Chapter 9. Even More Cool Stuff

ELASTO-PLASTIC PROBLEMS IN 2-D

Step 1 We begin by defining the yield criterion ($F(\sigma, q) = 0$ in principle stress space). This describes the stress level at which plastic deformation begins. Physically, the yield criterion should be independent of the coordinate system. This is typically accomplished by making it a function of the three stress invariants only which are defined as:

$$J_1 = \sigma_{ii} \qquad J_2 = \frac{1}{2}\sigma_{ij}\sigma_{ij} \qquad J_3 = \frac{1}{3}\sigma_{ij}\sigma_{jk}\sigma_{kl}$$

We note that typically, plastic deformation is independent of the hydrostatic pressure. As a result, the yield criterion depends on the second and third invariants of the deviatoric stress J_2' and J_3', which are determined from

$$\sigma_{ij}' = \sigma_{ij} - \frac{1}{3}\delta_{ij}\sigma_{kk}$$

For example, the VonMises yield criterion (1913) states that yielding occurs when J_2' reaches some critical value as:

$$\left(J_2'\right)^{1/2} = k(\kappa)$$

where the second invariant of the deviatoric stress can be explicitly written as:

$$J_2' = \frac{1}{2}\sigma_{ij}'\sigma_{ij}' = \frac{1}{6}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]$$

$$= \frac{1}{2}[\sigma_x'^2 + \sigma_y'^2 + \sigma_z'^2] + \tau_{xy}^2 + \tau_{yz}^2 + \tau_{xz}^2$$

Furthermore, the yield criterion can be written as:

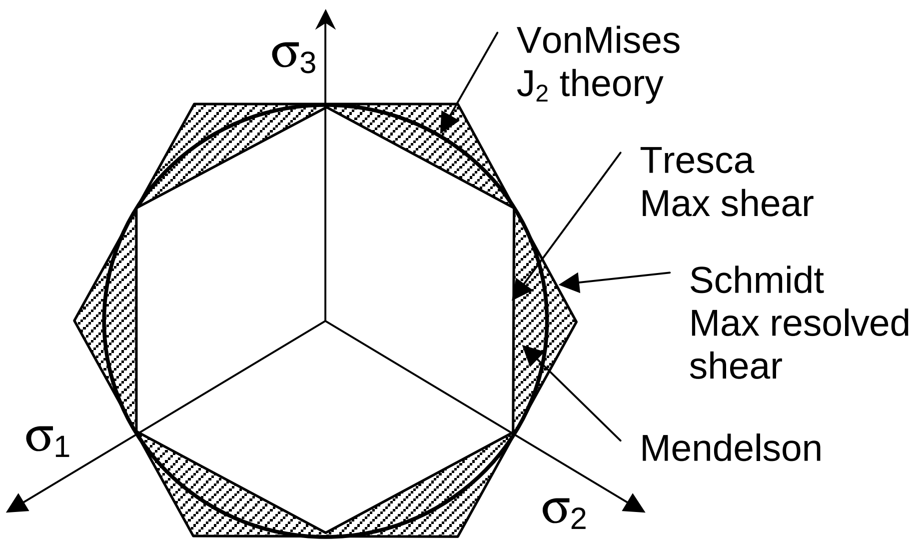
$$\overline{\sigma} = \sqrt{3}\left(J_2'\right)^{1/2}$$

where $\overline{\sigma}$ is called the *effective stress, generalized stress* or *physical stress*.

In general, we write the yield criterion as:

$$F(\sigma, q) = 0$$

The following figure shows several different representations of yield criteria plotted in the principle stress space. Note that there are many different yield criteria and even many different modifications to the ones shown below. For example, the Drucker-Prager (1952) yield criterion is a modification of the VonMises yield criterion to account for the influence of hydrostatic stress component.



If $\dot{F}(\sigma, q) < 0$ then we have elastic unloading.

If $\dot{F}(\sigma, q) = 0$ then we have plastic loading and the stress remains on the yield surface for a perfectly plastic material.

If $\dot{F}(\sigma, q) > 0$ then we have plastic loading for a hardening material and the stress remains on the yield surface, which changes with the amount of plastic deformation.

where

$$\dot{F} = \frac{\partial F}{\partial \boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}} = \frac{\partial F}{\partial \boldsymbol{\sigma}} : \mathbf{D} : (\dot{\boldsymbol{\varepsilon}} - \dot{\boldsymbol{\varepsilon}}_p)$$

At yielding

$$\dot{F} = \frac{\partial F}{\partial \boldsymbol{\sigma}} : \mathbf{D} : (\dot{\boldsymbol{\varepsilon}} - \dot{\boldsymbol{\varepsilon}}_p) = 0 = \mathbf{n}\dot{\boldsymbol{\sigma}}$$

$$= \mathbf{n} : \mathbf{D} : \dot{\boldsymbol{\varepsilon}} - \mathbf{n} : \mathbf{D} : \dot{\boldsymbol{\varepsilon}}_p$$

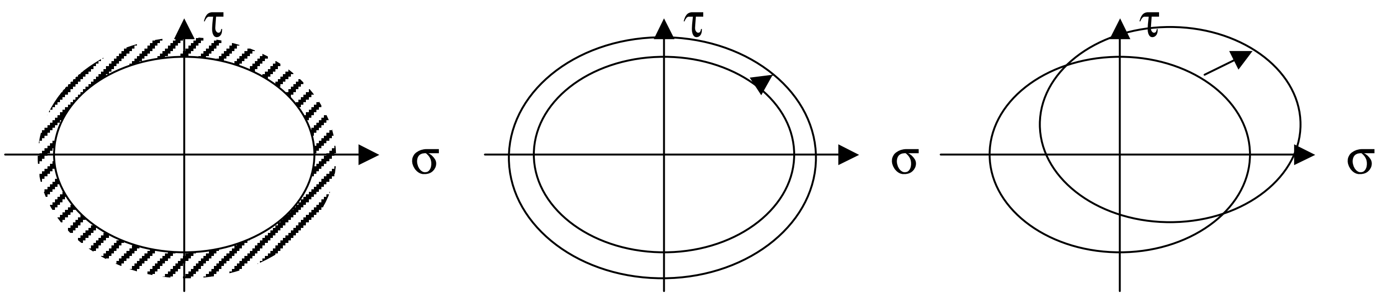
where $\mathbf{D} : \dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\sigma}}_{trial}$

In a strain controlled arrangement

i) Plastic loading when $\dot{\mathbf{n}} : \dot{\boldsymbol{\sigma}}_{trial} \geq 0$

ii) Elastic unloading when $\dot{\mathbf{n}} : \dot{\boldsymbol{\sigma}}_{trial} < 0$

Work and strain hardening – Even after initial yielding has occurred, the yield surface may change shape. This is called hardening. The yield surface may expand equally in all direction with increasing plastic deformation. This is call isotropic strain hardening. Or it may translate with increasing plastic deformation. This is called kinematic strain hardening. A perfectly plastic material does not experience strain hardening and the yield surface remains the same after further plastic deformation.



a) perfectly plastic b) isotropic hardening c) kinematic hardening

Note that some materials may not exhibit strain hardening but rather strain softening, a decrease in the size of the yield surface with plastic deformation, an example is soil.

After initial yielding, the material will experience a behavior that consist of an elastic part and a plastic part as:

$$\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}_e + \dot{\boldsymbol{\varepsilon}}_p$$

The elastic strain rate is related to the stress rate through

$$\dot{\sigma}_e = \mathbf{D}\dot{\boldsymbol{\varepsilon}}_e \qquad \text{or} \qquad \dot{\sigma}_{ij} = D_{ijkl}\dot{\varepsilon}^e_{kl}$$

Or splitting this into the hydrostatic and the deviatoric parts we get, in indicial notation

$$\dot{\varepsilon}^e_{ij} = \frac{\dot{\sigma}_{ij}}{2E} + \frac{1-2\upsilon}{E}\delta_{ij}\dot{\sigma}_{kk}$$

Next we look at the plastic strain rate. We assume that the plastic strain rate is proportional to the stress gradient of a term called the plastic potential *Q*. This equation is called the plastic flow rule and represents the evolution law that must be followed once yielding has occurred.

$$\dot{\boldsymbol{\varepsilon}}_p = \dot{\lambda}_p \frac{\partial Q}{\partial \sigma} = \dot{\lambda}\mathbf{m}$$

where $\dot{\lambda}_p$ is called the *plastic multiplier* and $\frac{\partial Q}{\partial \sigma} = \mathbf{m}$ is the direction. Here Q is called the *plastic potential* and represents the gradient of the flow rule. Thus, $\frac{\partial Q}{\partial F}$ would be the non-associativity, which characterizes the difference between the normal and the gradient of the flow rule. The physical argument behind the non-associated flow rule is to control the volumetric part, i.e. the dilatency. Note: for an associated flow rule $F = Q$ and hence, $\mathbf{n} = \mathbf{m}$.

Once we determine the stress level we apply the consistency condition to determine whether we are staying on the yield surface as:

$$\dot{F} = \frac{\partial F}{\partial \sigma}\dot{\sigma} + \frac{\partial F}{\partial q}\dot{q} = 0$$

Form this equation, we can determine the plastic multiplier as:

$$\dot{\lambda}_p = \frac{1}{\mathbf{H}_p + \mathbf{n} : \mathbf{D} : \mathbf{m}} \mathbf{m} : \mathbf{D} : \dot{\varepsilon}$$

Lastly, we can uses these results to determine the tangential stress-strain relation

$$\dot{\boldsymbol{\sigma}} = \mathbf{D}_{ep}\dot{\boldsymbol{\varepsilon}}$$

$$\mathbf{D}_{ep} = \mathbf{D} - \frac{1}{\mathbf{H}_p + \mathbf{n} : \mathbf{D} : \mathbf{m}} (\mathbf{D} : \mathbf{m} \otimes \mathbf{n} : \mathbf{D})$$

Physically, we think of a material as being elastic prior to yielding with stress strain relation characterized by the elastic modulus. After yielding, the stress strain relation is no longer linear where the local tangent to the stress strain curve varies. The slope of this curve is called the elasto-plastic tangent modulus $\mathbf{D}_{ep}$

Simplification: Associated Flow Without Hardening

Step 2 Adopt a plastic flow rule (Evolution law)

$$\dot{\boldsymbol{\varepsilon}}_p = \dot{\lambda}_p \frac{\partial F}{\partial \sigma} = \dot{\lambda}\mathbf{n}$$

where $\dot{\lambda}_p$ is the plastic multiplier and $\mathbf{n}$ is the direction. Note: the plastic strain rate is directed along the gradient of the flow rule, i.e. normality, (this is not valid for frictional materials).

Step 3 Apply consistency condition to determine the plastic multiplier

$$\dot{F} = \frac{\partial F}{\partial \sigma}\dot{\sigma} = 0$$

$$\dot{F} = \frac{\partial F}{\partial \boldsymbol{\sigma}} : \mathbf{D} : (\dot{\boldsymbol{\varepsilon}} - \dot{\boldsymbol{\varepsilon}}_p) = 0 = \mathbf{n}\dot{\boldsymbol{\sigma}}$$

$$= \mathbf{n} : \mathbf{D} : \dot{\boldsymbol{\varepsilon}} - \mathbf{n} : \mathbf{D} : \dot{\boldsymbol{\varepsilon}}_p = \mathbf{n} : \mathbf{D} : \dot{\boldsymbol{\varepsilon}} - \mathbf{n} : \mathbf{D} : (\dot{\lambda}\mathbf{n})$$

therefore

$$\dot{\lambda}_p = \frac{\mathbf{n} : \mathbf{D} : \dot{\varepsilon}}{\mathbf{n} : \mathbf{D} : \mathbf{n}}$$

Step 4 Determine the tangential stress-strain relation

$$\dot{\boldsymbol{\sigma}} = \mathbf{D}_{ep}\dot{\boldsymbol{\varepsilon}}$$

$$\mathbf{D}_{ep} = \mathbf{D} - \frac{1}{\mathbf{n} : \mathbf{D} : \mathbf{n}}(\mathbf{D} : \mathbf{n} \otimes \mathbf{n} : \mathbf{D})$$

1. Coaxiality – the plastic strain rate is parallel to the stress
2. Normality – the plastic strain rate is directed along the gradient of the flow rule

REALLY SIMPLE 1-D EXAMPLE

(Predictor-Corrector Method)

We decompose the

$$d\boldsymbol{\varepsilon} = d\boldsymbol{\varepsilon}_e + d\boldsymbol{\varepsilon}_p$$

and define the strain hardening parameter as

$$H' = \frac{d\sigma}{d\varepsilon_p}$$

This can be thought of as the slope of the stress- plastic strain curve

$$H' = \frac{d\sigma}{d(\varepsilon - \varepsilon_p)} = \frac{E_T}{1 - E_T / E}$$

When the response of the material is elastic, the applied stress is less than the yield stress, the force is related to the displacement for the element through:

$$\mathbf{F} = \mathbf{K_e u} \qquad \text{where} \qquad \mathbf{K}_e = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Suppose that this force is increased until yielding begins to occur. Then the change in length of the element due to a change in the load is:

$$du = (d\varepsilon_e + d\varepsilon_p) L$$

And the change in load is:

$$dF = A d\sigma = AH' d\varepsilon_p$$

The tangential stiffness of the material then becomes

$$K_{ep} = \frac{dF}{du} = \frac{AH' d\varepsilon_p}{L(Ed\sigma + d\varepsilon_p)}$$

Rewriting this we get

$$K_{ep} = \frac{EA}{L}\left(1 - \frac{E}{E + H'}\right)$$

Lastly, we can express the stiffness for elasto-plastic behavior of an element as:

$$\mathbf{K}_{ep} = \frac{EA}{L}\left(1 - \frac{E}{E + H'}\right)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Note that the first part of this represents the elastic stiffness and the second part represents a decrease in the elastic stiffness due to yielding

The solution procedure involves using the **tangential stiffness method** (or a slightly modified Newton Raphson method) where upon the application of a load increment, we do the following.

STEP 1 We compute the tangential stiffness at the trial displacement state $\mathbf{u}_n$ according to:

$$\mathbf{K}_{ep}^{n+1} = \frac{EA}{L}\left(1 - \frac{E}{E + H'}\right)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

STEP 2 We compute the residual force vector according to:

$$\mathbf{K}_{ep}^{n+1}\mathbf{u}_n - \mathbf{f}_n = \mathbf{\Psi}_n$$

STEP 3 We compute a correction to the trial value

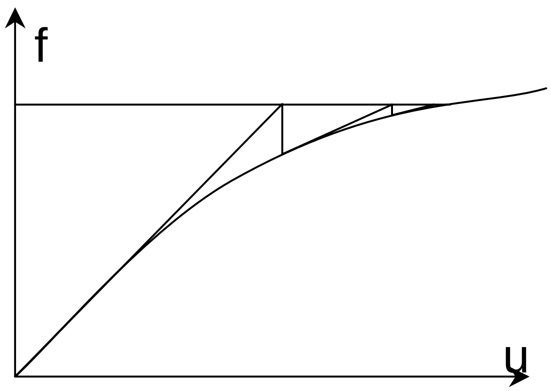
$$\Delta\mathbf{u}_n = (\mathbf{K}_{ep}^{n+1})^{-1}\mathbf{\Psi}_n$$

STEP 4 We update the solution

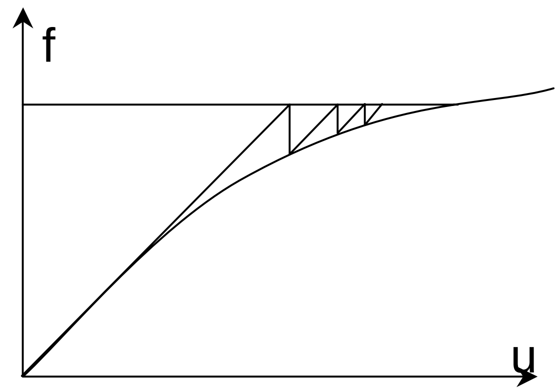
$$\mathbf{u}_{n+1} = \mathbf{u}_n + \Delta\mathbf{u}_n$$

STEP 5 Check for convergence, if not, then go to STEP 1.

(Note: Alternative techniques can be used such as the standard Newton-Raphson method or the Initial Stiffness method.)



Tangential Stiffness method



Initial Stiffness method

BACK TO 2-D ELASTO-PLASTICITY

(Elastic Predictor-Plastic Corrector)

Example: von Mises J2 theory

STEP 1 We compute the element stiffness and the tangential stiffness. For the first iteration, we basically assume that the material behavior is linear elastic. For subsequent iterations, we will update the stiffnesses to include the plastic effects which will cause a reduction in the elastic stiffness.
STEP 2 Solve the system of equations
STEP 3 Next we calculate the residual force vector as:
STEP 4 And the effective stress level is computed using:
STEP 5 Determine the flow vector as:
STEP 6 Lastly, we check to see whether the solution has converged. If not, return to step 1.
Note the difference in the approach here is that we will be using the stresses as the principle variable in the analysis. In the One-dimensional example presented earlier, we used the nodal displacements.