & 0. & 0. & -0.27 & 0.082 & 0.74 & -0.18 & -0.27 & 0.096 \\
0. & 0. & -0.18 & 0. & 0.18 & -0.55 & 0. & 0. & 0.096 & -0.096 & -0.18 & 0.74 & 0.082 & -0.096 \\
0. & 0. & 0. & 0. & 0. & -0.18 & -0.096 & 0.096 & 0. & 0. & -0.27 & 0.082 & 0.37 & 0. \\
0. & 0. & 0. & 0. & -0.18 & 0. & 0.082 & -0.27 & 0. & 0. & 0.096 & -0.096 & 0. & 0.37
\end{pmatrix}
\mathbf{u}_{red} =
\begin{pmatrix}
0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -2.5
\end{pmatrix}
$$

The solution is:

$$
\mathbf{u}_{red} =
\begin{pmatrix}
0.826276 \\ 3.40158 \\ 3.37213 \\ 1.48275 \\ 3.02932 \\ -2.28855 \\ 4.45799 \\ -7.35188 \\ 9.65709 \\ 1.79781 \\ 9.97215 \\ -3.31691 \\ 10.0838 \\ -15.164
\end{pmatrix}
$$

The following Mathematica code will solve any 2D plane stress or plane strain problem with an arbitrary number of 3-node CST elements

FEA Code to solve simple planar solid mechanics problems with constant strain triangles (CST) Prof. P. Labossiere 4/4/01

```
Off@General::spell1D;
Off@General::spellD;
deleteColumn@matrix_, column_D := Map@Delete@#, columnD &, matrixD;
deleteRow@matrix_, row_D :=
  Transpose@Map@Delete@#, rowD &, Transpose@matrixDDD;
```

INPUT

Nodal coordinates {nodenumber, xcoord,ycoord}

```
thickness = .005;
MatrixForm@
 nodeinf = 881, 0, 0<, 82, 1, 0<, 83, 2, 0<, 84, 0, 1<, 85, 1, 1<,
   86, 2, 1<, 87, 0, 2<, 88, 1, 2<, 89, 2, 2<<D
```

1	0	0
2	1	0
3	2	0
4	0	1
5	1	1
6	2	1
7	0	2
8	1	2
9	2	2

Connectivity {elementnumber, material number , node1, node2, node3}

```
MatrixForm@
 elcon = 881, 1, 1, 2, 5<, 82, 1, 1, 5, 4<, 83, 1, 2, 3, 6<, 84, 1, 2, 6, 5<,
   85, 1, 4, 5, 8<, 86, 1, 4, 8, 7<, 87, 1, 5, 6, 9<, 88, 1, 5, 9, 8<<D
```

1	1	1	2	5
2	1	1	5	4
3	1	2	3	6
4	1	2	6	5
5	1	4	5	8
6	1	4	8	7
7	1	5	6	9
8	1	5	9	8

Applied loads

```
Fg = Transpose@880 , 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 , -2.5<<D;
```

Fixed displacement boundary conditions (node number, dof 1=x, 2=y)

```
fixnode = 881, 1<, 81, 2<, 82, 2<, 83, 2<<;
numfix = Length@fixnodeD;
```

Material Properties (Plane stress, ps=1, plane strain, ps=2)

```
ps = 1;
E1 = 100;
ν1 = 0.3;
```

$$Dmatp\sigma = \frac{E1}{1-\nu 1^2}\ 981,\ \nu 1,\ 0<,\ 8\nu 1,\ 1,\ 0<,\ 90,\ 0,\ \frac{1-\nu 1}{2}==;$$

$$Dmatp\varepsilon = \frac{E1}{H1+\nu 1L\ H1-2\,\nu 1L}\ 981-\nu 1,\ \nu 1,\ 0<,\ 8\nu 1,\ 1-\nu 1,\ 0<,\ 90,\ 0,\ \frac{1-2\,\nu 1}{2}==;$$

```
MatrixForm@Dmat = Which@ps « 1, Dmatpσ, ps « 2, DmatpεDD
```

109.89	32.967	0
32.967	109.89	0
0	0	38.4615

Problem Formation

Construct and print the local stiffness matrices

```
numnode = Length@nodeinfD;
numel = Length@elconD;
Kg = Table@0., 8i, 2 numnode<, 8j, 2 numnode<D;
Clear@x, yD
DoA9x1 = nodeinf@@elcon@@elnum, 3DD, 2DD;
     x2 = nodeinf@@elcon@@elnum, 4DD, 2DD;
     x3 = nodeinf@@elcon@@elnum, 5DD, 2DD;
     y1 = nodeinf@@elcon@@elnum, 3DD, 3DD;
     y2 = nodeinf@@elcon@@elnum, 4DD, 3DD;
     y3 = nodeinf@@elcon@@elnum, 5DD, 3DD;
    xmatinv = Inverse@881, x1, y1<, 81, x2, y2<, 81, x3, y3<<D;
    A2 = Det@881, x1, y1<, 81, x2, y2<, 81, x3, y3<<D;
N1 = Hxmatinv@@1, 1DD + xmatinv@@2, 1DD x + xmatinv@@3, 1DD yL;
    N2 = Hxmatinv@@1, 2DD + xmatinv@@2, 2DD x + xmatinv@@3, 2DD yL;
    N3 = Hxmatinv@@1, 3DD + xmatinv@@2, 3DD x + xmatinv@@3, 3DD yL;
    Nmat = 88N1, 0, N2, 0, N3, 0<, 80, N1, 0, N2, 0, N3<<;
    Bmat = 88D@N1, xD, 0, D@N2, xD, 0, D@N3, xD, 0<,
      80, D@N1, yD, 0, D@N2, yD, 0, D@N3, yD<,
      8D@N1, yD, D@N1, xD, D@N2, yD, D@N2, xD, D@N3, yD, D@N3, xD<<;
    Print@A2D;
    Print@MatrixForm@NmatDD;
    Print@MatrixForm@BmatDD;
    Kmat = thickness A2/2 Transpose@BmatD.Dmat.Bmat;
    Do@
     Do@
      Do@
       Do@Kg@@i, jDD = Kg@@i, jDD +
          Kmat@@i + 2 − 2 elcon@@elnum, kDD + 2 Hk − 3L,
            j + 2 − 2 elcon@@elnum, lDD + 2 H l − 3LDD,
        8i, 2 elcon@@elnum, kDD − 1, 2 elcon@@elnum, kDD<D,
                      8j, 2 elcon@@elnum, lDD − 1, 2 elcon@@elnum, lDD<D,
      8k, 3, 5<D, 8l, 3, 5<D;

    Print@NumberForm@MatrixForm@KmatD, 3DD=, 8elnum, 1, numel<E;
Print@NumberForm@MatrixForm@KgD, 3DD
```

```
1
J 1−x   0   x−y   0   y  0 N
   0   1−x   0   x−y  0  y
i −1  0   1   0   0  0 y
|  0  0   0  −1   0  1 |
k  0 −1  −1   1   1  0 {
i  0.275     0.     −0.275   0.0824     0.    −0.0824 y
|    0.     0.0962   0.0962  −0.0962  −0.0962    0.    |
| −0.275   0.0962    0.371   −0.179   −0.0962  0.0824  |
|  0.0824  −0.0962  −0.179    0.371    0.0962  −0.275  |
|    0.    −0.0962  −0.0962   0.0962   0.0962    0.    |
k −0.0824    0.      0.0824  −0.275     0.     0.275  {
1
J 1−y   0   x  0  −x+y    0   N
   0   1−y  0  x    0    −x+y
i  0   0  1  0  −1   0 y
|  0  −1  0  0   0   1 |
k −1   0  0  1   1  −1 {
```

$$\begin{pmatrix} 0.0962 & 0. & 0. & -0.0962 & -0.0962 & 0.0962 \\ 0. & 0.275 & -0.0824 & 0. & 0.0824 & -0.275 \\ 0. & -0.0824 & 0.275 & 0. & -0.275 & 0.0824 \\ -0.0962 & 0. & 0. & 0.0962 & 0.0962 & -0.0962 \\ -0.0962 & 0.0824 & -0.275 & 0.0962 & 0.371 & -0.179 \\ 0.0962 & -0.275 & 0.0824 & -0.0962 & -0.179 & 0.371 \end{pmatrix}$$

1

$$\begin{pmatrix} 2-x & 0 & -1+x-y & 0 & y & 0 \\ 0 & 2-x & 0 & -1+x-y & 0 & y \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & -1 & -1 & 1 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0.275 & 0. & -0.275 & 0.0824 & 0. & -0.0824 \\ 0. & 0.0962 & 0.0962 & -0.0962 & -0.0962 & 0. \\ -0.275 & 0.0962 & 0.371 & -0.179 & -0.0962 & 0.0824 \\ 0.0824 & -0.0962 & -0.179 & 0.371 & 0.0962 & -0.275 \\ 0. & -0.0962 & -0.0962 & 0.0962 & 0.0962 & 0. \\ -0.0824 & 0. & 0.0824 & -0.275 & 0. & 0.275 \end{pmatrix}$$

1

$$\begin{pmatrix} 1-y & 0 & -1+x & 0 & 1-x+y & 0 \\ 0 & 1-y & 0 & -1+x & 0 & 1-x+y \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 & 1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 0.0962 & 0. & 0. & -0.0962 & -0.0962 & 0.0962 \\ 0. & 0.275 & -0.0824 & 0. & 0.0824 & -0.275 \\ 0. & -0.0824 & 0.275 & 0. & -0.275 & 0.0824 \\ -0.0962 & 0. & 0. & 0.0962 & 0.0962 & -0.0962 \\ -0.0962 & 0.0824 & -0.275 & 0.0962 & 0.371 & -0.179 \\ 0.0962 & -0.275 & 0.0824 & -0.0962 & -0.179 & 0.371 \end{pmatrix}$$

1

$$\begin{pmatrix} 1-x & 0 & 1+x-y & 0 & -1+y & 0 \\ 0 & 1-x & 0 & 1+x-y & 0 & -1+y \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & -1 & -1 & 1 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0.275 & 0. & -0.275 & 0.0824 & 0. & -0.0824 \\ 0. & 0.0962 & 0.0962 & -0.0962 & -0.0962 & 0. \\ -0.275 & 0.0962 & 0.371 & -0.179 & -0.0962 & 0.0824 \\ 0.0824 & -0.0962 & -0.179 & 0.371 & 0.0962 & -0.275 \\ 0. & -0.0962 & -0.0962 & 0.0962 & 0.0962 & 0. \\ -0.0824 & 0. & 0.0824 & -0.275 & 0. & 0.275 \end{pmatrix}$$

1

$$\begin{pmatrix} 2-y & 0 & x & 0 & -1-x+y & 0 \\ 0 & 2-y & 0 & x & 0 & -1-x+y \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 & 1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 0.0962 & 0. & 0. & -0.0962 & -0.0962 & 0.0962 \\ 0. & 0.275 & -0.0824 & 0. & 0.0824 & -0.275 \\ 0. & -0.0824 & 0.275 & 0. & -0.275 & 0.0824 \\ -0.0962 & 0. & 0. & 0.0962 & 0.0962 & -0.0962 \\ -0.0962 & 0.0824 & -0.275 & 0.0962 & 0.371 & -0.179 \\ 0.0962 & -0.275 & 0.0824 & -0.0962 & -0.179 & 0.371 \end{pmatrix}$$

1

$$\begin{pmatrix} 2-x & 0 & x-y & 0 & -1+y & 0 \\ 0 & 2-x & 0 & x-y & 0 & -1+y \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & -1 & -1 & 1 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0.275 & 0. & -0.275 & 0.0824 & 0. & -0.0824 \\ 0. & 0.0962 & 0.0962 & -0.0962 & -0.0962 & 0. \\ -0.275 & 0.0962 & 0.371 & -0.179 & -0.0962 & 0.0824 \\ 0.0824 & -0.0962 & -0.179 & 0.371 & 0.0962 & -0.275 \\ 0. & -0.0962 & -0.0962 & 0.0962 & 0.0962 & 0. \\ -0.0824 & 0. & 0.0824 & -0.275 & 0. & 0.275 \end{pmatrix}$$

$\frac{1}{J}$

2 − y	0	−1 + x	0	−x + y	0
0	2 − y	0	−1 + x	0	−x + y

0	0	1	0	−1	0
0	−1	0	0	0	1
−1	0	0	1	1	−1

0.0962	0.	0.	−0.0962	−0.0962	0.0962
0.	0.275	−0.0824	0.	0.0824	−0.275
0.	−0.0824	0.275	0.	−0.275	0.0824
−0.0962	0.	0.	0.0962	0.0962	−0.0962
−0.0962	0.0824	−0.275	0.0962	0.371	−0.179
0.0962	−0.275	0.0824	−0.0962	−0.179	0.371

0.371	0.	−0.275	0.0824	0.	0.	−0.0962	0.0962	0.	−0.179	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.371	0.0962	−0.0962	0.	0.	0.0824	−0.275	−0.179	0.	0.	0.	0.	0.	0.	0.	0.	0.
−0.275	0.0962	0.742	−0.179	−0.275	0.0824	0.	0.	−0.192	0.179	0.	−0.179	0.	0.	0.	0.	0.	0.
0.0824	−0.0962	−0.179	0.742	0.0962	−0.0962	0.	0.	0.179	−0.549	−0.179	0.	0.	0.	0.	0.	0.	0.
0.	0.	−0.275	0.0962	0.371	−0.179	0.	0.	0.	0.	−0.0962	0.0824	0.	0.	0.	0.	0.	0.
0.	0.	0.0824	−0.0962	−0.179	0.371	0.	0.	0.	0.	0.0962	−0.275	0.	0.	0.	0.	0.	0.
−0.0962	0.0824	0.	0.	0.	0.	0.742	−0.179	−0.549	0.179	0.	0.	−0.0962	0.0962	0.	−0.179	0.	0.
0.0962	−0.275	0.	0.	0.	0.	−0.179	0.742	0.179	−0.192	0.	0.	0.0824	−0.275	−0.179	0.	0.	0.
0.	−0.179	−0.192	0.179	0.	0.	−0.549	0.179	1.48	−0.357	−0.549	0.179	0.	0.	−0.192	0.179	0.	−0.179
−0.179	0.	0.179	−0.549	0.	0.	0.179	−0.192	−0.357	1.48	0.179	−0.192	0.	0.	0.179	−0.549	−0.179	0.
0.	0.	0.	−0.179	−0.0962	0.0962	0.	0.	−0.549	0.179	0.742	−0.179	0.	0.	0.	0.	−0.0962	0.0824
0.	0.	−0.179	0.	0.0824	−0.275	0.	0.	0.179	−0.192	−0.179	0.742	0.	0.	0.	0.	0.0962	−0.275
0.	0.	0.	0.	0.	0.	−0.0962	0.0824	0.	0.	0.	0.	0.371	−0.179	−0.275	0.0962	0.	0.
0.	0.	0.	0.	0.	0.	0.0962	−0.275	0.	0.	0.	0.	−0.179	0.371	0.0824	−0.0962	0.	0.
0.	0.	0.	0.	0.	0.	0.	−0.179	−0.192	0.179	0.	0.	−0.275	0.0824	0.742	−0.179	−0.275	0.0962
0.	0.	0.	0.	0.	0.	−0.179	0.	0.179	−0.549	0.	0.	0.0962	−0.0962	−0.179	0.742	0.0824	−0.0962
0.	0.	0.	0.	0.	0.	0.	0.	0.	−0.179	−0.0962	0.0962	0.	0.	−0.275	0.0824	0.371	0.
0.	0.	0.	0.	0.	0.	0.	0.	−0.179	0.	0.0824	−0.275	0.	0.	0.0962	−0.0962	0.	0.371

Apply the fixed displacment boundary condition by deleting the appropriate rows and columns

```
Kred = Kg;
Kred = Kg;
Do@
 Kred = deleteColumn@deleteRow@Kred, 2 fixnode@@k, 1DD − 2 + fixnode@@k, 2DDD,
   2 fixnode@@k, 1DD − 2 + fixnode@@k, 2DDD;
 Print@2 fixnode@@k, 1DD − 2 + fixnode@@k, 2DDD, 8k, numfix, 1, −1<D
NumberForm@MatrixForm@KredD, 2D
```

6

4

2

1

0.74	−0.27	0.	0.	−0.19	0.18	0.	−0.18	0.	0.	0.	0.	0.	0.
−0.27	0.37	0.	0.	0.	0.	−0.096	0.082	0.	0.	0.	0.	0.	0.
0.	0.	0.74	−0.18	−0.55	0.18	0.	0.	−0.096	0.096	0.	−0.18	0.	0.
0.	0.	−0.18	0.74	0.18	−0.19	0.	0.	0.082	−0.27	−0.18	0.	0.	0.
−0.19	0.	−0.55	0.18	1.5	−0.36	−0.55	0.18	0.	0.	−0.19	0.18	0.	−0.18
0.18	0.	0.18	−0.19	−0.36	1.5	0.18	−0.19	0.	0.	0.18	−0.55	−0.18	0.
0.	−0.096	0.	0.	−0.55	0.18	0.74	−0.18	0.	0.	0.	0.	−0.096	0.082
−0.18	0.082	0.	0.	0.18	−0.19	−0.18	0.74	0.	0.	0.	0.	0.096	−0.27
0.	0.	−0.096	0.082	0.	0.	0.	0.	0.37	−0.18	−0.27	0.096	0.	0.
0.	0.	0.096	−0.27	0.	0.	0.	0.	−0.18	0.37	0.082	−0.096	0.	0.
0.	0.	0.	−0.18	−0.19	0.18	0.	0.	−0.27	0.082	0.74	−0.18	−0.27	0.096
0.	0.	−0.18	0.	0.18	−0.55	0.	0.	0.096	−0.096	−0.18	0.74	0.082	−0.096
0.	0.	0.	0.	0.	−0.18	−0.096	0.096	0.	0.	−0.27	0.082	0.37	0.
0.	0.	0.	0.	−0.18	0.	0.082	−0.27	0.	0.	0.096	−0.096	0.	0.37

```
Fred = Fg;
Do@Fred = deleteRow@Fred, 2 fixnode@@k, 1DD − 2 + fixnode@@k, 2DDD,
 8k, numfix, 1, −1<D
MatrixForm@FredD
```

```
i  0  y
   0
   0
   0
   0
   0
   0
   0
   0
   0
   0
   0
   0
k −2.5 {
```

Solution

```
MatrixForm@Ured = Inverse@KredD.FredD
```

```
i 0.826276 y
   3.40158
   3.37213
   1.48275
   3.02932
  −2.28855
   4.45799
  −7.35188
   9.65709
   1.79781
   9.97215
  −3.31691
   10.0838
k −15.164 {
```

Postprocessing... **Recall that we has assumed a linear displacment function of the element hence, as the name suggest, CST elements (stands for Constant Strain Triangle) have constant strain (and also constant stress) over the entire surface of the element. This makes sense since the in-plane strain is related to the derivative of the in-plane displacement components.**

4-Noded brick elements

Before we get into 4 noded elements, let's introduce the concept of isoparametric formulation. As you will see, it makes things a lot easier.