Time-Domain Electromagnetic Plane Waves in Static and Dynamic Conducting Media: I

Joe LoVetri, *Member, IEEE*, and Joachim B. Ehrman

***Abstract*—Solutions are derived for the time-domain Maxwell equations for static ($J = \sigma E$) and dynamic ($\tau\partial J/\partial t + J = \sigma_0 E$) conducting media where the field is assumed to vary with respect to only one spatial direction, i.e., plane-wave propagation. The plane wave is introduced into the media via the imposition of an electric field boundary condition at the plane boundary of a half-space and it is assumed that the fields inside the half-space are initially zero. Solutions are derived directly from the first-order system of partial differential equations and it is shown that once the electric field at the plane boundary is imposed, the magnetic field is automatically determined for causal solutions. It is shown that the form of the Maxwell equations, without a magnetic conductivity term added, is sufficient to allow well and uniquely defined solutions of this problem.**

I. Introduction

THE PROBLEM of finding time-domain solutions to electromagnetic fields in a dissipative medium is not a new one. The reason this problem has not received as much attention, in the time domain, as compared to monochromatic excitations is probably due to the complicated analytic expressions which arise. If one searches the literature, under the disciplines where time-domain results are important, one finds that the subject has a substantive set of published work [1]–[9]. Work has continued in this area and, recently, a general formulation in terms of the singularity-expansion method (SEM), of the more complicated problem of scattering by perfectly conducting objects in lossy media has been given by Baum [10]. There seems to be a renewed interest in the more fundamental transient plane-wave problem, for generalized conducting media [11]–[13].

The reason for this renewed interest is due to a controversy over whether or not a solution to the electromagnetic fields in a homogeneous conducting medium, for the transient plane wave, can be obtained without the inclusion of a magnetic conductivity term in the Maxwell equations [11]–[13]. Recently, a challenge with regards to this problem, has been made to the electromagnetics community [14]. It is suggested in [14], that the subtle difficulties in finding a solution, although disputed, are somehow amplified when one deals with a dynamic conducting model, $\tau\partial J/\partial t + J = \sigma_0 E$, of the medium. In this paper, we hope to address most of the concerns raised by Harmuth and others. Since most of the concerns are subtle and mathematical in nature we will try to maintain mathematical rigor and thus may be repeating some of the results given in the older literature. We will attempt to reference similar results in previous literature.

Manuscript received June 28, 1993; revised January 14, 1994.
J. LoVetri is with the Department of Electrical Engineering, University of Western Ontario, London, Ont., Canada N6A 5B9.
J. B. Ehrman is with the Department of Applied Mathematics, University of Western Ontario, London, Ont., Canada, N6A 5B7.
IEEE Log Number 9401884.

It has been asserted by Harmuth that the Maxwell equations require modification in order to obtain the electric and magnetic fields of a nonmonochromatic wave in a lossy medium [11]. To support this view, he has advanced two entirely separate arguments. The first is that constitutive relations which use tensor, or more simply, scalar frequency-dependent ε and μ ([13, "Reply by Harmuth," eq. (6), p. 190]) are not derivable from the Maxwell equations, that they constitute a nontrivial modification of the Maxwell equations, and that they are insufficiently general. But constitutive relations are supplementary to the Maxwell equations and do not represent a modification of the Maxwell equations at all. It has never been claimed that the use of $\varepsilon(\omega)$ and $\mu(\omega)$ is absolutely general even for a uniform isotropic medium, since they assume a linear and spatially local relationship between the electric flux density $\boldsymbol{D}(\boldsymbol{x},t)$ and the electric field intensity $\boldsymbol{E}(\boldsymbol{x},t)$, and between the magnetic flux density $\boldsymbol{B}(\boldsymbol{x},t)$ and the magnetic field intensity $\boldsymbol{H}(\boldsymbol{x},t)$ (see Jackson, [15, p. 309]). However, scalars ε, μ, and σ dependent on ω only are applicable to a wide variety of media and to fields short of those magnitudes occurring in nonlinear optics. In fact, for sea water, which Harmuth suggests as a medium for testing the conflicting views on the need to modify the Maxwell equations, it is quite satisfactory to take ε, μ, and σ as frequency-independent constants provided that the only significant frequency components are $< 10^{10}$ [Hz]. If frequency components $> 10^{10}$ [Hz] are important, then constant ε, μ, and σ are unsatisfactory for sea water so that calculations such as those of Boules [16], using constant ε, μ, and σ, become invalid. (By the way, above 10^{10} [Hz], why not refer to sea water as simply water?)

The second, entirely different, argument of Harmuth claims that unless the Faraday Law is modified by the introduction of a magnetic current term, the Maxwell equations constitute an ill-posed problem in the case of a jump discontinuity in the initial/boundary conditions. In this view, Harmuth has recently been joined by Hillion [17]–[19], who points out that unless compatibility conditions analogous to [17, eq. (6)] or [19, eq. (9')] are satisfied, the equations are not solvable. This assertion of Hillion's, while strictly correct, does not at all invalidate the well-posedness of solutions of the Maxwell equations without the magnetic conductivity term in the Faraday Law. For example, if in the one-dimensional problem we take $E(x,0) = 0$ and $H(x,0) = 0$ for all $x > 0$ and $E(0,t) = E_0$ (where E_0 is a nonzero constant) for all $t > 0$, we really (physically) mean that $E(0,t)$ is a function equal to 0, and with its first two time derivatives equal to 0, at $t = 0$, and that $E(0,t)$ rises from 0 to its value E_0 in a time short compared to times of interest, but not short compared to 10^{-10} [s], since in considering ε, μ, and σ to be constant for sea water, we (and implicitly, also Harmuth and Boules) assume that frequency components above 10^{10} [Hz] are not appreciable. But if one then examines the functional dependence of $E(x,t)$, $H(x,t)$ on $E(0,t)$ (see (30) and (31) below), it is clear that the change from a function $E(0,t)$ with a continuous second derivative at $t = 0$ to a constant E_0 times a Heaviside unit step function will not change $E(x,t)$, $H(x,t)$ appreciably. Of course, if for mathematical convenience we use a step function for $E(0,t)$, this discontinuity will propagate along a characteristic into the interior of the quadrant, $x > 0$, $t > 0$ but off this characteristic, both $E(x,t)$ and $H(x,t)$ are well and uniquely defined. Contrary to Harmuth *et al.* [20, pp. 319–320], two solutions which differ only on one smooth curve in the quadrant $x > 0$, $t > 0$ may be considered identical. Harmuth's assertion that $E(x,t)$ can be uniquely obtained but not $H(x,t)$ when $E(0,t)$ is given as a boundary condition is not valid and both are explicitly given. With any time dependence, there are no electric or magnetic excitations as Harmuth would have it, only electromagnetic excitations, and the imposition of a time-dependent electric field at the boundary of a half-space problem as well as initial conditions inside the half-space automatically determines the form of the time-dependent magnetic field at the boundary (see (20) below).

Incidentally, it is not clear why [17, eq. (2)] of Hillion's, which follows from the use of a magnetic current density by Harmuth, should be less subject to the need of a compatibility condition at the origin to make the problem well-posed than [17, eq. (3)], which is obtained when no magnetic current density is introduced. Indeed, Hillion does not explicitly say, in [17], that there is a difference, though he implies it.

In this paper, the one-dimensional unmodified Maxwell equations are solved in the region $x > 0$, $t > 0$ subject to the initial conditions $E(x,0) = 0$, $H(x,0) = 0$ for $x > 0$, and the boundary condition $E(0,t) = E_0(t)$, an arbitrary function of time, for $t > 0$. It is assumed that ε, μ, and σ are constants which is satisfactory, e.g., for sea water at frequencies $< 10^{10}$ [Hz].

If Ohm's law is changed from $\boldsymbol{J} = \sigma \boldsymbol{E}$ to that of a dynamic conductivity [14], [21], $\tau \partial \boldsymbol{J}/\partial t + \boldsymbol{J} = \sigma_0 E$, then if only frequency components very small compared to $1/\tau$ are appreciable, the net effect is to replace ε by $\varepsilon - \tau\sigma_0$, and no qualitative change is made in the analysis as long as $\tau\sigma_0/\varepsilon < 1$. (In fact, for sea water, $\tau\sigma_0/\varepsilon \cong 10^{-6}$. For copper, to be sure, $\tau\sigma_0/\varepsilon \cong 10^{-4}$ and the analysis for this case will be dealt with in a later paper.)

II. Formulation of the Time-Domain Plane-Wave Problem

A. Homogeneous Static Conducting Medium: Ohms Law ($\boldsymbol{J} = \sigma \boldsymbol{E}$)

Consider a homogeneous conducting half-space with permittivity, permeability, and conductivity of the medium denoted as ε [F/m], μ [H/m], and σ [S/m], respectively. The electric and magnetic field intensities inside the half-space are denoted as $E_y = E(x,t)$ and $H_z = H(x,t)$, respectively. These fields are initially zero in the half-space (i.e., at time $t = 0$) and are excited by the imposition of a time-varying electric field which is constant over the plane $x = 0$ (i.e., the boundary of the half-space). Thus from the Maxwell equations, the fields can be shown [21], [22] to be related by the coupled partial differential equations

$$\begin{cases} \frac{\partial E}{\partial x} + \mu \frac{\partial H}{\partial t} = 0 & (1) \\ \frac{\partial H}{\partial x} + \varepsilon \frac{\partial E}{\partial t} + \sigma E = 0 & (2) \end{cases}$$

where it has been assumed that the current density in the region is given by Ohm's law, $J = \sigma E$, and that this is the only source of current density (i.e., no impressed currents). It should be mentioned that in practice, fields are usually produced by the imposition of sources in a region, i.e., either currents or charges, and that the use of the electric field boundary condition is only a mathematical convenience.

Upon differentiating (1) and (2) with respect to x and assuming that the order of differentiation with respect to x and t can be exchanged gives second-order wave equations

$$\frac{\partial^2 E}{\partial x^2} = \mu\varepsilon \frac{\partial^2 E}{\partial t^2} + \mu\sigma \frac{\partial E}{\partial t} \quad (3)$$

$$\frac{\partial^2 H}{\partial x^2} = \mu\varepsilon \frac{\partial^2 H}{\partial t^2} + \mu\sigma \frac{\partial H}{\partial t} \quad (4)$$

which are similar to the *telegraph* equations of the transmission-line problem. For this mixed initial boundary value problem these may be solved for the electric field E, say, and then the magnetic field H can be obtained by using either (1) or (2).

For the case where the boundary conditions on either of the field variables are discontinuous it is more convenient to solve the system of equations (1) and (2) directly. This is to allay doubts of the validity of the telegraph equation near the point of discontinuity due to the exchange of differentiation with respect to time and space. Note that the initial discontinuity will not remain on the boundary but will travel along a characteristic into the domain where a solution is being sought.

B. Homogeneous Conducting Medium: Dynamic Conductivity ($\tau \partial \boldsymbol{J}/\partial t + \boldsymbol{J} = \sigma_0 E$)

It has been suggested by Becker [21, p. 238] and others that the relationship between the current density in a medium which contains charge carriers and the electric field may be better represented by the equation

$$\tau \frac{\partial \boldsymbol{J}}{\partial t} + \boldsymbol{J} = \sigma_0 \boldsymbol{E} \quad (5)$$

where σ_0 [S/m] is the static or dc conductivity of the medium and τ [s] is a time constant which can be expressed as $\tau = \sigma_0 m/(Ne^2)$ where m is the mass of the charge carriers [kg], N is the density of the charge carriers [number of carriers per m^3], and $e = 1.6 \times 10^{-19}$ [C] is the value of one unit of charge. Typical values of this time constant are $\tau_c = 2.4 \times 10^{-14}$ [s] for copper and on the order of $\tau_s = 4 \times 10^{-15}$ [s] for sea water. For this case, the Maxwell equations are written in the more general form

$$\begin{cases} \frac{\partial E}{\partial x} + \mu \frac{\partial H}{\partial t} = 0 & (6) \\ \frac{\partial H}{\partial x} + \varepsilon \frac{\partial E}{\partial t} = -J & (7) \end{cases}$$

where the scalar function J is meant to represent the component of the current density vector having the same orientation as the electric field; in this case, $J(x,t) = J_y(x,t)$ and (5) is written as

$$\tau \frac{\partial J}{\partial t} + J = \sigma_0 E. \tag{8}$$

III. Solution of the System via the Laplace Transform

A. Initial Boundary Value Problem: Homogeneous Static Conducting Medium

The solution of the boundary value problem is now obtained through the use of the Laplace Transform technique. This problem is formulated mathematically by adding the initial conditions

$$E(x,0) = 0, \qquad H(x,0) = 0, \qquad x > 0 \tag{9}$$

and the boundary conditions

$$E(0,t) = E_0(t), \qquad H(0,t) = H_0(t), \qquad t > 0 \tag{10}$$

to the first-order equations (1) and (2) where $E_0(t)$ and $H_0(t)$ are specified functions of time. It should be noted that these boundary conditions cannot be specified independently of each other and that once one boundary field is specified, say $E_0(t)$, the remaining boundary field, $H_0(t)$, will be related to the previous in such a way as to satisfy the coupled partial differential equations with bounded field values throughout the half-space. It is assumed that solutions to the field variables will be required for the general case where $E_0(t)$ and $H_0(t)$ may be discontinuous functions of time.

Denoting the Laplace Transform with respect to the time variable of the field variables as $e(x,s) = L\{E(x,t)\}$, and $h(x,s) = L\{H(x,t)\}$, (1) and (2) transform to

$$\begin{cases} \frac{d}{dx} e(x,s) + \mu s h(x,s) = 0 & (11) \\ \frac{d}{dx} h(x,s) + \varepsilon s e(x,s) + \sigma e(x,s) = 0 & (12) \end{cases}$$

where the zero initial conditions have been used. The boundary conditions (10) transform to

$$e(0,s) = e_0(s) \qquad h(0,s) = h_0(s). \tag{13}$$

Thus the problem has become one of solving the ordinary differential equations (11) and (12) subject to the boundary conditions (13). This is most easily achieved by performing another Laplace transformation, this time with respect to the space variable x. Denoting

$$\tilde{e}(p,s) = \int_0^\infty e(x,s) e^{-\mathrm{px}}\, dx$$
$$\tilde{h}(p,s) = \int_0^\infty h(x,s) e^{-\mathrm{px}}\, dx \tag{14}$$

this will transform (11) and (12) to algebraic equations which in matrix notation are written as

$$\begin{bmatrix} p & \mu s \\ \varepsilon s + \sigma & p \end{bmatrix} \begin{bmatrix} \tilde{e}(p,s) \\ \tilde{h}(x,s) \end{bmatrix} = \begin{bmatrix} e_0(s) \\ h_0(s) \end{bmatrix}. \tag{15}$$

The solution of this algebraic system is easily found as

$$\begin{bmatrix} \tilde{e}(p,s) \\ \tilde{h}(x,s) \end{bmatrix} = \frac{1}{p^2 - k^2(s)} \begin{bmatrix} p & -\mu s \\ -(\varepsilon s + \sigma) & p \end{bmatrix} \begin{bmatrix} e_0(s) \\ h_0(s) \end{bmatrix} \tag{16}$$

where

$$k^2(s) \equiv (\varepsilon s + \sigma)\mu s. \tag{17}$$

This solution, in the (p,s) transform space, incorporates both the initial and the boundary conditions.

The inversion with respect to the p variable is easily accomplished via the two transform pairs

$$\frac{p}{p^2 - a^2} \leftrightarrow \cosh(ax)$$
$$\frac{a}{p^2 - a^2} \leftrightarrow \sinh(ax)$$

so that the solution in the (x,s) domain is found as

$$e(x,s) = e_0(s)\cosh(k(s)x) - h_0(s)\frac{\mu s}{k(s)}\sinh(k(s)x) \tag{18}$$

$$h(x,s) = h_0(s)\cosh(k(s)x) - e_0(s)\frac{(\varepsilon s + \sigma)}{k(s)}\sinh(k(s)x). \tag{19}$$

In order that a physical solution is obtained the field variables must become zero as x goes to infinity (for the case of no dissipation, i.e., where σ is zero, this condition would be changed to that of boundedness). This physical condition manifests itself mathematically in the statement that $\lim_{x\to\infty} e(x,s) = 0$, and $\lim_{x\to\infty} h(x,s) = 0$ for all values of s. If the hyperbolic functions in (18) and (19) are expressed as exponentials, then as x goes to infinity (18) and (19) are written as

$$\lim_{x\to\infty} e(x,s) = \lim_{x\to\infty}\left\{e_0(s)\left(\frac{e^{k(s)x} + e^{-k(s)x}}{2}\right) - h_0(s)\frac{\mu s}{k(s)}\left(\frac{e^{k(s)x} + e^{-k(s)x}}{2}\right)\right\} = 0$$

$$\lim_{x\to\infty} h(x,s) = \lim_{x\to\infty}\left\{h_0(s)\left(\frac{e^{k(s)x} + e^{-k(s)x}}{2}\right) - e_0(s)\frac{\varepsilon s + \sigma}{k(s)}\left(\frac{e^{k(s)x} + e^{-k(s)x}}{2}\right)\right\} = 0$$

and, choosing $\mathrm{Re}\{k(s)\} = \mathrm{Re}\{\sqrt{(\varepsilon s + \sigma \mu s)}\} > 0$ for $s > 0$, this implies that

$$\lim_{x\to\infty}\left\{e_0(s)\frac{e^{k(s)x}}{2} - h_0(s)\frac{\mu s}{k(s)}\frac{e^{k(s)x}}{2}\right\} = 0$$

$$\lim_{x\to\infty}\left\{h_0(s)\frac{e^{k(s)x}}{2} - e_0(s)\frac{\varepsilon s + \sigma}{k(s)}\frac{e^{k(s)x}}{2}\right\} = 0$$

or that

$$h_0(s) = \sqrt{\frac{\varepsilon s + \sigma}{\mu s}}\, e_0(s). \tag{20}$$

This relation between the boundary values of the magnetic and electric field variables restricts the solution to waves propagating in the positive x direction and is a result of the imposition of the zero initial conditions in the half-space as well as the physical requirement that the fields remain bounded as x goes to positive infinity (i.e., causal solutions). Substituting this equation into (18) and (19) results in the expressions

$$e(x,s) = e_0(s)e^{-k(s)x} \tag{21}$$

$$h(x,s) = e_0(s)\sqrt{\frac{\varepsilon s+\sigma}{\mu s}}e^{-k(s)x} = e_0(s)(\varepsilon s+\sigma)\frac{e^{-k(s)x}}{k(s)} \tag{22}$$

which must now be inverted with respect to the s variable back to the time variable t.

This inversion can be performed by the use of the integral (see [23, p. 179])

$$e^{-x\sqrt{as^2+bs+c}} = e^{-(b/(2\sqrt{a}))x}e^{-\sqrt{a}xs} + x\sqrt{\frac{d}{a}}\int_{\sqrt{a}x}^{\infty} e^{-st}\frac{e^{-bt/2a}}{\sqrt{t^2-ax^2}}I_1\left(\frac{\sqrt{d}}{a}\sqrt{t^2-ax^2}\right)dt$$

where

$$d = (b/2)^2 - ac.$$

For the case considered herein

$$a = \mu\varepsilon \qquad b = \mu s \qquad c = 0 \qquad d = \frac{1}{4}(\mu\sigma)^2$$

whence

$$e^{-x\sqrt{(\varepsilon s+\sigma)\mu s}} = e^{-\left(\frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}\right)x}e^{-\sqrt{\mu\varepsilon}xs} + x\left(\frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}\right) \times \int_{\sqrt{\mu\varepsilon}x}^{\infty} e^{-st}\frac{e^{-\left(\frac{\sigma}{2\varepsilon}\right)t}}{\sqrt{t^2-\mu\varepsilon x^2}} \times I_1\left(\frac{1}{2}\epsilon\sqrt{t^2-\mu\varepsilon x^2}\right)dt.$$

In order to simplify the above expression, the abbreviations

$$\sqrt{\mu\varepsilon} = \alpha \qquad \frac{\sigma}{2\varepsilon} = \beta \qquad \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}} = \gamma$$

are introduced and the integral expression becomes

$$\begin{aligned} e^{-x\sqrt{(\varepsilon s+\sigma)\mu s}} &= e^{-\gamma x}e^{-\alpha xs} + x\gamma\int_{\alpha x}^{\infty} e^{-st}e^{-\beta t}\frac{I_1(\beta\sqrt{t^2-\alpha^2x^2})}{\sqrt{t^2-\alpha^2x^2}}dt \\ &= e^{-\gamma x}e^{-\alpha xs} + L\{V_1(x,t)\} \\ &= e^{-\gamma x}e^{-\alpha xs} + v_1(x,s) \end{aligned} \tag{23}$$

where

$$V_1(x,t) = \begin{cases} 0, & 0 \le t < \alpha x \\ x\gamma e^{-\beta t}\frac{I_1(\beta\sqrt{t^2-\alpha^2x^2})}{\sqrt{t^2-\alpha^2x^2}}, & t > \alpha x. \end{cases} \tag{24}$$

The other Laplace Transform pair which is required is [24, p. 253, no. 63]

$$\frac{e^{-f\sqrt{(s+d)(s+e)}}}{\sqrt{(s+d)(s+e)}} \leftrightarrow \begin{cases} 0, & 0 < t < f \\ e^{-\frac{d+e}{2}t}I_0\left(\frac{d-e}{2}\sqrt{t^2-f^2}\right), & t > f. \end{cases}$$

Thus expressing

$$\frac{e^{-x\sqrt{(\varepsilon s+\sigma)\mu s}}}{(\varepsilon s+\sigma)\mu s} = \frac{e^{-x\sqrt{\mu\varepsilon}\sqrt{s^2+(s\frac{\sigma}{\varepsilon})}}}{\sqrt{\mu\varepsilon}\sqrt{s^2+(s\frac{\sigma}{\varepsilon})}} = \frac{e^{-x\sqrt{\mu\varepsilon}\sqrt{s(s+\frac{\sigma}{\varepsilon})}}}{\sqrt{\mu\varepsilon}\sqrt{s(s+\frac{\sigma}{\varepsilon})}}$$

and comparing with the transform pair with the abbreviations

$$f = x\sqrt{\mu\varepsilon} = x\alpha \qquad d = 0 \qquad e = \frac{\sigma}{\varepsilon}$$

the transform pair

$$\frac{e^{-k(s)x}}{k(s)} \leftrightarrow \begin{cases} 0, & 0 < t < \alpha x \\ \frac{1}{\sqrt{\mu\varepsilon}}e^{-\frac{1}{2}(\frac{\sigma}{\varepsilon})t}I_0\left(\frac{\sigma}{2\varepsilon}\sqrt{t^2-\alpha^2x^2}\right), & t > \alpha x \end{cases}$$

or in terms of the shorthand notation

$$\frac{e^{-k(s)x}}{k(s)} \leftrightarrow \begin{cases} 0, & 0 < t < \alpha x \\ \frac{1}{\alpha}e^{-\beta t}I_0(\beta\sqrt{t^2-\alpha^2x^2}), & t > \alpha x \end{cases}$$

where the fact that I_0 is an even function of its argument has been used. This transform pair can be written using the Heaviside step function, $\mathbf{1}(t-\alpha x)$, as

$$\frac{e^{-k(s)x}}{k(s)} \leftrightarrow \frac{1}{\alpha}\mathbf{1}(t-\alpha x)e^{-\beta t}I_0(\beta\sqrt{t^2-\alpha^2x^2}).$$

The solution for the electric and magnetic fields can now be written as

$$E(x,t) = L^{-1}\{e_0(s)e^{-\gamma x}e^{-\alpha xs} + e_0(s)L\{V_1(x,t)\}\}$$

$$H(x,t) = L^{-1}\left\{e_0(s)\varepsilon s\frac{e^{-k(s)x}}{k(s)} + e_0(s)\sigma\frac{e^{-k(s)x}}{k(s)}\right\}$$

which can be inverted for general $e_0(s)$ using operational transforms. Thus it is found that the fields are zero for $t < \alpha x$ and for $t > \alpha x$ the electric field intensity can be expressed as

$$\begin{aligned} E(x,t) &= e^{-\gamma x}E_0(t-\alpha x) + x\gamma\int_{\alpha x}^{t} E_0(t-\tau)e^{-\beta\tau}\frac{I_1(\beta\sqrt{\tau^2-\alpha^2x^2})}{\sqrt{(\tau^2-\alpha^2x^2)}}d\tau \\ &= e^{-\gamma x}E_0(t-\alpha x) + x\gamma\int_{t-\alpha x}^{0} E_0(\tau)e^{-\beta(t-\tau)}\frac{I_1(\beta\sqrt{(t-\tau)^2-\alpha^2x^2})}{\sqrt{((t-\tau)^2-\alpha^2x^2)}}d\tau \\ E(x,t) &= e^{-\gamma x}E_0(t-\alpha x) - x\gamma\int_{0}^{t-\alpha x} E_0(\tau)e^{-\beta(t-\tau)}\frac{I_1(\beta\sqrt{(t-\tau)^2-\alpha^2x^2})}{\sqrt{((t-\tau)^2-\alpha^2x^2)}}d\tau \end{aligned} \tag{25}$$

where it can be shown that this is identical to the solution obtained by Stratton [22, p. 320]. The magnetic field intensity is derived as follows. Application, again, of operational transforms leads to

$$H(x,t) = \varepsilon \int_0^t E_0(t-\tau)\frac{\partial}{\partial \tau}\left[\frac{1}{\alpha}\mathbf{1}(\tau-\alpha x)e^{-\beta\tau} \times I_0\left(\beta\sqrt{\tau^2-\alpha^2x^2}\right)\right] d\tau + \sigma \int_{\alpha x}^t E_0(t-\tau)\frac{1}{\alpha}e^{-\beta\tau}I_0\left(\beta\sqrt{\tau^2-\alpha^2x^2}\right) d\tau \tag{26}$$

which, when the partial derivative with respect to τ is performed inside the first integral and the sifting property of the Dirac delta function is used, becomes

$$H(x,t) = \frac{\varepsilon}{\alpha}\int_{\alpha x}^t E_0(t-\tau)\left[\frac{\beta\tau e^{-\beta\tau}}{\sqrt{\tau^2-\alpha^2x^2}}I_1\left(\beta\sqrt{\tau^2-\alpha^2x^2}\right) - \beta e^{-\beta\tau}I_0\left(\beta\sqrt{\tau^2-\alpha^2x^2}\right)\right] d\tau + \frac{\sigma}{\alpha}\int_{\alpha x}^t E_0(t-\tau)e^{-\beta\tau}I_0\left(\beta\sqrt{\tau^2-\alpha^2x^2}\right) d\tau + \frac{\varepsilon}{\alpha}E_0(t-\alpha x)e^{-\beta\alpha x}. \tag{27}$$

Grouping terms and simplifying slightly this can be written as

$$H(x,t) = \frac{\varepsilon}{\alpha}E_0(t-\alpha x)e^{-\gamma x} + \frac{\sigma}{2\alpha}\int_{\alpha x}^t E_0(t-\tau)e^{-\beta\tau} \times I_0\left(\beta\sqrt{\tau^2-\alpha^2x^2}\right) d\tau + \frac{\sigma}{2\alpha}\int_{\alpha x}^t E_0(t-\tau)\frac{\tau e^{-\beta\tau}}{\sqrt{\tau^2-\alpha^2x^2}} \times I_1\left(\beta\sqrt{\tau^2-\alpha^2x^2}\right)d\tau \tag{28}$$

and

$$H(x,t) = \frac{\varepsilon}{\alpha}E_0(t-\alpha x)e^{-\gamma x} + \frac{\sigma}{2\alpha}\int_t^{t-\alpha x} E_0(\tau)e^{-\beta(t-\tau)} \times I_0\left(\beta\sqrt{(t-\tau)^2-\alpha^2x^2}\right) d\tau + \frac{\sigma}{2\alpha}\int_0^{t-\alpha x} E_0(\tau)\frac{(t-\tau)e^{-\beta(t-\tau)}}{\sqrt{(t-\tau)^2-\alpha^2x^2}} \times I_1\left(\beta\sqrt{(t-\tau)^2-\alpha^2x^2}\right) d\tau. \tag{29}$$

The electric and magnetic fields can be written in terms of the media constitutive parameters as

$$E(x,t) = e^{-\left(\sigma/2\sqrt{\mu/\varepsilon}\right)x}E_0(t-x/c) + x\left(\frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}\right)\int_0^{t-x/c} E_0(\tau)e^{-(\sigma/2\varepsilon)(t-\tau)} \times \frac{I_1\left(\frac{\sigma}{2\varepsilon}\sqrt{(t-\tau)^2-(x/c)^2}\right)}{\sqrt{((t-\tau)^2-(x/c)^2)}} d\tau \tag{30}$$

$$H(x,t) = \sqrt{\frac{\varepsilon}{\mu}}E_0(t-x/c)e^{-\left(\sigma/2\sqrt{\mu/\varepsilon}\right)x} + \frac{\sigma c}{2}\int_0^{t-x/c} E_0(\tau)e^{-(\sigma/2\varepsilon)(t-\tau)} \times I_0\left(\frac{\sigma}{2\varepsilon}\sqrt{(t-\tau)^2-(x/c)^2}\right)d\tau + \frac{\sigma c}{2}\int_0^{t-x/c} E_0(\tau)\frac{(t-\tau)e^{-(\sigma/2\varepsilon)(t-\tau)}}{\sqrt{(t-\tau)^2-(x/c)^2}} \times I_1\left(\frac{\sigma}{2\varepsilon}\sqrt{(t-\tau)^2-(x/c)^2}\right) d\tau \tag{31}$$

where the constant $c = 1/\alpha = 1/\sqrt{\varepsilon\mu}$ has been introduced. Alternatively, starting with (26), and integrating by parts we find that the magnetic field intensity can be expressed in the more compact form

$$H(x,t) = \frac{\varepsilon}{\alpha}\left[E_0(t-\tau)\mathbf{1}(\tau-\alpha x)e^{-\beta\tau}I_0\left(\beta\sqrt{\tau^2-\alpha^2x^2}\right)\right]\Bigg|_0^t - \frac{\varepsilon}{\alpha}\int_0^t E_0'(t-\tau)\mathbf{1}(\tau-\alpha x)e^{-\beta\tau}I_0\left(\beta\sqrt{\tau^2-\alpha^2x^2}\right) d\tau + \sigma\int_{\alpha x}^t E_0(t-\tau)\frac{1}{\alpha}e^{-\beta\tau}I_0\left(\beta\sqrt{\tau^2-\alpha^2x^2}\right) d\tau$$

$$H(x,t) = \frac{\varepsilon}{\alpha}E_0(0)\mathbf{1}(\tau-\alpha x)e^{-\beta\tau}I_0\left(\beta\sqrt{\tau^2-\alpha^2x^2}\right) + \int_{\alpha x}^t\left(\frac{\sigma}{\alpha}E_0(t-\tau) - \frac{\varepsilon}{\alpha}E_0'(t-\tau)\right)e^{-\beta\tau} \times I_0\left(\beta\sqrt{\tau^2-\alpha^2x^2}\right) d\tau$$

$$H(x,t) = \frac{\varepsilon}{\alpha}E_0(0)\mathbf{1}(t-\alpha x)e^{-\beta\tau}I_0\left(\beta\sqrt{\tau^2-\alpha^2x^2}\right) d\tau + \int_0^{t-\alpha x}\left(\frac{\sigma}{\alpha}E_0(\tau) - \frac{\varepsilon}{\sigma}E_0'(\tau)\right)e^{-\beta(t-\tau)} \times I_0\left(\beta\sqrt{(t-\tau)^2-\alpha^2x^2}\right) d\tau \tag{32}$$

or zero for $t < x/c$ and for $t > x/c$

$$H(x,t) = \sqrt{\frac{\varepsilon}{\mu}}E_0(0)e^{-(\sigma/2\varepsilon)t}I_0\left(\frac{\sigma}{2\varepsilon}\sqrt{t^2-(x/c)^2}\right) + \int_0^{t-\alpha x}(c\sigma E_0(\tau) - \sqrt{\frac{\varepsilon}{\mu}}E_0'(\tau))e^{-(\sigma/2\varepsilon)(t-\tau)} \times I_0\left(\frac{\sigma}{2\varepsilon}\sqrt{(t-\tau)^2-(x/c)^2}\right). \tag{33}$$

B. Initial Boundary Value Problem: Homogeneous Dynamic Conducting Medium

For the solution to the initial boundary value problem where the conductivity of the medium is modeled via (8) the problem can be solved using the Laplace Transform method as well.

Equations (1), (2), and (8) are transformed as

$$\begin{cases} \frac{d}{dx}e(x,s) + \mu s h(x,s) = 0 & (34) \\ \frac{d}{dx}h(x,s) + \varepsilon s e(x,s) + j(x,s) = 0 & (35) \\ \tau s j(x,s) + j(x,s) = \sigma_0 e(x,s) & (36) \end{cases}$$

where $j(x,s)$ denotes the Laplace Transform of the current density $J(x,t)$ and in (36) the fact that $J(x,0) = 0$ has been used. Solving for $j(x,s)$ in (36) and substituting into (35) gives the coupled ordinary differential equations

$$\begin{cases} \frac{d}{dx}e(x,s) + \mu s h(x,s) = 0 & (37) \\ \frac{d}{dx}h(x,s) + (\varepsilon s + \frac{\sigma_0}{1+\tau s}e(x,s)) = 0 & (38) \end{cases}$$

which are to be solved subject to the boundary conditions given by (13). Using the same procedure as in the previous section of taking the Laplace Transform of these equations with respect to the space variable results in the algebraic system

$$\begin{bmatrix} p & \mu s \\ \varepsilon s + \frac{\sigma_0}{1+\tau s} & p \end{bmatrix} \begin{bmatrix} \tilde{e}(p,s) \\ \tilde{h}(p,s) \end{bmatrix} = \begin{bmatrix} e_0(s) \\ h_0(s) \end{bmatrix} \tag{39}$$

which is solved as

$$\begin{bmatrix} \tilde{e}(p,s) \\ \tilde{h}(p,s) \end{bmatrix} = \frac{1}{p^2 - \varsigma^2(s)} \begin{bmatrix} p & -\mu s \\ -\varepsilon s - \frac{\sigma_0}{1+\tau s} & p \end{bmatrix} \begin{bmatrix} e_0(s) \\ h_0(s) \end{bmatrix}$$

where we have defined

$$\varsigma^2(s) \equiv \left(\varepsilon s + \frac{\sigma_0}{1+\tau s}\right)\mu s. \tag{40}$$

This is easily transformed back from the (p,s) space to the (x,s) space as

$$e(x,s) = e_0(s)\cosh(\varsigma(s)x) - h_0(s)\frac{\mu s}{\varsigma(s)}\sinh(\varsigma(s)x) \tag{41}$$

$$h(x,s) = h_0(s)\cosh(\varsigma(s)x) - e_0(s)\left(\varepsilon s + \frac{\sigma}{1+\tau s}\right) \times \frac{\sinh(\varsigma(s)x)}{\varsigma(s)}. \tag{42}$$

Again using the conditions that

$$\lim_{x\to\infty} e(x,s) = 0$$

and

$$\lim_{x\to\infty} h(x,s) = 0$$

and given that for $s > 0$

$$\mathrm{Re}\{\varsigma(s)\} = \mathrm{Re}\sqrt{(\varepsilon s + \frac{\sigma_0}{1+\tau s})\mu s} > 0$$

implies that

$$\lim_{x\to\infty}\left\{e_0(s)\frac{e^{\varsigma(s)x}}{2} - h_0(s)\frac{\mu s}{\varsigma(s)}\frac{e^{\varsigma(s)x}}{2}\right\} = 0$$

$$\lim_{x\to\infty}\left\{h_0(s)\frac{e^{\varsigma(s)x}}{2} - e_0(s)\left(\varepsilon s + \frac{\sigma_0}{1+\tau s}\right)\frac{e^{\varsigma(s)x}}{2\varsigma(s)}\right\} = 0$$

or that the boundary value of the magnetic field must be related to the boundary value of the electric field by the relation

$$h_0(s) = \sqrt{\frac{1}{\mu s}\left(\varepsilon s + \frac{\sigma_0}{1+\tau s}\right)}e_0(s). \tag{43}$$

Notice that for the case $\sigma_0 = 0, h_0(s) = \sqrt{\varepsilon/\mu}e_0(s)$, which is what one would expect for the case of a plane wave propagating in the positive x direction. Substitution of (43) into (41) and (42) results in the equations

$$e(x,s) = e_0(s)e^{-\varsigma(s)x} \tag{44}$$

$$h(x,s) = e_0(s)\left(\varepsilon s + \frac{\sigma_0}{1+\tau s}\right)\frac{e^{\varsigma(s)x}}{\varsigma(s)} \tag{45}$$

which must be transformed back from the (x,s) domain to the (x,t) domain. If in these expressions we introduce the Taylor's expansion for the expression

$$\frac{1}{1+\tau s} = 1 - \tau s + (\tau s)^2 - \cdots \tag{46}$$

and neglect all but the first two terms, we have

$$\frac{1}{1+\tau s} \cong 1 - \tau s \tag{47}$$

which is certainly a very good approximation since t is on the order of 10^{-14} [s] and we are only interested in frequencies $< 10^{10}$ [Hz] due to the assumption of constant ε and μ. Then the function $\varsigma(s)$ becomes

$$\begin{aligned} \varsigma(s) &\cong \sqrt{(\varepsilon s + \sigma_0(1-\tau s))\mu s} \\ &= \sqrt{[\sigma_0 + (\varepsilon - \tau\sigma_0)s]\mu s}. \end{aligned} \tag{48}$$

Thus introducing, the effective permittivity $\hat{\varepsilon} = \varepsilon - \tau\sigma_0$ we have

$$\varsigma(s) = \sqrt{(\sigma_0 + \hat{\varepsilon}s)\mu s} \tag{49}$$

$$e(x,s) = e_0(s)e^{-\varsigma(s)x} \tag{50}$$

$$h(x,s) = e_0(s)(\sigma_0 + \hat{\varepsilon}s)\frac{e^{\varsigma(s)x}}{\varsigma(s)} \tag{51}$$

which are identical in form to (17), (21), and (22) which were solved for the static conductivity case. Thus we see that the field solutions for the case of the dynamic conductivity will be exactly as those given by (30) and (31) or (33) with ε replaced by $\hat{\varepsilon} = \varepsilon - \tau\sigma_0$ everywhere, provided $\hat{\varepsilon} > 0$. The case $\hat{\varepsilon} < 0$ will be dealt with in a later paper.

IV. Conclusion

It has been shown that electric field boundary conditions at $x = 0$ for $t > 0$, together with initial conditions for E and H at $t = 0$ for $x > 0$ (taken zero in our particular case) allow for a unique and well-posed (i.e., depending continuously on boundary condition) solution for both $E(x,t)$ and $H(x,t)$ in the $x > 0, t > 0$ quadrant. If a discontinuity is assumed for mathematical convenience at $(0,0)$, it propagates along the characteristic $x = t/\sqrt{\mu\varepsilon}$ (or $x = t/\sqrt{\mu(\varepsilon - \tau\sigma)}$ for $\tau > 0$ modification of the Ohm law) into the interior of the quadrant. (Note that, e.g., for sea water, $1/\sqrt{\mu(\varepsilon - \tau\sigma)}$ is smaller than the speed of light in a vacuum even though larger than $1/\sqrt{\mu\varepsilon}$.)

The results are in full agreement with Kuester [13] and others [1]–[9], and with Hallen's solution of the Telegraph equation for $R = 0$ when V, I, C, L, G are identified with E, H, ε, μ, and σ, respectively, [25, pp. 412–413]. We fully concur with Kuester's conclusions. There are two important points in this paper which we hope might help to settle the Harmuth–Kuester debate, and a third one which can be used to meet Harmuth's and Hussain's challenge to Wait. The first point is the care taken to derive the results from the first-order Maxwell equations rather than from the wave equations for E and H, thus justifying *a posteriori* the use of the wave equation in the neighborhood of the discontinuity at the origin. The second is the argument in the introduction which shows that Hillion's objections, while mathematically correct, have no serious consequences. The third item is the modification of Ohm's law to $\tau\partial J/\partial t + J = \sigma_0 E$, i.e., the introduction of a dynamic conductivity.

In a second paper (II), it will be assumed that there is a vacuum in the half-space $x < 0$, while the half-space $x > 0$ is filled with a material of constant ε_1, μ_1, and σ_1 (e.g., sea water below 10^{10} Hz), and that the source is an electromagnetic wave (nonsinusoidal, in general) impinging on the plane $x = 0$ from the vacuum, with prescribed incident electric field $E_0(t)$ for $t > 0$ and 0 for $t < 0$ (and, therefore, magnetic field $\sqrt{\varepsilon_0/\mu_0}E_0(t)$), leading, for $t > 0$, to reflected electric and magnetic fields $F(t)$ and $-\sqrt{\varepsilon_0/\mu_0}F(t)$ at $x = 0$. If $\tilde{E}(x,s)$, $\tilde{H}(x,s)$, $\tilde{E}_0(s)$, and $\tilde{F}(s)$ are the Laplace Transforms of $E(x,t)$, $H(x,t)$, $E_0(t)$, and $F(t)$ with respect to time, then

$$\tilde{F}(s) = \tilde{E}_0(s)\frac{1 - Q(s)}{1 + Q(s)}$$

where

$$Q(s) = \sqrt{\frac{\mu_0}{\varepsilon_0}\frac{\varepsilon_1}{\mu_1}}\sqrt{1 + \frac{\sigma_1}{\varepsilon_1}\frac{1}{(1 + \tau s)}}$$

or, when only frequencies $\ll 1/\tau$ are important,

$$Q(s) \cong \sqrt{\frac{\mu_0}{\varepsilon_0}\frac{\varepsilon_1}{\mu_1}}\sqrt{\left(1 - \frac{\tau\sigma_1}{\varepsilon_1}\right) + \frac{\sigma_1}{\varepsilon_1 s}}$$

and, for $x > 0$

$$\tilde{E}(x,s) = \left(\frac{2\tilde{E}_0(s)}{1 + Q(s)}\right)\exp\left(-x\sqrt{\mu_1\varepsilon_1 s^2 + \mu_1\sigma_1\frac{s}{1 + \tau s}}\right)$$

$$\tilde{H}(x,s) = \frac{1}{\mu_1 s}\left(\frac{2\tilde{E}_0(s)}{1 + Q(s)}\right)\sqrt{\mu_1\varepsilon_1 s^2 + \mu_1\sigma_1\frac{s}{1 + \tau s}} \times \exp\left(-x\sqrt{\mu_1\varepsilon_1 s^2 + \mu_1\sigma_1\frac{s}{1 + \tau s}}\right)$$

where for only frequency components $\ll 1/\tau$ appreciable, $1/(1 + \tau s)$ may again be replaced by $1 - \tau s$. In paper II, general expressions will first be given for $E(x,t)$ and $H(x,t)$ in terms of $\tilde{E}_0(s)$. Then, the results will be specialized to the case of sea water below 10^{10} Hz with an incident step function electric field, $\tilde{E}_0(s) = E_{\text{inc}}/s$, $\varepsilon_{r1} \cong 80$, $\mu_{r1} \cong 1$, $\sigma_1 \cong 4$ [S/m] (4×10^{10} [s^{-1}] in cgs units), and $\tau \cong 4 \times 10^{-15}$ [s]. Actually, the results for $\tau\sigma_1/\varepsilon_1$ so small are not much different from the case $\tau = 0$, but we shall use this positive τ in order to meet the Harmuth–Hussain challenge to Wait.

In a future paper, the same problem will be solved for the case where the only important frequency components are still $\ll 1/\tau$ but the conductivity is large enough so that $\tau\sigma_1/\varepsilon_1 > 1$, as is certainly true for copper.

REFERENCES

[1] J. R. Wait, "Transient electromagnetic propagation in a conducting medium," *Geophys.*, vol. XVI, no. 2, pp. 213–221, 1951.

[2] B. K. Bhattacharyya, "Propagation of transient electromagnetic waves in a conducting medium," *Geophys.*, pp. 959–961, 1955.

[3] ——, "Propagation of transient electromagnetic waves in a medium of finite conductivity," *Geophys.*, vol. XXII, no. 1, pp. 75–88, Jan. 1957.

[4] ——, "Propagation of an electric pulse through a homogeneous and isotropic medium," *Geophys.*, vol. XXII, no. 4, pp. 905–921, Oct. 1957.

[5] J. R. Wait, "Propagation of electromagnetic pulses in a homogeneous conducting earth," *Appl. Scient. Res.*, sect. B, vol. 8, pp. 213–253, 1960.

[6] J. R. Wait and K. P. Spies, "Transient fields for an electric dipole in a dissipative medium," *Can. J. Phys.*, vol. 48, pp. 1858–1862, 1970.

[7] C. R. Burroughs, "DC signaling in conducting media," *IRE Trans. Antennas Propagat.*, pp. 328–334, May 1962.

[8] C. R. Burroughs, "Transient response in an imperfect dielectric," *IRE Trans. Antennas Propagat.*, pp. 286–296, May 1963.

[9] J. S. Malik, "EM pulse fields in dissipative media," Theoretical Note 8, Apr. 1965.

[10] C. Baum, "The SEM representation of scattering from perfectly conducting targets in simply lossy media," Interaction Note 492, Apr. 21, 1993.

[11] H. F. Harmuth, "Correction of Maxwell's equations for signals I," *IEEE Trans. Electromagn. Compat.*, vol. EMC-28, no. 4, pp. 250–256, Nov. 1986.

[12] ——, "Correction of Maxwell's equations for signals II," *IEEE Trans. Electromagn. Compat.*, vol. EMC-28, no. 4, pp. 259–265, Nov. 1986.

[13] E. F. Kuester, "Comments on 'Correction of Maxwell's equations for signals I,' 'Correction of Maxwell's equations for signals II', and 'Propagation velocity of electromagnetic signals'," *IEEE Trans. Electromagn. Compat.*, vol. EMC-29, no. 2, pp. 187–191, May 1987.

[14] H. F. Harmuth and M. G. M. Hussain, "Response to a letter by J.R. Wait about electromagnetic waves in seawater," *IEEE Trans. Electromagn. Compat.*, vol. 34, no. 4, pp. 491–492, Nov. 1992.

[15] J. D. Jackson, *Classical Electrodynamics*, 2nd ed. New York: Wiley, 1975.

[16] R. N. Boules, "Absorption losses in seawater for a rectangular electromagnetic pulse," in *IEEE Int. Symp. on Electromagnetic Compatibility, Symp. Rec.* (Cherry Hill, NJ, Aug. 12–16, 1991), pp. 220–221.

[17] P. Hillion, "Remark on Harmuth's "Corrections of Maxwell's equations for signals I," *IEEE Trans. Electromagn. Compat.*, vol. 33, no. 2, p. 144, May 1991.

[18] P. Hillion, "Response to 'The magnetic conductivity and wave propagation'," *IEEE Trans. Electromagn. Compat.*, vol. 34, no. 3, pp. 376–377, Aug. 1992.

[19] P. Hillion, "A Further remark on Harmuth's problem," *IEEE Trans. Electromagn. Compat.*, vol. 34, no. 3, pp. 377–378, Aug. 1992.

[20] H. F. Harmuth, R. N. Boules, and M. G. M. Hussain, "Reply to Kuester's comments on the use of a magnetic conductivity," *IEEE Trans. Electromagn. Compat.*, vol. EMC-29, no. 4, pp. 318– 320, Nov. 1987.

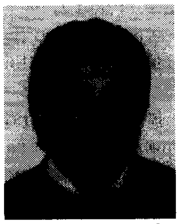
[21] R. Becker, *Electromagnetic Fields and Interactions*, vol. 1. New York: Blaisdell, 1964.

[22] J. A. Stratton, *Electromagnetic Theory.* New York: McGraw-Hill, 1941.

[23] G. Doetsch, *Guide to the Applications of Laplace Transforms.* London, England: Van Nostrand, 1961.

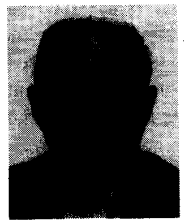
[24] G. E. Roberts and H. Kaufman, *Table of Laplace Transforms.* Philadelphia, PA: Saunders, 1966.

[25] E. Hallen, *Electromagnetic Theory.* London, England: Chapman & Hall, 1962.



Joe LoVetri (S'84–M'84–S'86–S'87–M'97–M'90) was born in Enna, Italy, in 1963. He received the B.Sc. (with distinction) and the M.Sc. degrees, both in electrical engineering, from the University of Manitoba, Winnipeg, Man., Canada, in 1984 and 1987, respectively, and the Ph.D. degree in electrical engineering from the University of Ottawa, Ottawa, Ont., Canada, in 1991.

From 1984 to 1986, he was EMI/EMC engineer at Sperry Defence Division in Winnipeg. From 1986 to 1988, he held the position of TEMPEST engineer at the Communications Security Establishment in Ottawa. From 1988 to 1991, he was a Research Officer at the Institute for Information Technology of the National Research Council of Canada. He is currently an Assistant Professor in the Department of Electrical Engineering of the University of Western Ontario, London, Ont., Canada. His main interests lie in the general area of algorithmic and nonalgorithmic computer modeling of physical processes, especially, electromagnetic interacions.



Joachim B. Ehrman was born in Nuremberg, Germany, in 1929. He received the A.B. degree from the University of Pennsylvania, Philadelphia, in 1948, the A.M. degree from Princeton University, Princeton, NJ, in 1949, and the Ph.D. degree, in 1954, also from Princeton University.

He worked at North American Aviation, Inc., from 1951 to 1953, Yale University, from 1954 to 1955, U.S. Naval Research Laboratory from 1955 to 1968 (Nucleonics Division from 1955 to 1966, Plasma Physics Division from 1966 to 1968), he was Visiting Research Scientist at the Princeton University Plasma Physics Laboratory from 1975 to 1976, and since 1968 has been Professor of Applied Mathematics at the University of Western Ontario, London, Ont., Canada. His main areas of research lie in plasma physics, electron ring accelerators, group theory, electromagnetic waves, electron penetration through matter, and nuclear physics.