Chapter 6.5 Thermo-Mechanical Analysis

All mechanical components are subject to thermal loading.

This is either due to processing (heat treating, welding or joining) or due to service conditions (climate changes) or on-off cycling (such as an automobile engine or the components in your computer).

Often it is important to account for the effects of thermal loading on the response of structures (you want your car to function the same whether it is –30^{o}C or +40^{o}C).

General Thermal Stress Problem

In general, a temperature change will cause a material to deform. In two-dimension, the thermal strain looks something like:

$$\boldsymbol{\varepsilon}_T = \begin{Bmatrix} \varepsilon_{xT} \\ \varepsilon_{yT} \\ \gamma_{xyT} \end{Bmatrix}$$

which is related to the change in temperature times the coefficient of thermal expansion. For an isotropic material

in the case of plane stress
$$\boldsymbol{\varepsilon}_T = \begin{Bmatrix} \alpha\Delta T \\ \alpha\Delta T \\ 0 \end{Bmatrix}$$

and in the case of plane strain
$$\boldsymbol{\varepsilon}_T = (1+\nu)\begin{Bmatrix} \alpha\Delta T \\ \alpha\Delta T \\ 0 \end{Bmatrix}$$

The total strain is a combination of strain due to mechanical loading and strain due to thermal loading

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_M + \boldsymbol{\varepsilon}_T = \mathbf{D}^{-1}\boldsymbol{\sigma} + \boldsymbol{\varepsilon}_T$$

which we can rewrite in terms of stress as:

$$\boldsymbol{\sigma} = \mathbf{D}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_T)$$

Let us now look at the potential energy

The total PE is

$$\pi_p = U + \Omega$$

The Internal Strain Energy is now

$$U = \iiint_V \frac{1}{2}\boldsymbol{\sigma}^T \boldsymbol{\varepsilon}\, dV$$

which becomes:

$$U = \iiint_V \frac{1}{2}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_T)^T \mathbf{D}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_T)\, dV$$

recall that we can express the strains in terms of the nodal degrees of freedom as: $\boldsymbol{\varepsilon} = \mathbf{Bu}$

Thus:

$$U = \iiint_V \frac{1}{2}(\mathbf{Bu} - \boldsymbol{\varepsilon}_T)^T \mathbf{D}(\mathbf{Bu} - \boldsymbol{\varepsilon}_T)\, dV$$

Which can be simplified as:

$$U = \iiint_V \frac{1}{2}\left[\mathbf{u}^T\mathbf{B}^T\mathbf{DBu} - \mathbf{u}^T\mathbf{B}^T\mathbf{D}\boldsymbol{\varepsilon}_T - \boldsymbol{\varepsilon}_T\mathbf{DBu} + \boldsymbol{\varepsilon}_T\mathbf{D}\boldsymbol{\varepsilon}_T)\right]dV$$

$$U_L = \iiint_V \frac{1}{2}\mathbf{u}^T\mathbf{B}^T\mathbf{DBu}\, dV$$

$$U_T = \iiint_V \mathbf{u}^T\mathbf{B}^T\mathbf{D}\boldsymbol{\varepsilon}_T\, dV$$

No **u**, will drop out

Lastly, we use the PMPE to obtain the stiffness equations as:

$$\underbrace{\iiint_V \mathbf{B}^T \mathbf{D}\mathbf{B}\, dV}_{\mathbf{K}} \underbrace{\mathbf{u}}_{\mathbf{u}} \underbrace{- \iiint_V \mathbf{B}^T \mathbf{D} \varepsilon_T \, dV - \mathbf{P} - \iiint_V \mathbf{N}^T \mathbf{X}_{body}\, dV - \iint_S \mathbf{N}^T \mathbf{T}_{tract}\, dS}_{\mathbf{f}} = 0$$

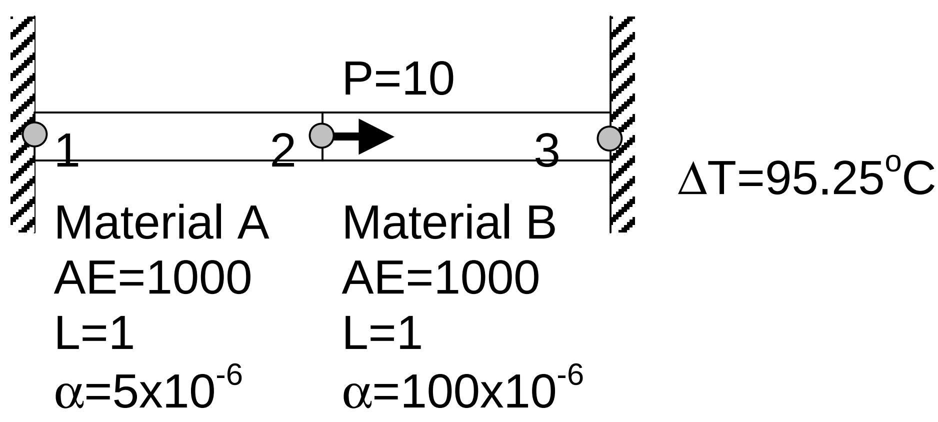
$$\mathbf{K} \quad \mathbf{u} \quad \mathbf{f} \quad = 0$$

Again, in order to evaluate integrals, we transform the integrals in the x-y plane to integrals over the s-t plane from –1 to 1 through the transformation and use Gaussian Quadrature to perform the integration

$$\int_A f(x)\,dxdy = \int_{-1}^{1}\int_{-1}^{1} f(s)\left|J\right| dsdt = 4\sum_{i=1}^{n} W_i \left(\mathbf{B}^T\mathbf{D}\mathbf{B}\right)_i \left|\mathbf{J}\right|_i$$

Example: 1-D 2-node trusses

Consider the simple 1-D truss structure shown below



The global stiffness matrix is:

$$\begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix} = \begin{bmatrix} \frac{A_A E_A}{L_A} & -\frac{A_A E_A}{L_A} & 0 \\ -\frac{A_A E_A}{L_A} & \frac{A_A E_A}{L_A} + \frac{A_B E_B}{L_B} & -\frac{A_B E_B}{L_B} \\ 0 & -\frac{A_B E_B}{L_B} & \frac{A_B E_B}{L_B} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

The force vector is now composed of two components

$$\mathbf{f} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix} = \mathbf{f}_P + \mathbf{f}_T = \begin{Bmatrix} R_{x1} \\ P \\ R_{x3} \end{Bmatrix} + \begin{Bmatrix} -E_A A_A \alpha \Delta T \\ E_A A_A \alpha \Delta T - E_B A_B \alpha \Delta T_2 \\ E_B A_B \alpha \Delta T \end{Bmatrix}$$

which upon substituting for the known quantities yields:

$$\begin{bmatrix} 1000 & -1000 & 0 \\ -1000 & 2000 & -1000 \\ 0 & -1000 & 1000 \end{bmatrix} \begin{Bmatrix} 0 \\ u_2 \\ 0 \end{Bmatrix} = \begin{Bmatrix} R_{x1} - .476 \\ 10 + .476 - 9.52 \\ R_{x3} + 9.52 \end{Bmatrix}$$

The solution is:

$$\mathbf{u} = \begin{Bmatrix} 0 \\ 0.00047 \\ 0 \end{Bmatrix} \qquad R_{x1} = 0 \qquad and \qquad R_{x3} = -10$$

For comparison, if we neglect thermal effects, then:

$$\begin{bmatrix} 1000 & -1000 & 0 \\ -1000 & 2000 & -1000 \\ 0 & -1000 & 1000 \end{bmatrix} \begin{Bmatrix} 0 \\ u_2 \\ 0 \end{Bmatrix} = \begin{Bmatrix} R_{x1} \\ 10 \\ R_{x3} \end{Bmatrix}$$

and the solution is:

$$\mathbf{u} = \begin{Bmatrix} 0 \\ 0.005 \\ 0 \end{Bmatrix} \qquad R_{x1} = -5 \qquad and \qquad R_{x3} = -5$$

Example: 4-node brick element with coupled thermo-mechanical loading

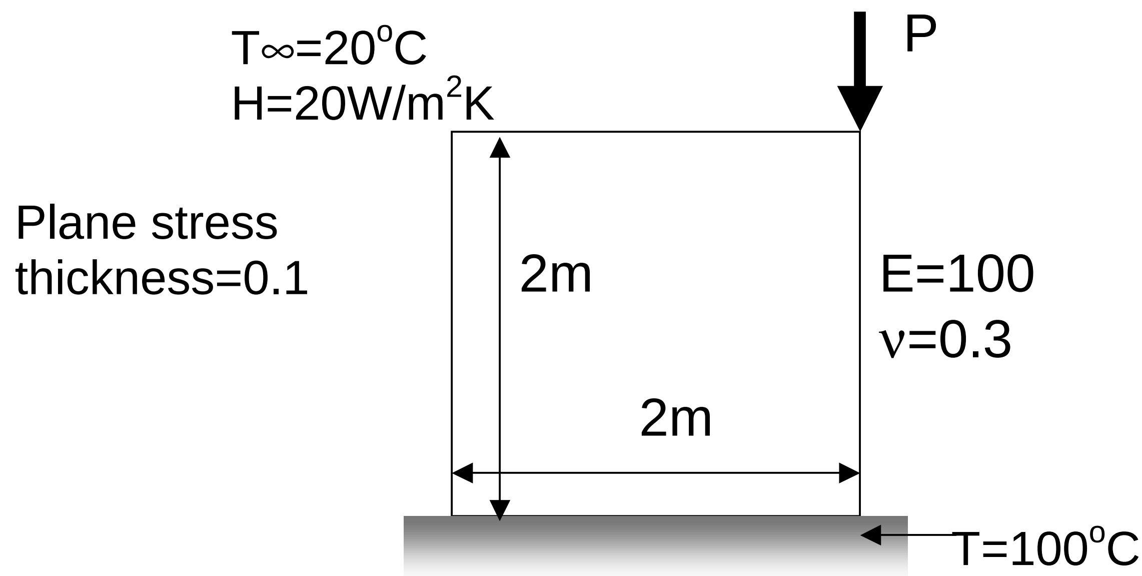
Now we consider a 2-D structure for which there is both mechanical and thermal loading.

The degrees of freedom at each node are now:

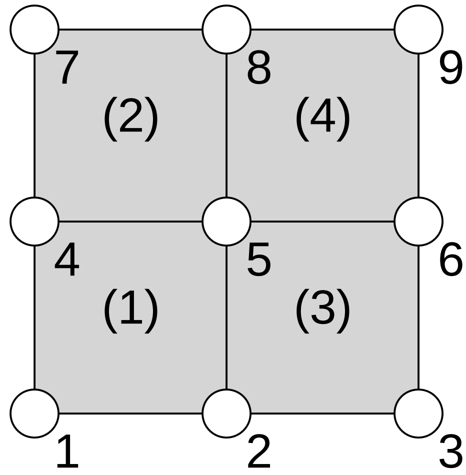
$$u_x, \qquad u_y, \qquad \text{and} \qquad T$$

We have to form both the Stiffness matrix and the Conduction matrix. Here the mechanical loading does not influence the thermal response of the structure; however, the thermal loading affects the mechanical response, thus we have to determine the temperature distribution and use this information to determine the strains resulting from the temperature change.

Consider the following structure
Consider the following simple 2-D problem



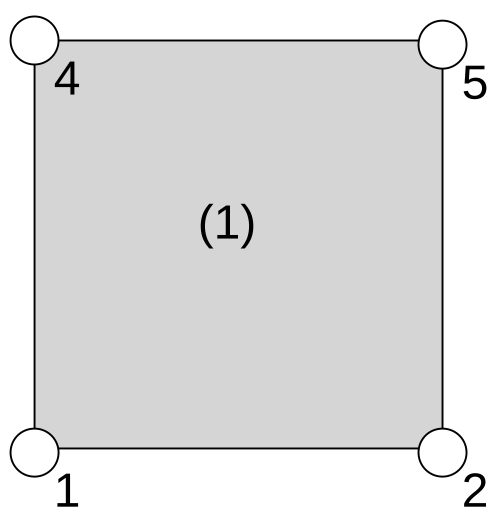
Start by breaking the surface up into elements and assigning node numbers and element numbers



Here we have 9 nodes (with 3 dof per node u_x, u_y and *T*)

Thus we would expect an 18by18 stiffness matrix, an 18by1 displacement vector, an 18by1 force vector and also a 9by9 conduction matrix, a 9by1 temperature vector, and a 9by1 flux vector.

Starting with Element 1



Using the isoparametric formulation, we map from the global coordinate system to the local coordinate system using the following relations for our geometry and nodal degrees of freedom:

$$x = N_i x_i$$
$$y = N_i y_i$$
$$T = N_i T_i$$
$$u_x = N_i u_{xi}$$
$$u_y = N_i u_{yi}$$

Here the shape functions are

$$N_1 = \frac{1}{4}(1-s)(1-t)$$
$$N_2 = \frac{1}{4}(1+s)(1-t)$$
$$N_5 = \frac{1}{4}(1+s)(1+t)$$
$$N_4 = \frac{1}{4}(1-s)(1+t)$$

For the thermal problem we have:

$$T(x,y) = \begin{bmatrix} N_1 & N_2 & N_5 & N_4 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_5 \\ T_4 \end{Bmatrix} \quad \text{or} \quad T = \mathbf{N}_T \mathbf{t}$$

and for the mechanical problem we have

$$\begin{Bmatrix} u_x(x,y) \\ u_y(x,y) \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_5 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_5 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_5 \\ y_5 \\ x_4 \\ y_4 \end{Bmatrix} \quad \text{or} \quad \mathbf{\Psi} = \mathbf{N}_M \mathbf{u}$$

The element conduction equations are formed using the result of Chapter 4.5

$$\left[\iiint_V \mathbf{B}_T{}^T \mathbf{D}_T \mathbf{B}_T \; dv + \iint_{Sh} h(\mathbf{N}_T{}^T \mathbf{N}_T) ds_h\right]\mathbf{t} =$$

$$= \left\{\iiint_V \mathbf{N}_T{}^T Q dV + \iint_{Sq} \mathbf{N}_T{}^T q^* \, ds_q - \iint_{Sh} \frac{1}{2} h(\mathbf{N}_T{}^T T_\infty) ds_h\right\}$$

Once we determine the temperature distribution, we can determine the corresponding mechanical response. Here the element stiffness equations are formed using

$$\left[\iiint_V \mathbf{B}^T \mathbf{D} \mathbf{B} dA\right]\mathbf{u} =$$

$$= \left\{\iiint_V \mathbf{B}^T \mathbf{D} \varepsilon_T dV + \mathbf{P} + \iiint_V \mathbf{N}^T \mathbf{X}_{body} dV + \iint_S \mathbf{N}^T \mathbf{T}_{tract} dS\right\}$$