Chapter 8. Other Cool Stuff

LINEAR ELASTO-DYNAMICS

$$\mathbf{M\ddot{u}} + \mathbf{Ku} = \mathbf{f}_{\varepsilon_o} + \mathbf{f}_b + \mathbf{f}_t + \mathbf{P}$$

where

$$u(x,y,t) = \mathbf{Nu}$$

$$\varepsilon(x,y,t) = \mathbf{Bu}$$

$$\sigma(x,y,t) = \mathbf{D}(\mathbf{Bu} - \boldsymbol{\varepsilon}_o)$$

and the stiffness and mass matrices are:

$$\mathbf{K} = \iiint_V \mathbf{B}^T \mathbf{DB}\,dv \qquad \mathbf{M} = \iiint_V \mathbf{N}^T \rho \mathbf{N}\,dv$$

Note that **M** and **K** have the same form (due to connectivity)

MODAL ANALYSIS

We will start by looking at the free vibration problem

$$\mathbf{M\ddot{u}} + \mathbf{Ku} = 0 \qquad (\mathbf{f} = 0)$$

let $u(t) = \hat{\mathbf{u}}\sin(\omega t + \theta)$ where $\hat{\mathbf{u}}$ are the mode shapes, then by substitution, we have:

$$(\mathbf{K} - \omega^2\mathbf{M})\hat{\mathbf{u}} = 0$$

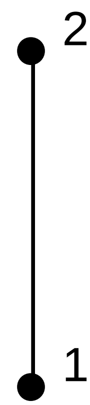
for a nontrivial solution to exist, we require

$$\det(\mathbf{K} - \omega^2\mathbf{M}) = 0$$

from which we can solve for the frequencies and then the corresponding mode shapes.

EXAMPLE: 1-D BAR

Recall that we had:



$$u(s) = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

where

$$N_1 = \frac{1-s}{2} \qquad N_2 = \frac{1+s}{2}$$

and the strains are related to the displacements through:

$$\boldsymbol{\varepsilon}_s = \begin{bmatrix} \frac{-1}{L} & \frac{1}{L} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \mathbf{Bu}$$

And the stress/strain relation as:

$$\boldsymbol{\sigma}_s = E\mathbf{B}\,\mathbf{u}$$

We apply Newton's second law of motion to the 2 nodes

$$F = ma$$

$$f_{1ext} - f_{1\text{int}} = m_1 \frac{\partial^2 u_1}{\partial t^2}$$

$$f_{2ext} - f_{2\text{int}} = m_2 \frac{\partial^2 u_2}{\partial t^2}$$

where $m_1 = \rho \frac{AL}{2}$ and $m_1 = \rho \frac{AL}{2}$ (called the lumped mass)

Writing out the equations, we have:

$$\begin{Bmatrix} f_{1ext} \\ f_{2ext} \end{Bmatrix} = \begin{Bmatrix} f_{1\text{int}} \\ f_{2\text{int}} \end{Bmatrix} + \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix}$$

Consistent Mass matrix

Starting with d'Alembert's principle

$$\mathbf{X}^e = -\rho \ddot{\mathbf{u}}(x, y)$$

where $\mathbf{X}^e$ is the effective body force due to the mass of the element. We can then convert this body force to nodal forces through

$$\mathbf{f}_b = \iiint_V \mathbf{N}^T \mathbf{X}^e \, dv$$

Making the substitution for $\mathbf{X}^e$ and knowing that $\ddot{\mathbf{u}}(x, y) = \mathbf{N}\ddot{\mathbf{u}}$, we have

$$\mathbf{f}_b = \iiint_V \mathbf{N}^T \rho \mathbf{N} \, dv \ddot{\mathbf{u}}$$

ELASTODYNAMICS INCLUDING INELASTICITY

Starting with:

$$\mathbf{M}\ddot{\mathbf{u}} + \underbrace{\mathbf{K}\mathbf{u}}_{\mathbf{s}} = \mathbf{f}$$

where

$$\mathbf{s} = \iiint_V \mathbf{B}^T \boldsymbol{\sigma} \, dv$$

Now including rate effects $\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon} + \mathbf{V}\dot{\boldsymbol{\varepsilon}}$

Therefore

$$\mathbf{s} = \iiint_V \mathbf{B}^T \mathbf{D} \mathbf{B} \, dv \mathbf{u} + \iiint_V \mathbf{B}^T \mathbf{V} \mathbf{B} \, dv \dot{\mathbf{u}}$$

So we have

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{f}$$

SPECIAL CASES

1) No damping-Linear elasto-dynamics

$$\mathbf{M\ddot{u}} + \mathbf{Ku} = \mathbf{f}$$

2) No inertial effects-elasto-statics

$$\mathbf{Ku} = \mathbf{f}$$

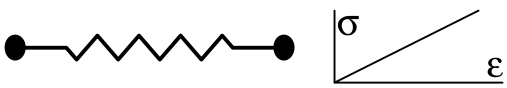
3) Quasi-static-Visco-elasto statics

$$\mathbf{C\dot{u}} + \mathbf{Ku} = \mathbf{f}$$

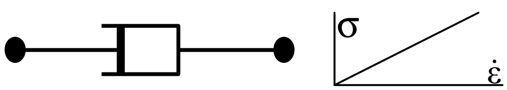
4) Viscous flow (Newtonian fluids)

$$\mathbf{C\dot{u}} = \mathbf{f}$$

RHEOLOGICAL MODELS

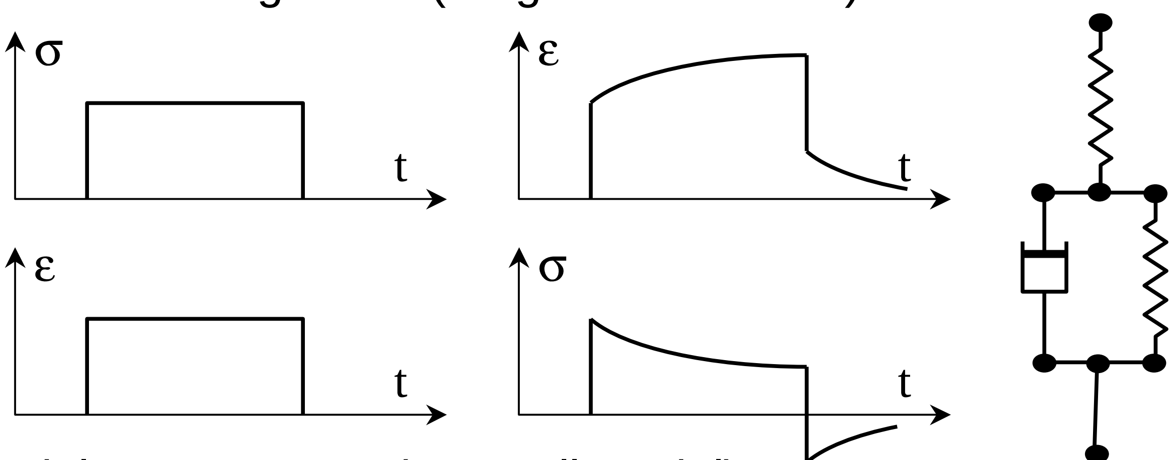


Hookian $\sigma = E\varepsilon$

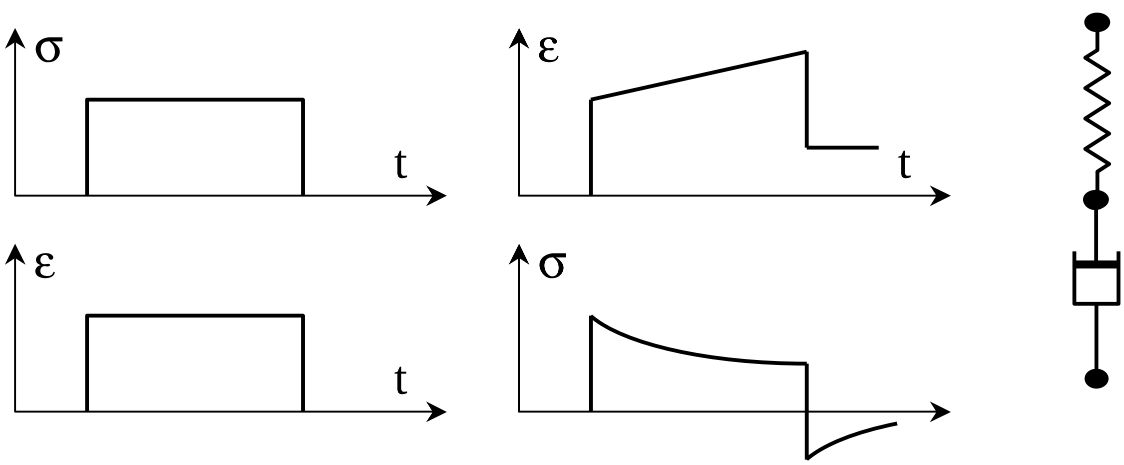


Newtonian $\sigma = \eta\dot{\varepsilon}$

Parallel Arrangement (Voigt-Kelvin Model)



Serial Arrangement (Maxwell model)



During the creep response, the material represented by these models undergoes

1. Initial instantaneous response
2. Non-linear delayed elastic response
3. Instantaneous elastic recovery
4. Delayed elastic-viscoelastic recovery
5. Permanent deformation

For the Maxwell model, the relationship between the deflection and the applied load is:

$$\dot{u} = \frac{\dot{P}}{k} + \frac{P}{\eta}, \qquad u(0) = \frac{P(0)}{k}$$

The resulting creep function for a unit step is:

$$c(t) = \left(\frac{1}{k} + \frac{1}{\eta} t \right) U(t)$$

and the relaxation function is:

$$r(t) = k e^{-(k/\eta)t} U(t)$$

For the Voigt model (slightly different than above), the relationship between the deflection and the applied load is:

$$P = ku + \eta \dot{u}, \qquad u(0) = 0$$

The resulting creep function for a unit step is:

$$c(t) = \frac{1}{k}\left(1 - e^{-(k/\eta)t}\right) U(t)$$

and the relaxation function is:

$$r(t) = \eta \delta(t) + kU(t)$$

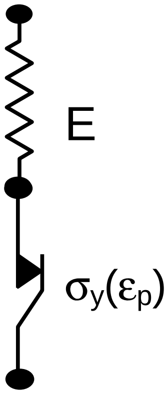
Viscoelastic Overstress – Delayed Elasticity

This element is associated with some threshold condition

$\sigma_y(\varepsilon_p)$

ELASTO-PLASTICITY

(we will look at this in more detail later)



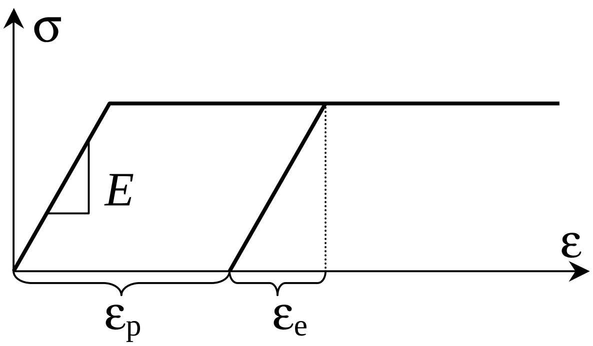
Total Format (Henke)

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_e + \boldsymbol{\varepsilon}_p$$

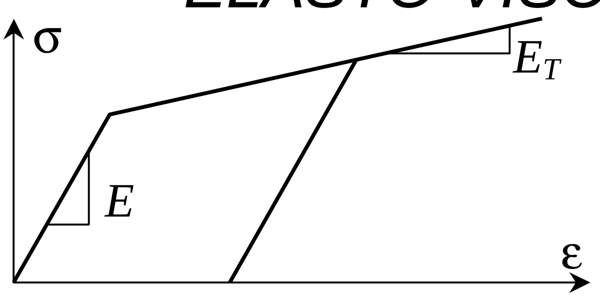
Rate Format (St Venant VonMises Prandtl-Reuss)

$$\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}_e + \dot{\boldsymbol{\varepsilon}}_p$$

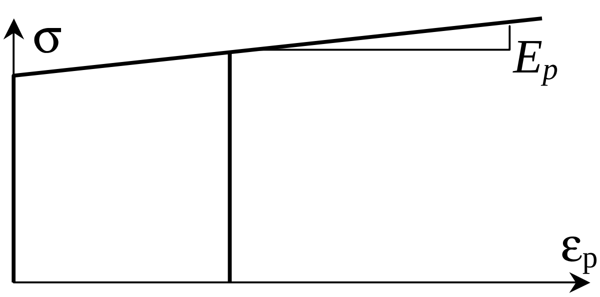
And our stress-strain relation becomes: $\dot{\boldsymbol{\sigma}} = \mathbf{D} : (\dot{\boldsymbol{\varepsilon}} - \dot{\boldsymbol{\varepsilon}}_p)$



ELASTO-VISCO-PLASTICITY



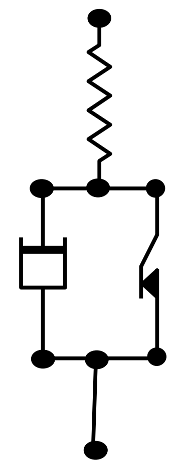
$$E_T = \frac{\partial \sigma}{\partial \varepsilon} = \frac{E\ E_p}{E + E_p}$$



$$E_p = \frac{\partial \sigma}{\partial \varepsilon_p}$$

This is a combination of the three rheological models. Solution procedure is much more complicated than for linear elasticity

Example One Dimensional Viscoplasticity



Consider the basic one-dimensional viscoplastic model. The total strain n the model can be expressed as the sum of the elastic and the viscoelstic components as:

$$\varepsilon = \varepsilon_e + \varepsilon_{vp}$$

The applied stress is related to the elastic strain by:

$$\sigma_e = \sigma = E\boldsymbol{\varepsilon}_e$$

The stress in the dashpot is related to the viscoplastic strain by:

$$\sigma_d = \eta\dot{\boldsymbol{\varepsilon}}_{vp}$$

And the stress in the friction slider is:

$$\sigma_p = \sigma \quad if \quad \sigma_p < Y$$

$$\sigma_p = Y \quad if \quad \sigma_p \geq Y$$

where Y represents the threshold stress which is a function of some yield stress and some strain hardening as:

$$Y = \sigma_y + H'\varepsilon_{vp}$$

Prior to the onset of viscoplastic yielding, $\boldsymbol{\varepsilon}_{vp} = 0$ giving $\sigma_d = 0$ thus $\sigma_p = \sigma$. Combining stresses in the dashpot and the friction slider gives:

$$\sigma = \sigma_y + H'\varepsilon_{vp} + \eta\dot{\varepsilon}_{vp}$$

Using $\varepsilon_{vp} = \varepsilon - \varepsilon_e$ and $\sigma_e = \sigma = E\boldsymbol{\varepsilon}_e$ gives

$$H'E\varepsilon + \eta E\dot{\varepsilon} = H'\sigma + E(\sigma - \sigma_y) + \eta\dot{\sigma}$$

which is a first order ODE, Rearranging, we get

$$\dot{\varepsilon} = \frac{\dot{\sigma}}{E} + \frac{1}{\eta}[\sigma - (\sigma_y + H'\varepsilon_{vp})]$$

or

$$\dot{\varepsilon} = \dot{\varepsilon}_e + \dot{\varepsilon}_{vp}$$

Considering the case when we apply a constant stress to the model

$$H'E\varepsilon + \eta E\dot{\varepsilon} = H'\sigma_A + E(\sigma_A - \sigma_y)$$

The solution is:

$$\varepsilon(t) = \frac{\sigma_A}{E} + \frac{(\sigma_A - \sigma_y)}{H'}[1 - e^{-H't/\eta}]$$

Note: the solution to ODE's of the form $y' + p\,y = r$ is

$y(t) = e^{-h}\int e^h r dt + C$ where: $h = \int p dt$

In the case of a perfectly viscoplastic material, ($H' = 0$) then we have (by applying L'Hopital's rule):

$$\varepsilon(t) = \frac{\sigma_A}{E} + \frac{(\sigma_A - \sigma_y)t}{\eta}$$

Viscoplasticity is a transient phenomena, thus the solution involves taking a time incremental (time-stepping) approach. The simplest approach is to use Euler's rule where we extrapolate the value at some time t_{n+1} in terms of the quantities at time t.

Using this approach, we can define the viscoplastic strain increment over time step $\Delta t_n = t_{n+1} - t_n$ as:

$$\Delta\varepsilon_{vp}^n = \dot{\varepsilon}_{vp}^n \Delta t_n$$

The change in length of the element due to the strain increment is:

$$\Delta u^n = \Delta\varepsilon_{vp}^n L$$

and adding this to the change in length due to the applied loading gives:

$$\Delta u^n = \Delta \varepsilon_{vp}^n L + \frac{L}{AE} \Delta P_n$$

and rewriting in matrix form gives:

$$\Delta u^n = \begin{Bmatrix} \Delta u_1^n \\ \Delta u_2^n \end{Bmatrix} = \mathbf{K}^{-1} \Delta F_n$$

Where

$$\Delta F_n = AE \dot{\varepsilon}_{vp}^n \Delta t_n \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} + \Delta P_n$$

and

$$\mathbf{K} = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Rewritting in the standard form

$$\mathbf{K} = \int_V \mathbf{B}^T \mathbf{D} \mathbf{B} dv$$

$$\Delta F_n = \int_V \mathbf{B}^T \mathbf{D} \varepsilon dv + \Delta P_n$$

The updated displacements at obtained as:

$$u^{n+1} = u^n + \Delta u^n$$

The stress increment is:

$$\Delta \sigma^n = E \Delta \varepsilon_e^n = E(\Delta \varepsilon^n - \Delta \varepsilon_{vp}^n)$$

$$= E\left(\frac{\Delta u_1^n - \Delta u_2^n}{L} - \dot{\varepsilon}_{vp}^n \Delta t_n \right)$$

The stress at time t_{n+1} is:

$$\sigma^{n+1} = \sigma^{n} + \Delta\sigma^{n}$$

The viscoplastic strain is:

$$\varepsilon_{vp}^{n+1} = \varepsilon_{vp}^{n} + \Delta\varepsilon_{vp}^{n}$$

And lastly, the viscoplastic strain rate is:

$$\dot{\varepsilon}_{vp}^{n+1} = \frac{1}{\eta}[\sigma^{n+1} - (\sigma_y + H'\varepsilon_{vp}^{n+1})]$$

The out of balance residual forces as expressed as the sum of the applied nodal loads and the nodal forces equivalent to the elemental stresses are:

$$\Psi_{n+1} = A\sigma^{n+1}\begin{Bmatrix} 1 \\ -1 \end{Bmatrix} + P_{n+1}$$

These residual forces are added to the pseudo forces to give the forces for the next time increment as:

$$\Delta F_{n+1} + = AE\dot{\varepsilon}_{vp}^{n+1}\Delta t_{n+1}\begin{Bmatrix} 1 \\ -1 \end{Bmatrix} + \Delta P_{n+1} + \Psi_{n+1}$$

This step is repeated until the solution for the desired time duration or steady state conditions are achieved (when the viscoplastic strain rate becomes negligible).

There is a limit on the time step that one can use for the viscoplastic solution. For the one-dimensional case considered here, the limiting value is (Cormeau proposed this one and there are many different values that have been proposed):

$$\Delta t \le \frac{\eta\sigma_y}{E}$$

Computational Implementation of 1-D Viscoplasticity

STEP 1 At time $t=t_n$, we compute that following quantities using the standard approach

$$\sigma^n, \quad \varepsilon^n, \quad \varepsilon_{vp}^n, \quad f^n, \text{ and } u^n \quad \text{known}$$

and compute the viscoplastic strain rate for each element as:

$$\dot{\varepsilon}_{vp}^n = \frac{1}{\eta}[\sigma^n - (\sigma_y + H'\varepsilon_{vp}^n)]$$

STEP 2 Compute the displacement increment according to:

$$\Delta u^n = \begin{Bmatrix} \Delta u_1^n \\ \Delta u_2^n \end{Bmatrix} = \mathbf{K}^{-1}\Delta F_n$$

where

$$\Delta F_n = AE\dot{\varepsilon}_{vp}^n \Delta t_n \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} + \Delta P_n$$

and for each element, the stiffness matrix is

$$\mathbf{K} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

STEP 3 Compute the stress increment and the viscoplastic strain increment for each element as:

$$\Delta\sigma^n = E\left(\frac{\Delta u_1^n - \Delta u_2^n}{L} - \dot{\varepsilon}_{vp}^n \Delta t_n\right)$$

$$\Delta\varepsilon_{vp}^n = \dot{\varepsilon}_{vp}^n \Delta t_n$$

STEP 4 Determine the total displacements, stresses and the viscoplastic strains

$$u^{n+1} = u^n + \Delta u^n$$

$$\sigma^{n+1} = \sigma^n + \Delta\sigma^n$$

$$\varepsilon_{vp}^{n+1} = \varepsilon_{vp}^n + \Delta\varepsilon_{vp}^n$$

STEP 5 Compute the Viscoplastic strain rate for each element for the next time step

$$\dot{\varepsilon}_{vp}^{n+1} = \frac{1}{\eta}[\sigma^{n+1} - (\sigma_y + H'\varepsilon_{vp}^{n+1})]$$

STEP 6 Evaluate the residual forces by applying equilibrium correction to each element

$$\Psi_{n+1} = A\sigma^{n+1}\begin{Bmatrix} 1 \\ -1 \end{Bmatrix} + \Delta P_{n+1}$$

and add these into the vector of pseudo nodal loads to be used in the next time step

$$\Delta F_{n+1} = AE\dot{\varepsilon}_{vp}^{n+1}\Delta t_{n+1}\begin{Bmatrix} 1 \\ -1 \end{Bmatrix} + \Delta P_{n+1} + \Psi_{n+1}$$

STEP 7 Check if the viscoplastic strain rate in each element has become negligibly small. If so, then steady state conditions are said to have been reached. If not, return to STEP 1 and repeat the entire process for the next time step. One way to check for convergence is as follows:

$$\frac{\sum_{i=1}^{M}\left|\left(\Delta\varepsilon_{vp}^{n}\right)_i\right|}{\sum_{i=1}^{M}\left|\left(\Delta\varepsilon_{vp}^{1}\right)_i\right|} \times 100 \le TOLERANCE$$