Time-Domain Electromagnetic Plane Waves in Static and Dynamic Conducting Media II

Joachim B. Ehrman and Joe LoVetri, *Member, IEEE*

***Abstract*—The electromagnetic field inside a lossy half-space for the case of a transient electromagnetic plane wave impinging on the half-space from free space is derived. The losses in the half-space are modeled by assuming either a static ($J = \sigma E$) or a dynamic ($\tau \partial J/\partial t + J = \sigma_0 E$) conducting medium. Solutions are derived directly from the first order system of partial differential equations, i.e., the Maxwell equations. Plots for the total fields at the half-space boundary are given and expressions for the fields anywhere inside the half-space based on these boundary fields are given. Asymptotic formulae for late and early times are derived for the case of a step function as well as a square pulse plane wave.**

I. Introduction

In a previous paper [1], we considered the initial boundary value problem where the electric field was given at the plane boundary of a lossy half-space. This lossy half-space was modeled by assuming either a static or a dynamic conductivity [2], [3]. It was shown that the inclusion of a magnetic conductity term, in the Maxwell equations, as originally suggested by Harmuth [4], [5], was unnecessary for the solution of this problem. Although the problem was first thus formulated in the literature [6], the specification of the total electric field at the boundary is less suggestive of how the field is to be produced physically than a formulation in terms of a plane transient wave impinging from a free space region onto a lossy half-space. The formulation by specification of the total electric field is the more common one in the literature [6], [5], but there exist various research notes [7]–[11] as well as a conference paper by Barnes and Tesche [12] which study the reflection of a plane wave from and the transmission into a lossy half-space. (References [7]–[12] were brought to our attention by one of the reviewers.)

This approach has been used in the present paper to complete the arguments of our previous paper [1] of the possibility of a unique solution without the inclusion of a magnetic conductivity term in the Maxwell equations. References [7]–[12] are more general than this paper in that we deal only with normal incidence of the plane wave on the boundary. On the other hand, the present paper considers not only static but also dynamic conductivity [2] for the lossy medium, in order to answer the challenge of Harmuth and Hussain [3]. In addition, the results given herein are considerably simpler than those of references [7]–[12], and we derive a useful long time asymptotic formula for the total electric field at the boundary and other pertinent physical quantities. All of the above references as well as the present work assume a frequency independent dielectric constant, which is valid for water below 10^{10} [Hz] but must be modified at higher frequencies.

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J. B. Ehrman is with the Department of Applied Mathematics, The University of Western Ontario, London, ON, N6A 5B7 Canada.

J. LoVetri is with the Department of Electrical Engineering, The University of Western Ontario, London, ON, N6A 5B9 Canada.

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II. Mathematical Formulation of the Problem

The problem of a transient plane wave impinging on a lossy-half space can be formulated mathematically via the Maxwell equations, in each region. That is

$$\frac{\partial E}{\partial x} + \mu_0 \frac{\partial H}{\partial t} = 0 \tag{1}$$

$$\frac{\partial H}{\partial x} + \varepsilon_0 \frac{\partial E}{\partial t} = 0 \tag{2}$$

in the free space region, $x < 0$, and

$$\frac{\partial E}{\partial x} + \mu_1 \frac{\partial H}{\partial t} = 0, \tag{3}$$

$$\frac{\partial H}{\partial x} + \varepsilon_1 \frac{\partial E}{\partial t} + J = 0 \tag{4}$$

$$\frac{\partial J}{\partial t} + \frac{J}{\tau} = \frac{\sigma_0 E}{\tau} \tag{5}$$

in the lossy half-space, $x > 0$, where $E = E_y(x,t)$ [V/m], $H = H_z(x,t)$ [A/m], represent the y-component of the electric and the z-component of the magnetic field intensity vectors respectively, $J = J_y(x,t)$ [A/m^2] is the y-component of the current density vector, σ_0 [S/m] is the static or dc conductivity of the medium, and τ [sec] is the time constant of the medium related to the type and density of charge carriers in the medium [2]. Of course, ε_1 [F/m] and μ_1 [H/m] are the permittivity and permeability of the half-space.

We now assume that the impinging electromagnetic field is such that at the half-space boundary, $x = 0^-$,

$$E_i(0^-,t) = \begin{cases} E_0(t) & t > 0 \\ 0 & t < 0 \end{cases} \tag{6}$$

$$H_i(0^-,t) = \begin{cases} \sqrt{\dfrac{\varepsilon_0}{\mu_0}} E_0(t) & t > 0 \\ 0 & t < 0 \end{cases} \tag{7}$$

and we denote the reflected fields at the point $x = 0^-$ as

$$E_r(0^-,t) = \begin{cases} F(t) & t > 0 \\ 0 & t < 0 \end{cases} \tag{8}$$

$$H_r(0^-,t) = \begin{cases} -\sqrt{\dfrac{\varepsilon_0}{\mu_0}} F(t) & t > 0 \\ 0 & t < 0 \end{cases}. \tag{9}$$

The total field is given by the sum of the incident and reflected components and therefore, due to the continuity of the tangential fields across the plane interface, we have that the total fields at the point $x = 0^+$ are given by

$$E_{\text{Total}}(t) = E(0^+, t) = \begin{cases} E_0(t) + F(t) & t > 0 \\ 0 & t < 0 \end{cases} \tag{10}$$

$$H_{\text{Total}}(t) = H(0^+, t) = \begin{cases} \sqrt{\frac{\varepsilon_0}{\mu_0}}(E_0(t) - F(t)) & t > 0 \\ 0 & t < 0 \end{cases}. \tag{11}$$

The initial conditions in the lossy half-space are given as

$$\begin{aligned} E(x, 0^+) &= 0, \quad H(x, 0^+) = 0, \\ J(x, 0^+) &= 0, \quad x > 0 \end{aligned} \tag{12}$$

where it should be noted that the current density at the boundary ($x = 0$) is also initially equal to zero (at least for the case where $\tau > 0$) even for the case of a discontinuous electric field arriving at the boundary. These equations define the initial boundary value problem which we must solve for all values of x and for positive time.

III. Solution Via the Laplace Transform

A. Expressions in the Laplace Domain

We denote the Laplace transform with respect to the time variable of the electric field intensity as $e(x, s) = L\{E(x, t)\}$, with similar notation for the transforms of the magnetic field, $h(x, s)$, and the current density, $j(x, s)$. Thus, (1)–(5) transform to

$$\frac{d}{dx}e(x, s) + \mu_0 s h(x, s) = 0 \tag{13}$$

$$\frac{d}{dx}h(x, s) + \varepsilon_0 s e(x, s) = 0 \tag{14}$$

in the free space region, $x < 0$, and

$$\frac{d}{dx}e(x, s) + \mu_1 s h(x, s) = 0 \tag{15}$$

$$\frac{d}{dx}h(x, s) + \varepsilon_1 s e(x, s) + j(x, s) = 0 \tag{16}$$

$$\tau s j(x, s) + j(x, s) = \sigma_0 e(x, s) \tag{17}$$

in the lossy half-space, $x > 0$. Solving for $j(x, s)$ in (17) and substituting into (16) we have

$$\frac{d}{dx}h(x, s) + \left(\varepsilon_1 s + \frac{\sigma_0}{1 + \tau s}\right)e(x, s) = 0. \tag{18}$$

The boundary conditions of (10) and (11) transform as

$$e(0^+, s) = e_0(s) + f(s) \tag{19}$$

$$h(0^+, s) = \sqrt{\frac{\varepsilon_0}{\mu_0}}[e_0(s) - f(s)] \tag{20}$$

and thus, (15), and (18)–(20) represent a boundary value problem in the (x, s) domain. Taking the Laplace transform with respect to the x variable, by defining

$$\begin{aligned} \tilde{e}(p, s) &= \int_0^\infty e(x, s) e^{-px}\, dx, \\ \tilde{h}(p, s) &= \int_0^\infty h(x, s) e^{-px}\, dx \end{aligned} \tag{21}$$

we get the algebraic system

$$\begin{bmatrix} p & \mu_1 s \\ \varepsilon_1 s + \frac{\sigma_0}{1 + \tau s} & p \end{bmatrix} \begin{bmatrix} \tilde{e}(p, s) \\ \tilde{h}(x, s) \end{bmatrix} = \begin{bmatrix} e_0(s) + f(s) \\ \sqrt{\frac{\varepsilon_0}{\mu_0}}[e_0(s) - f(s)] \end{bmatrix}. \tag{22}$$

This system is easily solved as

$$\begin{bmatrix} \tilde{e}(p, s) \\ \tilde{h}(p, s) \end{bmatrix} = \frac{1}{p^2 - \varsigma^2(s)} \begin{bmatrix} p & -\mu_1 s \\ -\varepsilon_1 s - \frac{\sigma_0}{1 + \tau s} & p \end{bmatrix} \cdot \begin{bmatrix} e_0(s) + f(s) \\ \sqrt{\frac{\varepsilon_0}{\mu_0}}[e_0(s) - f(s)] \end{bmatrix},$$

where we have defined

$$\varsigma^2(s) \equiv \left(\varepsilon_1 s + \frac{\sigma_0}{1 + \tau s}\right)\mu_1 s. \tag{23}$$

Transforming back from the (p, s) space to the (x, s) space yields the expressions

$$e(x, s) = [e_0(s) + f(s)] \cosh(\varsigma(s)x) - \sqrt{\frac{\varepsilon_0}{\mu_0}}[e_0(s) - f(s)]\frac{\mu_1 s}{\varsigma(s)} \sinh(\varsigma(s)x) \tag{24}$$

$$\begin{aligned} h(x, s) = {}& \sqrt{\frac{\varepsilon_0}{\mu_0}}[e_0(s) - f(s)] \cosh(\varsigma(s)x) \\ & - [e_0(s) + f(s)]\left[\varepsilon_1 s + \frac{\sigma_0}{1 + \tau s}\right] \cdot \frac{\sinh(\varsigma(s)x)}{\varsigma(s)} \end{aligned} \tag{25}$$

which, as in [1], using the conditions that $\lim_{x\to\infty} e(x, s) = 0$, and $\lim_{x\to\infty} h(x, s) = 0$ for all values of s results in

$$\lim_{x\to\infty} \left\{ [e_0(s) + f(s)]\frac{e^{\varsigma(s)x}}{2} - \sqrt{\frac{\varepsilon_0}{\mu_0}}[e_0(s) - f(s)]\frac{\mu_1 s}{\varsigma(s)}\frac{e^{\varsigma(s)x}}{2} \right\} = 0,$$

$$\lim_{x\to\infty} \left\{ \sqrt{\frac{\varepsilon_0}{\mu_0}}[e_0(s) - f(s)]\frac{e^{\varsigma(s)x}}{2} - [e_0(s) + f(s)]\left[\varepsilon_1 s + \frac{\sigma_0}{1 + \tau s}\right]\frac{e^{\varsigma(s)x}}{2\varsigma(s)} \right\} = 0.$$

Both these expressions require that the reflected field boundary value be implicitly related to the incident field boundary value via the relation

$$\begin{aligned}\frac{e_0(s)-f(s)}{e_0(s)+f(s)} &= \sqrt{\frac{\mu_0}{\varepsilon_0}}\sqrt{\frac{1}{\mu_1 s}\left(\varepsilon_1 s+\frac{\sigma_0}{1+\tau s}\right)}\\ &= \sqrt{\frac{\mu_0\varepsilon_1}{\varepsilon_0\mu_1}}\sqrt{1+\frac{\sigma_0}{\varepsilon_1 s(1+\tau s)}}\\ &= \frac{\varsigma(s)}{\mu_1 s}\sqrt{\frac{\mu_0}{\varepsilon_0}} = Q(s). \end{aligned} \tag{26}$$

It is a simple matter to derive the explicit relation

$$f(s) = \frac{1-Q(s)}{1+Q(s)}e_0(s) \tag{27}$$

which when substituted back into (24) and (25) produces

$$\begin{aligned} e(x,s) &= \left[e_0(s)\frac{2}{1+Q(s)}\right]\frac{e^{-\varsigma(s)x}}{2}\\ &\quad+\sqrt{\frac{\varepsilon_0}{\mu_0}}\left[e_0(s)\frac{2Q(s)}{1+Q(s)}\right]\frac{\mu_1 s}{\varsigma(s)}\frac{e^{-\varsigma(s)x}}{2}\\ e(x,s) &= e_0(s)\frac{2}{1+Q(s)}e^{-\varsigma(s)x} = e_0(s)A(s)e^{-\varsigma(s)x} \end{aligned} \tag{28}$$

and

$$\begin{aligned} h(x,s) &= \sqrt{\frac{\varepsilon_0}{\mu_0}}\left[e_0(s)\frac{2Q(s)}{1+Q(s)}\right]\frac{e^{-\varsigma(s)x}}{2}\\ &\quad+\left[e_0(s)\frac{2}{1+Q(s)}\right]\left[\varepsilon_1 s+\frac{\sigma_0}{1+\tau s}\right]\\ &\quad\cdot\frac{1}{\varsigma(s)}\frac{e^{-\varsigma(s)x}}{2}\\ h(x,s) &= \sqrt{\frac{\varepsilon_0}{\mu_0}}\left[e_0(s)\frac{2Q(s)}{1+Q(s)}\right]e^{-\varsigma(s)x}\\ &= \sqrt{\frac{\varepsilon_0}{\mu_0}}e_0(s)Q(s)A(s)e^{-\varsigma(s)x} \end{aligned} \tag{29}$$

where $A(s)$ is given by

$$A(s) = \frac{2}{1+Q(s)}. \tag{30}$$

B. Inversion of Laplace Domain Expressions

In order to complete the solution of the total electric and magnetic fields inside the lossy half-space region it is now required that we take the inverse Laplace transform of the expressions given by (28) and (29). The expressions for $Q(s)$ and $\zeta(s)$ can be simplified by assuming that

$$\frac{1}{1+\tau s} \cong 1-\tau s \tag{31}$$

which is valid for $\tau s \ll 1$ so that we can write

$$\begin{aligned} Q(s) &= \sqrt{\frac{\mu_0\varepsilon_1}{\varepsilon_0\mu_1}}\sqrt{1+\frac{\sigma_0}{\varepsilon_1 s(1+\tau s)}}\\ &\cong \sqrt{\frac{\mu_0\varepsilon_1}{\varepsilon_0\mu_1}}\sqrt{\left(1-\frac{\tau\sigma_0}{\varepsilon_1}\right)+\frac{\sigma_0}{s\varepsilon_1}}\\ &= Q_0\sqrt{1+\frac{a}{s}} \end{aligned} \tag{32}$$

and

$$\begin{aligned} \varsigma(s) &= \sqrt{\left(\varepsilon_1 s+\frac{\sigma_0}{1+\tau s}\right)\mu_1 s}\\ &\cong \sqrt{\mu_1(\varepsilon_1-\sigma_0\tau)}\sqrt{s(s+a)}\\ &= \sqrt{\mu_1 s(\hat{\varepsilon}s+\sigma_0)} \end{aligned} \tag{33}$$

where

$$Q_0 = \sqrt{\frac{\mu_0\hat{\varepsilon}}{\mu_1\varepsilon_0}}, \qquad a = \frac{\sigma_0}{\hat{\varepsilon}}$$

and

$$\hat{\varepsilon} = \varepsilon_1 - \tau\sigma_0. \tag{34}$$

Note that for seawater, $\tau \cong 4\times 10^{-15}$ [sec], $\epsilon_1 \cong 80\epsilon_0$, $\sigma_0 \cong 4$ [S/m] and thus, $Q_0 \cong 9$ and $a \cong 6\times 10^9$ [sec^{-1}].

As can be seen from the previous section, the expressions for the field inside the lossy half-space are quite formidable. The solution can be simplified by first finding the inverse of $f(s)$ given by (27). To obtain $F(t)$ we first wish to obtain

$$L^{-1}\left[\frac{1-Q(s)}{1+Q(s)}\right] = L^{-1}\left[\frac{1-Q_0\sqrt{1+\frac{a}{s}}}{1+Q_0\sqrt{1+\frac{a}{s}}}\right]. \tag{35}$$

It can be shown [13], [14] that, for $Q_0 \neq 1$,

$$\begin{aligned} L^{-1}\left[\frac{1-Q_0\sqrt{1+\frac{a}{s}}}{1+Q_0\sqrt{1+\frac{a}{s}}}\right] &= \frac{1-Q_0}{1+Q_0}\delta(t-0^+)\\ &\quad+\frac{2Q_0}{Q_0^2-1}e^{-\Theta t}W(t) \end{aligned} \tag{36}$$

where

$$\Theta = \frac{aQ_0^2}{Q_0^2-1} = \frac{\sigma_0\mu_0}{\mu_0\hat{\varepsilon}-\mu_1\varepsilon_0} \tag{37}$$

and

$$\begin{aligned} W(t) &= \frac{aQ_0}{Q_0^2-1}+\frac{a^2Q_0^2}{(Q_0^2-1)^2}\\ &\quad\cdot\int_0^t \exp\left(\frac{Q_0^2+1}{Q_0^2-1}\frac{a}{2}y\right)I_0\left(\frac{a}{2}y\right)dy\\ &\quad+\frac{a}{2}\exp\left(\frac{Q_0^2+1}{Q_0^2-1}\frac{a}{2}t\right)\\ &\quad\cdot\left[I_1\left(\frac{a}{2}t\right)-\frac{Q_0^2+1}{Q_0^2-1}I_0\left(\frac{a}{2}t\right)\right]. \end{aligned} \tag{38}$$

Since

$$F(t) = E_0(t)\otimes L^{-1}\left[\frac{1-Q(s)}{1+Q(s)}\right],$$

where $\otimes$ denotes the convolution operator, $F(t)$ can be obtained if $E_0(t)$ is specified. In particular, consider the case $E_0(t) = E_0\mathbf{1}(t)$, where E_0 is a constant and $\mathbf{1}(t)$ is the

Heaviside step function (i.e. equals 1 for $t > 0$ and 0 for $t < 0$). Then

$$F(t) = -E_0 \frac{Q_0 - 1}{Q_0 + 1} + \frac{2Q_0 E_0}{Q_0^2 - 1} \int_0^t e^{-\Theta y} W(y)\, dy \tag{39}$$

and it can be shown (see Appendix A) that, for $t > 0$,

$$\begin{aligned} F(t) = &-E_0 - \frac{2E_0}{Q_0^2 - 1} e^{-\Theta t} + \frac{2Q_0 E_0}{Q_0^2 - 1} e^{-at/2} I_0\left(\frac{at}{2}\right) \\ &- \frac{2\Theta E_0}{Q_0(Q_0^2 - 1)} e^{-\Theta t} \int_0^t \exp\left(\frac{Q_0^2 + 1}{Q_0^2 - 1} \frac{ay}{2}\right) \\ &\cdot I_0\left(\frac{ay}{2}\right) dy. \end{aligned} \tag{40}$$

Note that the total electric field and magnetic field at $x = 0$ (for $t > 0$) are given by (10) and (11) respectively. These can now be substituted into any of the formulae given in [6], [15], or in [4] for the fields inside a lossy slab given the total electric field at $x = 0$. For convenience, we repeat the expressions of [1] here

$$\begin{aligned} E(x,t) = &e^{-(ax/2c_1)} E_{\text{Total}}(t - x/c_1) \\ &+ e^{-(at/2)} \frac{ax}{2c_1} \int_0^{t - x/c_1} E_{\text{Total}}(y) e^{ay/2} \\ &\cdot \frac{I_1\left(\frac{a}{2}\sqrt{(t-y)^2 - (x/c_1)^2}\right)}{\sqrt{(t-y)^2 - (x/c_1)^2}}\, dy \end{aligned} \tag{41}$$

$$\begin{aligned} H(x,t) = \sqrt{\frac{\hat{\varepsilon}}{\mu_1}} \Bigg[&e^{-(ax/2c_1)} E_{\text{Total}}(t - x/c_1) \\ &+ \frac{a}{2} e^{-(at/2)} \int_0^{t - x/c_1} E_{\text{Total}}(y) e^{ay/2} \\ &\cdot I_0\left(\frac{a}{2}\sqrt{(t-y)^2 - (x/c_1)^2}\right) dy \\ &+ \frac{a}{2} e^{-(at/2)} \int_0^{t - x/c_1} E_{\text{Total}}(y) e^{ay/2} \\ &\cdot \frac{t - y}{\sqrt{(t-y)^2 - (x/c_1)^2}} \\ &\cdot I_1\left(\frac{a}{2}\sqrt{(t-y)^2 - (x/c_1)^2}\right) dy \Bigg] \end{aligned} \tag{42}$$

where $c_1 = (\mu_1 \hat{\varepsilon})^{-1/2}$ is the speed of propagation in the half-space.

IV. Asymptotic Expansion of the Fields

A. Step Function Incidence

Using the asymptotic expansion for the modified Bessel function it can be shown (see Appendix B) that for $t \gg 1/a$(i.e. $t \gg 2 \times 10^{-10}$ [sec] for seawater)

$$F(t) \cong E_0 \left\{ -1 + \frac{2}{Q_0 \sqrt{\pi a t}} + \frac{Q_0^2 - 2}{2\sqrt{\pi} Q_0^3 (at)^{3/2}} + \cdots \right\} \tag{43}$$

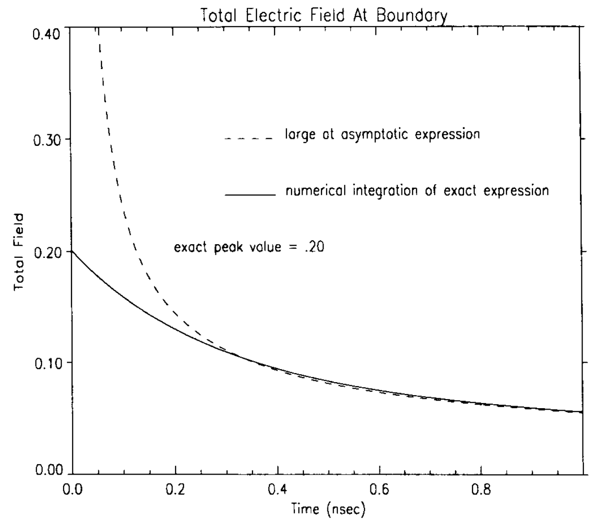

Fig. 1. Total electric field at boundary of lossy half-space (sea water).

so that the total electric field at the boundary is given by

$$\begin{aligned} E_{\text{Total}}(t) &= E(0^+, t) \\ &\cong E_0 \left\{ \frac{2}{Q_0 \sqrt{\pi a t}} + \frac{Q_0^2 - 2}{2\sqrt{\pi} Q_0^3 (at)^{3/2}} + \cdots \right\} \end{aligned} \tag{44}$$

while the total magnetic field at the boundary is given by

$$\begin{aligned} H_{\text{Total}}(t) &= H(0^+, t) \\ &\cong E_0 \sqrt{\frac{\varepsilon_0}{\mu_0}} \left\{ 2 - \frac{2}{Q_0 \sqrt{\pi a t}} \right. \\ &\qquad \left. - \frac{Q_0^2 - 2}{2\sqrt{\pi} Q_0^3 (at)^{3/2}} - \cdots \right\}. \end{aligned} \tag{45}$$

Thus, we see that $E_{\text{Total}}(t) \to 0$ proportional to $1/\sqrt{at}$ as $t \to \infty$ (i.e., the electric field is "shorted out" by the conducting medium) while $H_{\text{Total}}(t) \to 2E_0\sqrt{\varepsilon_0/\mu_0}$ as $t \to \infty$ (i.e. the magnetic field becomes a constant twice the value of the incident magnetic field). This can be compared to the case considered in [1], [6], [4] where the total electric field at the boundary of the lossy half-space is a constant and the magnetic field goes to infinity. A plot of the asymptotic expansion for the electric field, (44), as well as the numerical integration of the exact expression given by (40) is shown in Fig. 1.

For times such that $at \ll 1$, that is for the early time behavior, it is easy to show by Taylor series expansion that

$$F(t) \cong E_0 \left\{ -\frac{Q_0 - 1}{Q_0 + 1} - \frac{Q_0}{(Q_0 + 1)^2} at + O(a^2 t^2) \right\} \tag{46}$$

from which

$$E_{\text{Total}}(t) \cong E_0 \left\{ \frac{2}{Q_0 + 1} - \frac{Q_0}{(Q_0 + 1)^2} at + O(a^2 t^2) \right\} \tag{47}$$

However, for seawater, this formula is much less interesting than the large at formulae of (44) and (45) because $t \ll 1/a \cong 2 \times 10^{-10}$ [sec], but the assumption that ϵ_1 is frequency independent becomes invalid for frequencies $>10^{10}$ [Hz], which means that a step function really is meant to model a function which does not change appreciably in times $<10^{-10}$ [sec]. Hence, the range of t over which the small at formula of (46) is valid, for seawater, is negligible. For a somewhat poorer conductor than seawater, for which $1/a$ is quite a bit longer than 2×10^{-10} [sec], the small at formula might be useful.

Let us now compute the dominant term in the asymptotic time dependence of $E(x,t)$, $H(x,t)$ for fixed x as $t \to \infty$. For the function $E_{\text{Total}}(t)$ in the integrands of the second terms in (41) and (42) we shall not use the complicated exact expression derivable from (40) and (10), but rather

$$E_{\text{Tot}}(t) = \begin{cases} \dfrac{2E_0}{Q_0\sqrt{\pi P}} & 0 < t \le P/a \\ \dfrac{2E_0}{Q_0\sqrt{\pi a t}} & P/a \le t \end{cases} \tag{48}$$

where

$$P = \frac{1}{\pi}\left[\frac{Q_0+1}{Q_0}\right]^2 \cong 0.39$$

for seawater. This expression gives the dominant behavior correctly for $at \gg 1$, the correct value $2E_0/(Q_0+1)$, for $t = 0$,and is continuous for all t. It will be shown below that those results which depend on the precise value of P (rather than the assumption that $P \sim 1$) do not contribute to the leading term in the asymptotic expansion, which justifies this simplification for $E_{\text{Total}}(t)$ *a posteriori*. Of course, coefficients of higher terms in the asymptotic expansion cannot be validly calculated using this simplification and require the use of the exact expression for $E_{\text{Total}}(t)$, or at least a closer (and more complicated) approximation to it.

Inserting $E_{\text{Total}}(t)$ into the second term of (41) for $E(x,t)$ gives

$$\begin{aligned} E_{\text{s.t.}}(x,t) = {} & \frac{axE_0}{Q_0 c_1\sqrt{\pi P}} \int_0^{P/a} e^{-a/2(t-y)} \\ & \cdot \frac{I_1\left(\frac{a}{2}\sqrt{(t-y)^2-(x/c_1)^2}\right)}{\sqrt{((t-y)^2-(x/c_1)^2)}}\, dy \\ & + \frac{axE_0}{Q_0 c_1}\int_{P/a}^{t-x/c_1} \frac{e^{-a/2(t-y)}}{\sqrt{\pi a y}} \\ & \cdot \frac{I_1\left(\frac{a}{2}\sqrt{(t-y)^2-(x/c_1)^2}\right)}{\sqrt{(t-y)^2-(x/c_1)^2}}\, dy. \end{aligned} \tag{49}$$

Defining $X = ax/2c_1$, $u = c_1t/x$, so that $uX = at/2$ and introducing in lieu of y the variable of integration where $v = (c_1/x)(t-y)$, then

$$\begin{aligned} E_{\text{s.t.}}(x,t) = {} & X^2\frac{2E_0}{Q_0\sqrt{\pi P}}\int_{u(1-P/at)}^{u} e^{-vX} \\ & \cdot \frac{I_1(X\sqrt{v^2-1})}{X\sqrt{v^2-1}}\, dv \\ & + X^{3/2}\frac{\sqrt{2}E_0}{Q_0\sqrt{\pi}}\int_1^u \frac{e^{-vX}}{\sqrt{u-v}} \\ & \cdot \frac{I_1(X\sqrt{v^2-1})}{X\sqrt{v^2-1}}\, dv \\ & - X^{3/2}\frac{\sqrt{2}E_0}{Q_0\sqrt{\pi}}\int_{u(1-P/at)}^{u} \frac{e^{-vX}}{\sqrt{u-v}} \\ & \cdot \frac{I_1(X\sqrt{v^2-1})}{X\sqrt{v^2-1}}\, dv \end{aligned} \tag{50}$$

where the first line comes from the first integral of (49) while the second and third lines come from the second integral in (49). Defining these three lines as E_{st1}, E_{st2}, and E_{st3} respectively we now evaluate the leading term in an asymptotic expansion for E_{st2} (for $u \gg 1$) and show that E_{st1} and E_{st3} are both negligible compared to this leading term.

We define

$$I_E(X,u) = \int_1^u \frac{e^{-vX}}{\sqrt{u-v}} \frac{I_1(X\sqrt{v^2-1})}{X\sqrt{v^2-1}}\, dv \tag{51}$$

which for u large and X fixed

$$I_E(X,u) \approx f(X)u^{-1/2} + O(u^{-3/2}) \tag{52}$$

and we want to determine $f(X)$. The function $f(X)$ is determined as follows (see [16], #8, p. 200)

$$\begin{aligned} f(X) &= \lim_{u\to\infty}(u^{1/2}I_E(X,u)) \\ &= \lim_{u\to\infty}\int_1^u \frac{e^{-vX}}{\sqrt{1-v/u}} \frac{I_1(X\sqrt{v^2-1})}{X\sqrt{v^2-1}}\, dv \\ &= \int_1^\infty e^{-vX}\frac{I_1(X\sqrt{v^2-1})}{X\sqrt{v^2-1}}\, dv = \frac{1-e^{-X}}{X^2} \end{aligned} \tag{53}$$

and hence,

$$\begin{aligned} E_{st2}(x,t) &= \sqrt{\frac{2}{\pi}}\frac{E_0}{Q_0}\frac{1-e^{-X}}{X^{1/2}}u^{-1/2} + O(u^{-3/2}) \\ &= \frac{2E_0}{Q_0\sqrt{\pi a t}}(1-e^{-X}) + O(t^{-3/2}). \end{aligned} \tag{54}$$

To estimate E_{st1}, we replace the integrand by

$$e^{-uX}\frac{I_1(X\sqrt{u^2-1})}{X\sqrt{u^2-1}} \approx \frac{1}{\sqrt{2\pi}X^{3/2}u^{3/2}} \tag{55}$$

to give

$$E_{st1} \approx \frac{1}{\pi\sqrt{2}}\frac{E_0\sqrt{P}}{Q_0 u^{3/2}\sqrt{X}} \tag{56}$$

which is small compared to E_{st2} for $u \gg 1$.

Similarly, in E_{st3}, we replace v by u in all factors of the integrand except in the factor $1/\sqrt{u-v}$, and again use the approximation of (55). Doing this, we have

$$E_{st3} \approx -\frac{\sqrt{2}}{\pi}\frac{E_0\sqrt{P}}{Q_0 u^{3/2}\sqrt{X}} \tag{57}$$

which is of the same order of magnitude as E_{st1}. Hence the second term

$$E_{st} \approx \frac{2E_0}{Q_0\sqrt{\pi a t}}(1-e^{-X}) + O(t^{-3/2}) \tag{58}$$

but from (41), the first term, E_{ft} of E is

$$\begin{aligned} E_{ft} &\approx e^{-X} E_{\mathrm{Tot}}(t - x/c_1) \\ &= e^{-X} \frac{2E_0}{Q_0\sqrt{\pi a(t - x/c_1)}} \\ &= e^{-X} \frac{2E_0}{Q_0\sqrt{\pi at}} + O(t^{-3/2}) \end{aligned} \tag{59}$$

so that

$$E(x,t) = \frac{2E_0}{Q_0\sqrt{\pi at}} + O(t^{-3/2}) \tag{60}$$

for *all* $x > 0$, t large enough. (Of course, it has been assumed that $u = t/(x/c_1) \gg 1$, not merely that $at \gg 1$.

Now, inserting E_{Tot} of (48) into the second and third terms, which we denote H_{st}, of (42) for $H(x,t)$, and proceeding similarly, it can be shown that

$$H_{st}(x,t) = H_{st1} + H_{st2} + H_{st3} \tag{61}$$

where

$$\begin{aligned} H_{st1}(x,t) = \sqrt{\frac{\hat{\varepsilon}}{\mu_1}} \frac{2XE_0}{Q_0\sqrt{\pi P}} \int_{u(1-P/at)}^{u} e^{-vX} &\cdot \left[\frac{v}{\sqrt{v^2-1}} I_1(X\sqrt{v^2-1}) \right. \\ &\left. + I_0(X\sqrt{v^2-1})\right] dv \end{aligned} \tag{62}$$

$$\begin{aligned} H_{st2}(x,t) = \sqrt{\frac{\hat{\varepsilon}}{\mu_1}} \sqrt{\frac{2}{\pi}} \frac{\sqrt{X}E_0}{Q_0} \int_1^u \frac{e^{-vX}}{\sqrt{u-v}} &\cdot \left[\frac{v}{\sqrt{v^2-1}} I_1(X\sqrt{v^2-1}) \right. \\ &\left. + I_0(X\sqrt{v^2-1})\right] dv \end{aligned} \tag{63}$$

$$\begin{aligned} H_{st3}(x,t) = -\sqrt{\frac{\hat{\varepsilon}}{\mu_1}} \sqrt{\frac{2}{\pi}} \frac{\sqrt{X}E_0}{Q_0} \int_{u(1-P/at)}^{u} \frac{e^{-vX}}{\sqrt{u-v}} &\cdot \left[\frac{v}{\sqrt{v^2-1}} I_1(X\sqrt{v^2-1}) \right. \\ &\left. + I_0(X\sqrt{v^2-1})\right] dv. \end{aligned} \tag{64}$$

Again, we begin by obtaining the asymptotic behavior of the important term H_{st2}. Defining

$$\begin{aligned} I_H(X,u) = \int_1^u \frac{e^{-vX}}{\sqrt{u-v}} &\cdot \left[\frac{v}{\sqrt{v^2-1}} I_1(X\sqrt{v^2-1}) \right. \\ &\left. + I_0(X\sqrt{v^2-1})\right] dv \end{aligned} \tag{65}$$

for X fixed and u large, $I_H(X,u)$ approaches a nonzero limit as $u \to \infty$. For all but a negligible portion of the v interval $[1, u]$, we may make the approximation

$$\begin{aligned} &e^{-vX}\left[\frac{v}{\sqrt{v^2-1}} I_1(X\sqrt{v^2-1}) + I_0(X\sqrt{v^2-1})\right] \\ &\approx \frac{2}{\sqrt{2\pi Xv}} \end{aligned} \tag{66}$$

that is, each term is approximated by $1/(\sqrt{2\pi Xv})$. Hence, for $u \to \infty$,

$$\begin{aligned} I_H(X,u) &= \sqrt{\frac{2}{\pi X}} \int_1^u \frac{1}{\sqrt{(u-v)v}} dv \\ &\approx \sqrt{\frac{2}{\pi X}} \left(\frac{\pi}{2} - \sin^{-1}(-1 + 2/u)\right) \\ &\to \sqrt{\frac{2\pi}{X}}. \end{aligned} \tag{67}$$

To estimate H_{st1} and H_{st3}, one can make the approximation

$$\begin{aligned} &e^{-vX}\left[\frac{v}{\sqrt{v^2-1}} I_1(X\sqrt{v^2-1}) + I_0(X\sqrt{v^2-1})\right] \\ &\approx e^{-uX}\left[\frac{u}{\sqrt{u^2-1}} I_1(X\sqrt{u^2-1}) + I_0(X\sqrt{u^2-1})\right] \\ &\approx \frac{2}{\sqrt{2\pi Xu}} \end{aligned} \tag{68}$$

and these terms are small, $O(t^{-1/2})$, as $t \to \infty$. The same is true of the first term

$$H_{ft}(x,t) = \sqrt{\frac{\hat{\varepsilon}}{\mu_1}} e^{-X} \frac{2E_0}{Q_0\sqrt{\pi a(t - x/c_1)}}. \tag{69}$$

Hence, for $x > 0$ and $c_1 t/x \gg 1$, as well as $at \gg 1$, $H(x,t)$ becomes

$$\begin{aligned} H(x,t) &\approx H_{st2}(x,t) \\ &= \sqrt{\frac{\hat{\varepsilon}}{\mu_1}} \sqrt{\frac{2}{\pi}} E_0 \frac{\sqrt{X}}{Q_0} I_H(X,u) \\ &\approx \sqrt{\frac{\hat{\varepsilon}}{\mu_1}} \sqrt{\frac{2}{\pi}} E_0 \frac{\sqrt{X}}{Q_0} \sqrt{\frac{2\pi}{X}} \\ &= \frac{2E_0\sqrt{\hat{\varepsilon}}}{Q_0\sqrt{\mu_1}} = 2E_0\sqrt{\frac{\varepsilon_0}{\mu_0}} \end{aligned} \tag{70}$$

since $\sqrt{\hat{\varepsilon}/\mu_1} = Q_0\sqrt{\epsilon_0/\mu_0}$ and we see that this agrees with the leading term for $H_{\mathrm{Total}}(0,t)$ of (45), as it must.

We now derive expressions for the early times after arrival at any fixed spatial position. That is, we assume $(a/2)(t - (x/c_1)) > 0$ but $(a/2)(t - (x/c_1)) \ll 1$ which gives the field just above the characteristic line $x = c_1 t$. (Note that we do not assume $at/2 \ll 1$ nor $ax/2c_1 \equiv X \ll 1$, only that the difference, $at/2 - X$, while >0, is $\ll 1$.) Let us then compute $E(x,t)$, $H(x,t)$ up to and including the first power of $at/2 - X$, but neglecting quantities of order $(at/2 - X)^2$. Then it is necessary to use the expansion given in (47) in the first term of the expressions for $E(x,t)$, $H(x,t)$ in (41) and (42) but only

$$E_{\mathrm{Total}}(t) = \frac{2E_0}{Q_0 + 1} + O\left(\frac{at}{2}\right) \tag{71}$$

in the other terms. Carrying out the Taylor series expansion consistently in powers of the small parameter $at/2 - X$ gives then

$$E_{ft}(x,t) = \frac{2E_0 e^{-X}}{Q_0 + 1} - \frac{2E_0 Q_0 e^{-X}}{(Q_0+1)^2}\left(\frac{at}{2} - X\right)$$

$$+O\left(\left(\frac{at}{2}-X\right)^2\right) \tag{72}$$

$$E_{st}(x,t)=\frac{XE_0e^{-X}}{Q_0+1}\left(\frac{at}{2}-X\right)+O\left(\left(\frac{at}{2}-X\right)^2\right) \tag{73}$$

and

$$E(x,t)=E_0e^{-X}\left[\frac{2}{Q_0+1}+\left(\frac{at}{2}-X\right)\cdot\left(\frac{-2Q_0}{(Q_0+1)^2}+\frac{X}{Q_0+1}\right)+O\left(\left(\frac{at}{2}-X\right)^2\right)\right]. \tag{74}$$

For $H(x,t)$, using $\sqrt{\epsilon/\mu_1}=Q_0\sqrt{\epsilon_0/\mu_0}$

$$H_{ft}(x,t)=\sqrt{\frac{\varepsilon_0}{\mu_0}}Q_0E_0e^{-X}\left[\frac{2}{Q_0+1}-\frac{2Q_0}{(Q_0+1)^2}\cdot\left(\frac{at}{2}-X\right)+O\left(\left(\frac{at}{2}-X\right)^2\right)\right] \tag{75}$$

$$H_{st}(x,t)=\sqrt{\frac{\varepsilon_0}{\mu_0}}Q_0E_0e^{-X}\left[\frac{X+2}{Q_0+1}\cdot\left(\frac{at}{2}-X\right)+O\left(\left(\frac{at}{2}-X\right)^2\right)\right] \tag{76}$$

and

$$H(x,t)=\sqrt{\frac{\varepsilon_0}{\mu_0}}Q_0E_0e^{-X}\left[\frac{2}{Q_0+1}+\left(\frac{at}{2}-X\right)\cdot\left(\frac{-2Q_0}{(Q_0+1)^2}+\frac{X+2}{Q_0+1}\right)+O\left(\left(\frac{at}{2}-X\right)^2\right)\right]. \tag{77}$$

B. Finite Square Pulse Incidence

If instead of a step function we take the incident field to be a square pulse which can be represented mathematically as

$$E_0(t)=E_0[\mathbf{1}(t)-\mathbf{1}(t-T)]=\begin{cases}0 & t<0\\ E_0 & 0<t<T\\ 0 & t>T\end{cases} \tag{78}$$

then, instead of (39), we get (taking the corresponding functions of t, $t-T$, and subtracting) for $t>T$

$$F(t)=\frac{2Q_0E_0}{Q_0^2-1}\int_{t-T}^{t}e^{-\Theta y}W(y)\,dy. \tag{79}$$

(Note that, for $0<t<T$, the formula of (39) still holds.) Hence as in (40), for $t>T$

$$\begin{aligned}F(t)=&-\frac{2E_0}{Q_0^2-1}[e^{-\Theta t}-e^{-\Theta(t-T)}]+\frac{2Q_0E_0}{Q_0^2-1}\cdot\left[e^{-(at/2)}I_0\left(\frac{at}{2}\right)-e^{-[a(t-T)/2]}I_0\left(\frac{a(t-T)}{2}\right)\right]\\&-\frac{2\Theta E_0}{Q_0(Q_0^2-1)}e^{-\Theta t}\cdot\int_0^t\exp\left(\frac{Q_0^2+1}{Q_0^2-1}\frac{ay}{2}\right)I_0\left(\frac{ay}{2}\right)dy\\&+\frac{2\Theta E_0}{Q_0(Q_0^2-1)}e^{-\Theta(t-T)}\cdot\int_0^{(t-T)}\exp\left(\frac{Q_0^2+1}{Q_0^2-1}\frac{ay}{2}\right)I_0\left(\frac{ay}{2}\right)dy\end{aligned} \tag{80}$$

and asymptotically, for the case $[a(t-T)]\gg 1$ and $T\ll t$,

$$F(t)\cong E_0\left\{\frac{2}{Q_0\sqrt{\pi a}}[t^{-1/2}-(t-T)^{-1/2}]+\frac{Q_0^2-2}{2\sqrt{\pi a^3Q_0^3}}[t^{-3/2}-(t-T)^{-3/2}]+\cdots\right\} \tag{81}$$

or

$$F(t)\cong\frac{-E_0aT}{Q_0\sqrt{\pi}}(at)^{-3/2}\cdot\left\{1+\frac{3}{4}\left(aT+1-\frac{2}{Q_0^2}\right)\frac{1}{at}+O(a^{-2}t^{-2})\right\}. \tag{82}$$

In this case, for $t>T$,

$$E_{\text{Total}}(0,t)=F(t) \tag{83}$$

and

$$H_{\text{Total}}(0,t)=-\sqrt{\frac{\varepsilon_0}{\mu_0}}F(t). \tag{84}$$

V. Conclusion

The problem of a transient plane wave impinging normally on a lossy half-space, where the losses are modeled by a dynamic conductivity has been solved in terms of a simple integral for the reflected electric field at the boundary (40). This has been used to obtain even simpler asymptotic expressions for the total electric and magnetic fields at the boundary in the case of step function incidence ((44) and (45)) and of square pulse incidence ((82)–(84)). For step funcion incidence, we have also derived the dominant terms of the total electric and magnetic fields in the interior of the lossy medium for large times ((60) and (70)) and for short times after arrival ((74) and (77)) of the wave. For seawater, these results have been shown to be valid for $t\gg 2\times 10^{-10}$ [sec].

The treatment of dynamic condutivity [2] becomes more difficult if frequencies so high that (31) cannot be justified become important. In the case of water, the assumption of a frequency-independent dielectric constant requires restiction to frequencies $\leq 10^{10}$ [Hz], and then, not only is (31) well-justified, but the (dynamic) correction to the static conductivity of seawater becomes very small. It was kept there in order to meet the point raised by Harmuth and Hussain [3].

The next obvious extension of this work is the inclusion of the Lorentz dispersion model into the formulation of the problem in order to properly model the half-space. Then, with a frequency-dependent dielectric constant, the restriction to frequencies $\leq 10^{10}$ [Hz] for water, for example, could be dropped.

APPENDIX A

We now derive the simplified form of $F(t)$ given in (40) from (39). Equation (39) is re-written as

$$\frac{F(t)}{E_0} = -\frac{Q_0-1}{Q_0+1} + \frac{2Q_0}{Q_0^2-1}\int_0^t e^{-\Theta y} W(y)\,dy \tag{85}$$

and the three terms in (38) are denoted as

$$W(t) = W_1(t) + W_2(t) + W_3(t) \tag{86}$$

such that

$$W_1(t) = \frac{aQ_0}{Q_0^2-1},$$
$$W_2(t) = \frac{a^2Q_0^2}{(Q_0^2-1)^2}\int_0^t \exp\left(\frac{Q_0^2+1}{Q_0^2-1}\frac{a}{2}y\right) I_0\left(\frac{a}{2}y\right) dy,$$

and

$$W_3(t) = \frac{a}{2}\exp\left(\frac{Q_0^2+1}{Q_0^2-1}\frac{a}{2}t\right) \cdot \left[I_1\left(\frac{a}{2}t\right) - \frac{Q_0^2+1}{Q_0^2-1} I_0\left(\frac{a}{2}t\right)\right]$$

The second term of (85) is now evaluated for each of these terms. We have, for $W_1(t)$,

$$\frac{2Q_0}{Q_0^2-1}\int_0^t e^{-\Theta y} W_1(y)\,dy = \frac{2\Theta}{(Q_0^2-1)}\int_0^t e^{-\Theta y}\,dy = \frac{2}{Q_0^2-1}[1-e^{-\Theta t}]. \tag{87}$$

For the term containing $W_3(t)$,

$$\begin{aligned}\frac{2Q_0}{Q_0^2-1}\int_0^t e^{-\Theta y} W_3(y)\,dy &= \frac{aQ_0}{Q_0^2-1}\int_0^t e^{-ay/2} \cdot \left[I_1\left(\frac{a}{2}y\right) - \frac{Q_0^2+1}{Q_0^2-1} I_0\left(\frac{a}{2}y\right)\right] dy \\ &= \frac{2Q_0}{Q_0^2-1}\int_0^{at/2} e^{-z} \left[I_1(z) - \frac{Q_0^2+1}{Q_0^2-1} I_0(z)\right] dz.\end{aligned} \tag{88}$$

Integrating by parts,

$$\int_0^{at/2} e^{-z} I_1(z)\,dz = e^{-at/2} I_0(at/2) - 1 + \int_0^{at/2} e^{-z} I_0(z)\,dz \tag{89}$$

so that this third term in $F(t)/E_0$ becomes

$$\begin{aligned}&\frac{2Q_0}{Q_0^2-1}\left[e^{-at/2} I_0(at/2) - 1 + \frac{2}{Q_0^2-1}\int_0^{at/2} e^{-z} I_0(z)\,dz\right] \\ &= \frac{2Q_0}{Q_0^2-1} e^{-at/2} I_0(at/2) - \frac{2Q_0}{Q_0^2-1} - \frac{4Q_0}{(Q_0^2-1)^2}\int_0^{at/2} e^{-z} I_0(z)\,dz.\end{aligned} \tag{90}$$

Finally, the term in $F(t)/E_0$ due to $W_2(t)$ is

$$\frac{2a^2Q_0^3}{(Q_0^2-1)^3}\int_0^t e^{-\Theta y}\int_0^y \exp\left(\frac{Q_0^2+1}{Q_0^2-1}\frac{a}{2}y'\right) \cdot I_0\left(\frac{a}{2}y'\right) dy'\,dy \tag{91}$$

which becomes, when the order of integration is interchanged

$$\begin{aligned}&\frac{2a^2Q_0^3}{(Q_0^2-1)^3}\int_0^t \exp\left(\frac{Q_0^2+1}{Q_0^2-1}\frac{a}{2}y'\right) \cdot I_0\left(\frac{a}{2}y'\right)\int_{y'}^t e^{-\Theta y}\,dy\,dy' \\ &= \frac{2aQ_0}{(Q_0^2-1)^2}\int_0^t \exp\left(\frac{Q_0^2+1}{Q_0^2-1}\frac{a}{2}y'\right) \cdot I_0\left(\frac{a}{2}y'\right)[e^{-\Theta y'} - e^{-\Theta t}]\,dy' \\ &= \frac{4Q_0}{(Q_0^2-1)^2}\int_0^{at/2} e^{-z} I_0(z)\,dz - \frac{2aQ_0}{(Q_0^2-1)^2} e^{-\Theta t}\int_0^t \exp\left(\frac{Q_0^2+1}{Q_0^2-1}\frac{a}{2}y'\right) \cdot I_0\left(\frac{a}{2}y'\right) dy'.\end{aligned} \tag{92}$$

Now collecting the terms in $F(t)/E_0$, the time independent terms total -1, and the coefficients of

$$\int_0^{at/2} e^{-z} I_0(z)\,dz$$

total zero and thus the result of (40) is obtained.

APPENDIX B

Using the asymptotic expansion for the modified Bessel function

$$I_0(x) \sim \frac{e^x}{\sqrt{2\pi x}}\left(1 + \frac{1}{8x} + \cdots\right), x \to \infty \tag{93}$$

in (40) for $F(t)$, and assume that $at/2 \gg 1$, we see that, of the four terms in the expression for $F(t)$, the second one decays exponentially for large positive *at*, and may be ignored in the large *at* asymptotic series. Using the first two terms in the above asymptotic expansion for $I_0(x)$ gives for the third term in (40)

$$\frac{2Q_0E_0}{Q_0^2-1}\exp\left(-\frac{at}{2}\right) I_0\left(\frac{at}{2}\right)$$

$$\approx \frac{2Q_0E_0}{Q_0^2-1}\frac{1}{\sqrt{\pi at}}\left(1+\frac{1}{4at}+\cdots\right),\quad at\to\infty. \tag{94}$$

In the integral in the fourth term, since both factors grow exponentially with the variable of integration, y, (note $Q_0 > 1$), the asymptotic contribution of the integral for large at comes entirely from the region of the upper endpoint, so that

$$\begin{aligned}\int_0^t &\exp\left(\frac{Q_0^2+1}{Q_0^2-1}\frac{ay}{2}\right)I_0\left(\frac{ay}{2}\right)dy\\ &\cong \int^t e^{\Theta y}\left(\frac{1}{\sqrt{\pi}(ay)^{1/2}}+\frac{1}{4\sqrt{\pi}(ay)^{3/2}}+\cdots\right)dy\\ &= \frac{1}{\Theta}\int^{\Theta t} e^v\left(\frac{1}{\sqrt{\pi(Q_0^2-1)/Q_0^2}v^{1/2}}\right.\\ &\quad\left.+\frac{1}{4\sqrt{\pi(Q_0^2-1)^3/Q_0^6}v^{3/2}}+\cdots\right)dv\end{aligned} \tag{95}$$

where we have made the substitution $v = \Theta y$.

Now using the first two terms in the integral

$$\int^U \frac{e^v}{v^{1/2}}dv = e^U\left(U^{-1/2}+\frac{1}{2}U^{-3/2}+\cdots\right) \tag{96}$$

which has been obtained by integrating by parts, but only the first term in

$$\int^U \frac{e^v}{v^{3/2}}dv = e^U(U^{-3/2}+\cdots) \tag{97}$$

gives (with $U \equiv \Theta t$) for the approximation of (95)

$$\begin{aligned}&\left[\frac{\sqrt{Q_0^2-1}}{aQ_0\sqrt{\pi}}\left(U^{-1/2}+\frac{1}{2}U^{-3/2}\right)\right.\\ &\quad\left.+\frac{Q_0}{4a\sqrt{\pi}\sqrt{Q_0^2-1}}U^{-3/2}\right]e^U\\ &= \frac{e^{\Theta t}}{a\sqrt{\pi}}\left\{\frac{Q_0^2-1}{Q_0^2\sqrt{at}}+\frac{1}{4}\frac{(3Q_0^2-2)(Q_0^2-1)}{Q_0^4(at)^{3/2}}\right\}\end{aligned} \tag{98}$$

so that after multiplying by

$$-\frac{2aQ_0}{(Q_0^2-1)^2}\exp(-\Theta t),$$

the fourth term in $F(t)/E_0$ becomes asymptotically

$$-\frac{2}{Q_0(Q_0^2-1)\sqrt{\pi at}}-\frac{(3Q_0^2-2)}{2Q_0^3(Q_0^2-1)\sqrt{\pi}(at)^{3/2}}. \tag{99}$$

Adding to the corresponding contribution from the third term in $F(t)/E_0$, i.e., (94), and recalling the first term, -1, (and ignoring the exponentially small second term) gives the expression of (43).

References

[1] J. LoVetri and J. B. Ehrman, "The one dimensional time domain Maxwell equations in static and dynamic conducting media I," *IEEE Trans. Electromagn. Compat.*, vol. 36, no. 3, pp. 221–228, Aug. 1994.

[2] R. Becker, *Electromagnetic Fields and Interactions*. vol. 1, New York: Blaisdell Pub. Co.,1964.

[3] H. F. Harmuth and M. G. M. Hussain, "Response to a letter by J. R. Wait about electromagnetic waves in seawater," *IEEE Trans. Electromagn. Compat.*, vol. 34, no. 4, pp. 491–492, Nov. 1992.

[4] H. F. Harmuth, "Correction of Maxwell's equations for signals I," *IEEE Trans. Electromagn. Compat.*, vol. 28, no. 4, pp. 250–256, Nov. 1986.

[5] H. F. Harmuth, "Correction of Maxwell's equations for signals II," *IEEE Trans. Electromag. Compat.*, vol. 28, no. 4, pp. 259–265, Nov. 1986.

[6] J. A. Stratton, *Electromagnetic Theory*. New York: McGraw-Hill Book Co., 1941.

[7] C. E. Baum, "The reflection of pulsed waves from the surface of a conducting dielectric," *EMP Theoret. Notes*, Note XXV, AFWL, 18 Feb. 1967.

[8] H. P. Neff, Jr., and D. A. Reed, "Plane wave (EMP) incidence on an infinitely conducting plane earth with the magnetic field intensity parallel to the earth's surface," *Theoret. Notes*, Note 351, AFWL, 8 Feb. 1984.

[9] J. J. A. Klaasen, "Time-domain analysis of one-dimensional electromagnetic scattering by lossy media," *Theoret. Notes*, Note 357, AFWL, June 1990.

[10] P. R. Barnes, and F. M. Tesche, "On the direct calculation of a transient plane wave reflected from a finitely conducting half-space," *Theoret. Notes*, Note 358, 5 Dec. 1990.

[11] K. D. Leuthauser, "Representation of reflection and transmission functions of canonical electromagnetic waveforms at a conducting half-space," *Theoret. Notes*, Note 359, Dec. 1990.

[12] P. R. Barnes, and F. M. Tesche, "On the time-domain transient electric field reflected from a finitely conducting earth," presented at *Nuclear EMP Meeting*, Albuquerque, NM, May 21–24, 1990.

[13] G. Doetsch, *Guide to the Applications of Laplace Transforms*. London: D. Van Nostrand Company Ltd., 1961.

[14] G. E. Roberts and H. Kaufman, *Table of Laplace Transforms*. Philadelphia, PA: W. B. Saunders Co., 1966.

[15] E. Hallen, *Electromagnetic Theory*. London: Chapman & Hall, 1962.

[16] Bateman manuscript project: *Table of Integral Transforms*, A. Erdelyi W. Magnus, F. Oberhettinger, and F. G. Tricomi, Eds., vol. 1. New York: McGraw-Hill, 1954.

Joachim B. Ehrman, photograph and biography not available at the time of publication.

Joe LoVetri (S'84–M'84–S'84–S'86–S'87–M'87–M'90), photograph and biography not available at the time of publication.