The Invariance of the Moment of Inertia of Magic Squares.

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Abstract

Magic squares are characterised by having the sum of the elements of all rows, columns, and main diagonals having the same sum. A new "physical" invariance for magic squares is reported for the "moment of inertia" of these squares which depends only on the order of the square. The numbers in the magic square are replaced by corresponding multiples of a unit mass placed on a square unit lattice.

1 Introduction

Classical magic squares of the whole numbers, $1...N^2$, have the same line sum (magic constant) for each row, column, and main diagonals [1]:

$$C_N = \frac{N}{2}(N^2 + 1) \tag{1}$$

This line sum invariance depends only on the order, N, of the magic square. Here we report a second invariance which was found by considering the analogue of the physical moment of inertia of a square array of masses which are taken to be proportional to the numbers in the magic square. In the mathematical context this would be called the second moment.

The smallest and earliest is the ancient Chinese 3×3 Lo-shu magic square of the sequence 1..9, with a magic constant of 15, as shown below:

$$\boxed{\begin{array}{ccc} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{array}} \tag{2}$$

Define the moment of inertia, I_N, of a magic square of order N about an axis perpendicular to its centre by summing mr^2 for each cell, where m is the number centred in a cell and r is the distance of the centre of that cell from the centre of the square measured in units of the nearest neighbour distance. The corner cells then have their centres at a distance of $\sqrt{2}$ from the axis. We can now calculate the sum for the 3×3:

$$I_3 = [1 + 3 + 7 + 9]\,(1)^2 + [2 + 4 + 6 + 8]\left(\sqrt{2}\right)^2 = 60 \tag{3}$$

The moments of inertia about the horizontal and vertical axes through the centre are each 30, reminding us of the perpendicular axis theorem [2], which says that their sum gives the moment of inertia about the axis through the centre and perpendicular to the plane.

One can calculate the moment of inertia for larger magic squares in a similar manner, but while the 3×3 is unique, the number of squares grows very rapidly, numbering 880 for the 4×4's, $275, 305, 224$ for the 5×5's, and no exact count for the 6×6's where a recent Monte Carlo simulated annealing computation [3] gives an estimate of $(0.17745 \pm 0.00016) \times 10^{20}$. In 1998 Ollerenshaw and Brée [4] achieved a remarkable breakthrough by finding the first formula for counting a special class of magic squares, the so-called pandiagonal most-perfect ones which have doubly even order. Recently we used a data file [5] for the complete set of the 4×4's and we were **surprised** to find that they all gave $I_4 = 340$. Since Dudeney's classification [4] groups them into 12 classes, we had expected some variety due to the different distribution of the numbers. One of these 4×4 squares is:

$$\begin{array}{|cccc|} \hline 16 & 2 & 3 & 13 \\ 5 & 11 & 10 & 8 \\ 9 & 7 & 6 & 12 \\ 4 & 14 & 15 & 1 \\ \hline \end{array} \tag{4}$$

The perpendicular axis theorem means that we only have to calculate the moment of inertia about a horizontal, I_x (or vertical, I_y) axis in the plane of the square through its centre of mass (located at the centre of the square), and then double it to obtain I_z. Since each row (or column) has the same mass (line sum, C_N) the calculation of I_x reduces to summing the squared distances of the rows from the centre of the square. Furthermore if we calculate I_x about the top (or bottom) edge of the square,

$$I_{edge} = C_N \left[0^2 + 1^2 + 2^2 + ... + (N-1)^2 = \frac{1}{6}\,(N-1)\,N\,(2N-1)\right] \tag{5}$$

then the parallel axis theorem enables us to obtain I_x through the centre of the square by subtracting the mass of the square (NC_N) times the square of distance between the centre and an edge, $(N-1)/2$, so that

$$I_N \equiv I_z = 2\left(I_{edge} - NC_N\left[(N-1)/2\right]^2\right) \tag{6}$$

which reduces to:

$$I_N = \frac{1}{12}N^2(N^4 - 1) \tag{7}$$

This remarkably simple formula [6] recovers the results for $N = 3, 4$ above and is valid for arbitrary order. The derivation of the formula only depends on the row and column properties, and not on the diagonals of magic squares, so that it actually applies to the larger class of **semi-magic** squares which lack one or both diagonal magic sums of magic squares. Semi-magic squares include the magic squares as a special case. The formula for I_N (7) is consistent for large N with the moment of inertia of a uniform (continuous) square plate of mass M and side L, $I = \frac{1}{6}ML^2$, where $M = NC_N$ and $L = N$ [2].

Recently the author has constructed a $12,544 \times 12,544$ magic square with W. Chan [7] for which one could calculate its moment of inertia with the new result above with much greater ease than with a brute force approach.

1.1 Acknowledgment

The referee suggested the present derivation which simplifies my original approach which needed separate analysis for even and for odd orders, for the same final result.

References

[1] K. Ollerenshaw and H. Bondi, *Magic squares of order four.* Philosophical Transactions of the Royal Society of London A306, 443-532 (1982).

[2] E.g. P. Tipler, *Physics*, Worth Publishers, New York, NY (1976), 313-316, or K. R. Symon, *Mechanics*, 3rd edition, Addison-Wesley (1971).

[3] K. Pinn and C. Wieczerkowski, *Number of Magic Squares from Parallel Tempering Monte Carlo.* International Journal of Modern Physics C, Vol. 9, No. 4, 541-546 (1998).

[4] K. Ollerenshaw and D. S. Brée, *Most-perfect pandiagonal magic squares: their construction and enumeration.* Southend-on-Sea, U.K.: The Institute of Mathematics and its Applications (1998).

[5] H. Heinz, www.geocities.com/~harveyh/Downloads/MS4_List-Group.zip

[6] We have assumed the sequence $1 \ldots N^2$, but note that mathematical studies often use the sequence $0 \ldots \left(N^2 - 1\right)$, in which case the inertia formula is: $\frac{N^2}{12}\left(N^2 - 1\right)^2$.

[7] W. Chan and P. D. Loly, *Iterative compounding of square matrices to generate large-order magic squares.* Mathematics Today (IMA), Vol. 38(4), 113-118, August (2002).