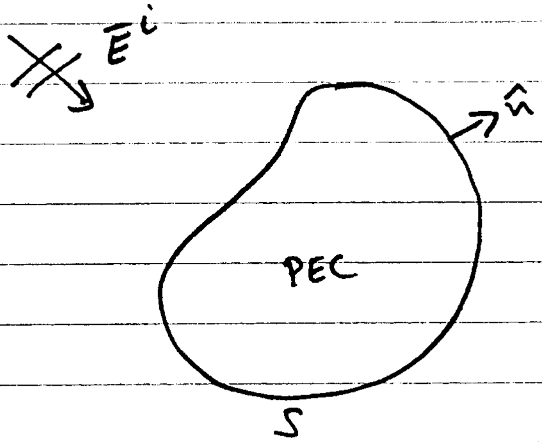
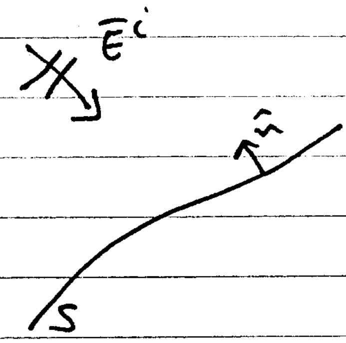
The Electric Field Integral Equation and the Rao-Wilton-Glisson Elements - PEC Scatterer

Formulation of the EFIE: ($e^{j\omega t}$ time dependence)



closed surface scatterer



(also PEC)

open surface scatterer

incident field $\bar{E}^i$ is the fied due to an impressed source in the absence of the scatterer.

$\bar{E}^i$ induces surface currents $\bar{J}$ on S.

NB: if S is open then $\bar{J}$ is regarded as the vector sum of the surface currents on opposite sides of S

the magnetic vector potential produced by the induced surface currents can be calculated as:

$$\bar{A}(\bar{r}) = \frac{\mu}{4\pi} \int_S \bar{J} \, \frac{e^{-jKR}}{R} \, ds' \quad \Big\} \quad (1)$$

and the salar potential as:

$$\phi(\bar{r}) = \frac{1}{4\pi\varepsilon} \int_S \sigma \, \frac{e^{-jKR}}{R} \, ds' \quad \Big\} \quad (2)$$

where:

- $K = \frac{2\pi}{\lambda} = \omega\sqrt{\mu\varepsilon}$ is the wavenumber
- λ — wavelength
- $\omega = 2\pi f$ — frequency
- $R = |\bar{r} - \bar{r}'|$
- $\bar{r}$ — observation point } both defined
- $\bar{r}'$ — source point on S } w.r.t a global origin O.

σ — surface charge density where:

$$\nabla_s \cdot \bar{J} = -j\omega\sigma \quad \Big\} \quad (3)$$

∇_s — surface divergence $\quad (\nabla_s = \nabla - \hat{n} \cdot \nabla)$

the sattered electric field can be calculated from $\bar{A}$ and ϕ as:

$$\bar{E}^s = -j\omega \bar{A} - \nabla\phi \qquad (5)$$

An integrodifferential equation is derived for $\bar{J}$ by enforcing the B.C. that the total tangential electric field be 0 on the surface of the scatterer:

$$(4) \qquad \boxed{-\bar{E}^i_{tan} = (-j\omega\bar{A} - \nabla\phi)_{tan} \qquad \text{with } \bar{r} \text{ on } S.}$$

Substituting (1), (2), and (3) into (4) we have the EFIE which is to be solved for the surface current $\bar{J}$. Once $\bar{J}$ is found, we can use (1), (2) and (5) to determine the scattered field anywhere.

Because of the need to take the surface divergence of $\bar{J}$ in (3) and the gradient of ϕ in (5) care must be taken in choosing the expansion and testing functions in the MOM.

Basis Functions on Triangular Patches.

the surface of an arbitrary PEC body can be broken up into triangular patches as shown below:

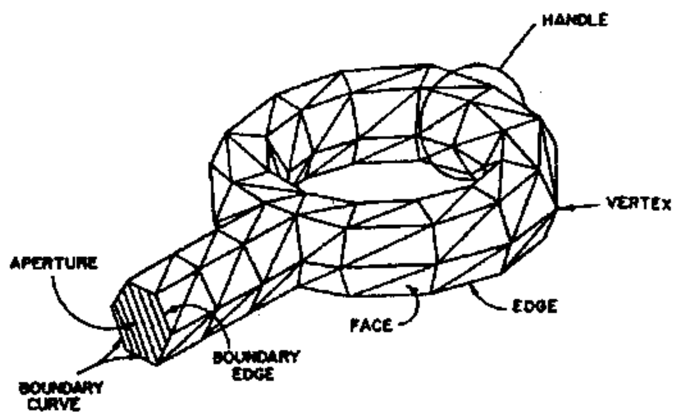


Fig. 1. Arbitrary surface modeled by triangular patches.

basis functions are now associated with edges between two triangles (i.e. interior edges):

(6) $$\bar{f}_n(\bar{r}) = \begin{cases} \dfrac{l_n}{2A_n^+}\,\bar{\rho}_n^+ & \bar{r} \text{ in } T_n^+ \\[2ex] \dfrac{l_n}{2A_n^-}\,\bar{\rho}_n^- & \bar{r} \text{ in } T_n^- \\[2ex] 0 & \text{otherwise} \end{cases}$$

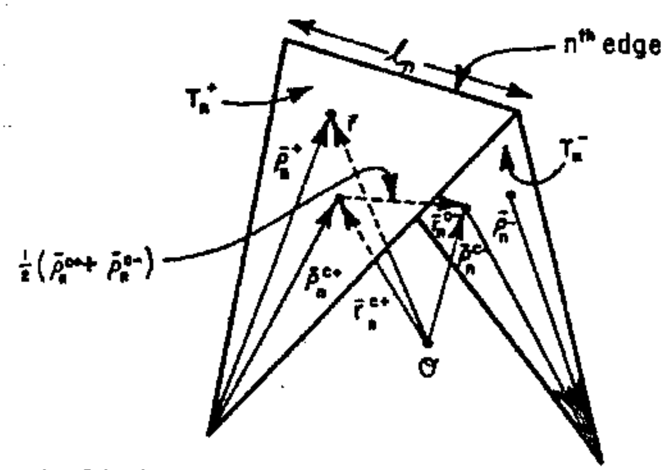


Fig. 2. Triangle pair and geometrical parameters associated with interior edge.

- l_n - length of edge n
- T_n^+, T_n^- - the 2 triangles connected via edge n
- A_n^+, A_n^- - the areas of triangles T_n^+ & T_n^-
- $\bar{r}$ - position vector w.r.t. global origin O.
- $\bar{\rho}_n^+, \bar{\rho}_n^-$ - position vectors w.r.t. free nodes of A_n^+ & A_n^-

Note: one way to decide which of the two triangles associated with an edge to name T_n^+ and T_n^- is as follows:

given that an edge is specified, in the connection matrix specified by the user, as

$n \quad N_i \quad N_j$ (edge node node)

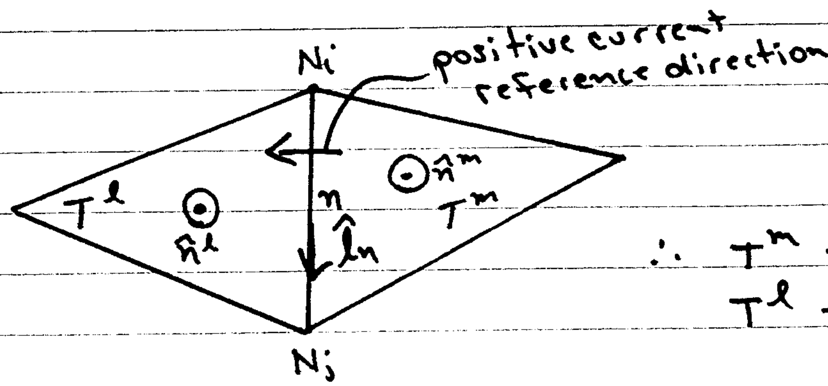
construct an edge vector in the following way: $\bar{l}_n = \bar{r}_{N_j} - \bar{r}_{N_i}$

then the unit edge vector will be given by: $\hat{l}_n = \dfrac{\bar{l}_n}{|\bar{l}_n|} = \dfrac{\bar{r}_{N_j} - \bar{r}_{N_i}}{|\bar{r}_{N_j} - \bar{r}_{N_i}|}$

if triangles T^l and T^m are connected via edge n and they have outward normals given by $\hat{n}^l$ and $\hat{n}^m$ then

$$\hat{l}_n \times \hat{n}^l \text{ and } \hat{l}_n \times \hat{n}^m$$

are the positive current reference directions in each triangle l and m respectively.

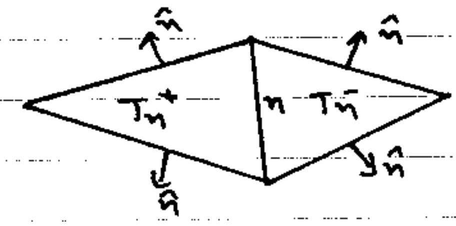


$$\therefore \left.\begin{array}{l} T^m \rightarrow T_n^+ \\ T^l \rightarrow T_n^- \end{array}\right\} \text{for edge } n$$

the basis functions $\bar{f}_n$ are used to approximate the surface current on the scatterer.

Properties of $\bar{f}_n$:

1) $\bar{f}_n$ has no component normal to the boundary formed by the two triangles T_n^+ and T_n^-



$\hat{n} \cdot \bar{f}_n = 0$

2) component of $\bar{f}_n$ normal to edge n is constant and continuous across the edge

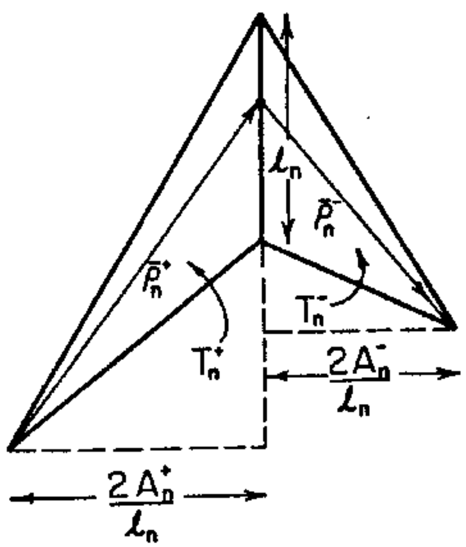
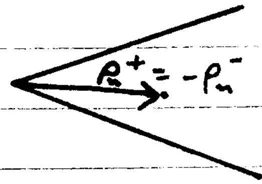


Fig. 3. Geometry for construction of component of basis function normal to edge.

at the edge, normal components of $\bar{\rho}_n^+$ and $\bar{\rho}_n^-$ are $\frac{2A_n^+}{l_n}$ and $\frac{2A_n^-}{l_n}$ respectively but $\bar{f}_n$ will

therefore have unity normal component on both sides of the edge. $\Rightarrow$ no line charges.

3). We can calculate the surface divergence by assuming we are in cylindrical coordinates:



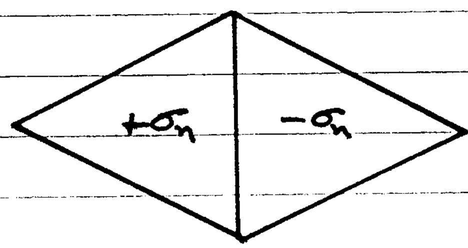
$$\nabla_s \cdot \bar{f}_n = \pm \frac{1}{\rho_n^\pm} \frac{\partial (\rho_n^\pm f_{n\rho})}{\partial \rho_n}$$

where $f_{n\rho}$ is the mag. of $\bar{f}_n$ in the ρ direction

we get : $f_{n\rho} = \frac{l_n}{2A_n^\pm} \rho_n^\pm$

(7) $$\therefore \quad \nabla_s \cdot \bar{f}_n = \begin{cases} \dfrac{l_n}{2A_n^+} & \bar{r} \text{ in } T_n^+ \\ -\dfrac{l_n}{A_n^-} & \bar{r} \text{ in } T_n^- \\ 0 & \text{otherwise} \end{cases}$$

$\therefore$ the charge density is constant in each triangle but opposite in sign with the total charge being zero $\Rightarrow$ this is the form of "<u>pulse doublets</u>"



4) "moment" of $\bar{f}_n$ is given by $(A_n^+ + A_n^-)\bar{f}_n^{avg}$

$$\left.\begin{aligned} (A_n^+ + A_n^-)\,\bar{f}_n^{avg} &\equiv \int_{T_n^+ + T_n^-} \bar{f}_n \, ds \\ &= \frac{l_n}{2}\left(\bar{\rho}^{c+} + \bar{\rho}_n^{c-}\right) \\ &= l_n\left(\bar{r}_n^{c+} - \bar{r}_n^{c-}\right) \end{aligned}\right\} \quad (8)$$

where "c" denotes the centroid of the triangle

With these basis functions, we can now approximate the surface current on S as:

$$\bar{J} \approx \sum_{n=1}^{N} I_n \bar{f}_n(\bar{r}) \quad (9)$$

N = the number of interior edges in the mesh.

Note: up to 3 basis functions may have non-zero value on one triangle.

Since the normal component of $\bar{f}_n$ at the n^{th} edge is unity, each coefficient I_n in the expansion above may be interpreted as the normal component of current density flowing past the n^{th} edge.

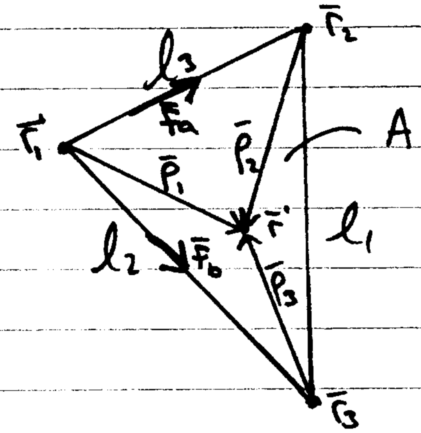
Now consider a triangle composed of 3 interior edges $\Rightarrow$ that 3 basis functions are defined over the surface of the triangle.

Assume that the current reference direction are out of the triangle for each edge (i.e the triangle is a positive triangle for each edge):

$$\bar{f}_1 = \frac{l_1}{2A} \bar{\rho}_1$$

$$\bar{f}_2 = \frac{l_2}{2A} \bar{\rho}_2$$

$$\bar{f}_3 = \frac{l_3}{2A} \bar{\rho}_3$$



$\bar{f}_a$ & $\bar{f}_b$ form a basis for any vector on the surface

let: $$\bar{f}_a = l_2 \bar{f}_1 - l_1 \bar{f}_2 = \frac{l_1 l_2}{2A} \left(\bar{\rho}_1 - \bar{\rho}_2 \right)$$

$$\bar{f}_b = l_3 \bar{f}_1 - l_1 \bar{f}_3 = \frac{l_1 l_3}{2A} \left(\bar{\rho}_1 - \bar{\rho}_3 \right)$$

$\therefore$ since $\bar{f}_a$ & $\bar{f}_b$ form a basis $\bar{f}_1, \bar{f}_2$ & $\bar{f}_3$ can be used to represent any vector on the surface of the triangle.

Now the Galerkin method is used. Thus, once the expansion in terms of $\bar{f}_n$ is substituted into the integrodifferential equation, the same functions are chosen as testing functions.

The inner product used is:

$$\langle \bar{f}, \bar{g} \rangle \triangleq \int_S \bar{f} \cdot \bar{g} \, ds \qquad (10)$$

testing (4) with $\bar{f}_m$:

$$\therefore \quad \langle \bar{E}^i, \bar{f}_m \rangle = j\omega \langle \bar{A}, \bar{f}_m \rangle + \langle \nabla\phi, \bar{f}_m \rangle \qquad (11)$$

the last term can be written as

$$\langle \nabla\phi, \bar{f}_m \rangle = -\int \phi \, \nabla_s \cdot \bar{f}_m \, ds \qquad (12)$$

using (7), this is written as:

$$\int_S \phi \, \nabla_s \cdot \bar{f}_m \, ds = l_m \left(\frac{1}{A_m^+} \int_{T_m^+} \phi \, ds - \frac{1}{A_m^-} \int_{T_m^-} \phi \, ds \right)$$

if the average of ϕ over each triangle is approximated by the value of ϕ at the triangle centroids, $\bar{r}_m^{c+}$ and $\bar{r}_m^{c-}$, we have:

$$\int_S \phi \, \nabla_s \cdot \bar{f}_m \, ds \cong l_m \left[\phi(\bar{r}_m^{c+}) - \phi(\bar{r}_m^{c-}) \right] \qquad (13)$$

Similarly:

$$\left\langle \begin{Bmatrix} \bar{E}^i \\ \bar{A} \end{Bmatrix}, \bar{f}_m \right\rangle = l_m \left[\frac{1}{2A_m^+} \int_{T_m^+} \begin{Bmatrix} \bar{E}^i \\ \bar{A} \end{Bmatrix} \cdot \bar{\rho}_m^+ \, ds + \frac{1}{2A_m^-} \int_{T_m^-} \begin{Bmatrix} \bar{E}^i \\ \bar{A} \end{Bmatrix} \cdot \bar{\rho}_m^- \, ds \right]$$

$$\cong \frac{l_m}{2} \left[\begin{Bmatrix} \bar{E}^i(\bar{r}_m^{c+}) \\ \bar{A}(\bar{r}_m^{c-}) \end{Bmatrix} \cdot \bar{\rho}_m^{c+} + \begin{Bmatrix} \bar{E}^i(\bar{r}_m^{c-}) \\ \bar{A}(\bar{r}_m^{c-}) \end{Bmatrix} \cdot \bar{\rho}_m^{c-} \right] \quad (14)$$

Putting these results (approximations) back into (11) we get:

$$j\omega \frac{l_m}{2} \left[\bar{A}(\bar{r}_m^{c+}) \cdot \bar{\rho}_m^{c+} + \bar{A}(\bar{r}_m^{c-}) \cdot \bar{\rho}_m^{c-} \right] + l_m \left[\phi(\bar{r}_m^{c+}) - \phi(\bar{r}_m^{c-}) \right]$$

$$= \frac{l_m}{2} \left[\bar{E}^i(\bar{r}_m^{c+}) \cdot \bar{\rho}_m^{c+} + \bar{E}^i(\bar{r}_m^{c-}) \cdot \bar{\rho}_m^{c-} \right] \quad (15)$$

and this equation is enforced at each triangle edge, that is: $m = 1, 2, \ldots N$.

Once the current expansion, (9), is substituted into (15) we will have N equations in the N unknowns, I_n.

$$Z\, I = V$$

$$Z = [Z_{mn}] \in \mathbb{C}^{N \times N} \qquad I = [I_n] \in \mathbb{C}^N \qquad V = [V_m] \in \mathbb{C}^N$$

$$Z_{mn} = l_m \left[j\omega \left(\bar{A}^{+}_{mn} \cdot \frac{\bar{\rho}^{c+}_m}{2} + \bar{A}^{-}_{mn} \cdot \frac{\bar{\rho}^{c-}_m}{2} \right) + \phi^{-}_{mn} - \phi^{+}_{mn} \right] \quad (17)$$

$$V_m = l_m \left(\bar{E}^{+}_m \cdot \frac{\bar{\rho}^{c+}_m}{2} + \bar{E}^{-}_m \cdot \frac{\bar{\rho}^{c-}_m}{2} \right) \quad (18)$$

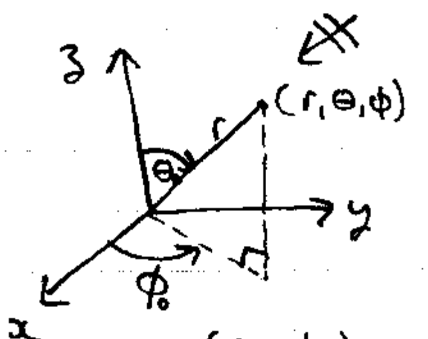
$$\bar{A}^{\pm}_{mn} = \frac{\mu}{4\pi} \int_S \bar{f}_n(\bar{r}') \frac{e^{-jkR^{\pm}_m}}{R^{\pm}_m} \, ds' \quad (19)$$

$$\phi^{\pm}_{mn} = -\frac{1}{4\pi j\omega\varepsilon} \int_S \nabla_s' \cdot \bar{f}_n(\bar{r}') \frac{e^{-jkR^{\pm}_m}}{R^{\pm}_m} \, ds' \quad (20)$$

$$R^{\pm}_m = \left| \bar{r}^{c\pm}_m - \bar{r}' \right|$$

(observation point is denoted by m)
(n denotes the source triangle over which the integration is performed)

$$\bar{E}^{\pm}_m = \bar{E}^i\left(\bar{r}^{c\pm}_m\right) \quad (21)$$



In spherical coordinates:

$$\bar{E}^i(\bar{r}) = (E_\theta \hat{a}_{\theta_0} + E_\phi \hat{a}_{\phi_0}) e^{j\bar{k}\cdot\bar{r}}$$

$(\theta_0, \phi_0) \rightarrow$ angles of arrival

$$\bar{k} = k(\sin\theta_0 \cos\phi_0 \, \hat{a}_x + \sin\theta_0 \sin\phi_0 \, \hat{a}_y + \cos\theta_0 \, \hat{a}_z)$$

(propagation vector). $\left(k = \frac{2\pi}{\lambda} = \frac{\omega}{c}\right)$

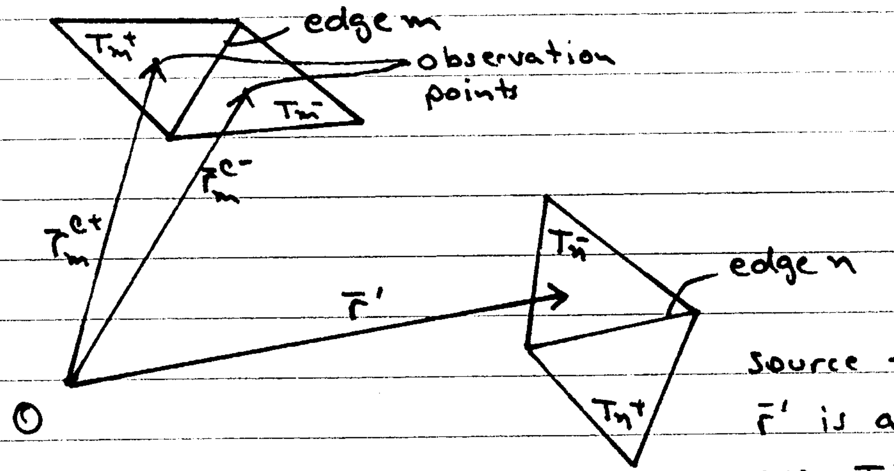
examples: $\bar{k} = -k\,\hat{a}_x \quad \left(\phi_0 = \pi, \quad \theta_0 = \frac{\pi}{2}\right)$

$E_\theta = 1 \quad E_\phi = 0$

$$\bar{E}^i(\bar{r}) = \hat{a}_{\theta_0} e^{-jkx} = -\hat{a}_z e^{-jkx}$$

Efficient Numerical Evaluation of Matrix Elements

a matrix element Z_{mn} is associated with internal edges m and n

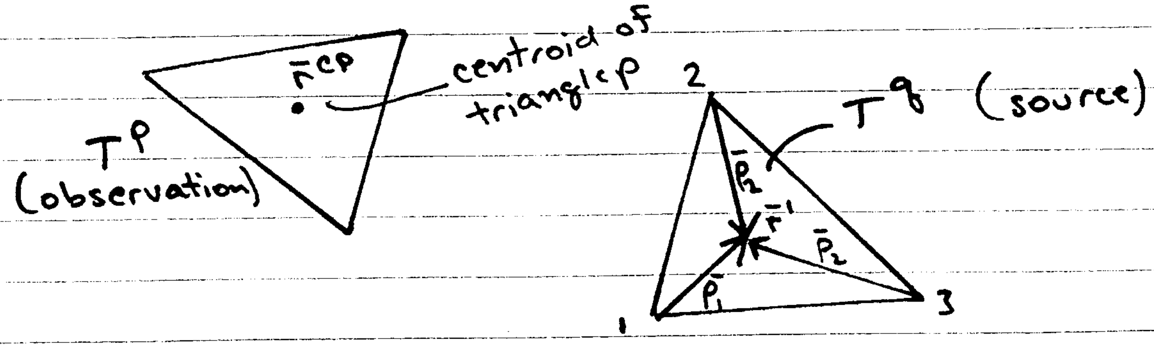


source triangles
$\bar{r}'$ is allowed to vary over T_n^- and T_n^+ for integration

integration over T_n^- (as source location) with observation point at $\bar{r}_m^{c-}$ will happen 9 times!
(once for each edge of T_n^- paired up with each edge of T_m^-)

$\therefore$ we should evaluate the integrals with regard to face pairs instead of edge pairs.

Consider the evaluation of vector and scalar potential integrals (19 + 20) for a given source / observation face combination



using local indices 1,2,3 :

$$\bar{\rho}_i = \pm(\bar{r}' - \bar{r}_i) \qquad i = 1,2,3 \tag{22}$$

(each vertex of source triangle)

Note: '+' if positive current reference is out of T^q
'−' if positive current reference is into T^q

we want to evaluate the 2 integrals:

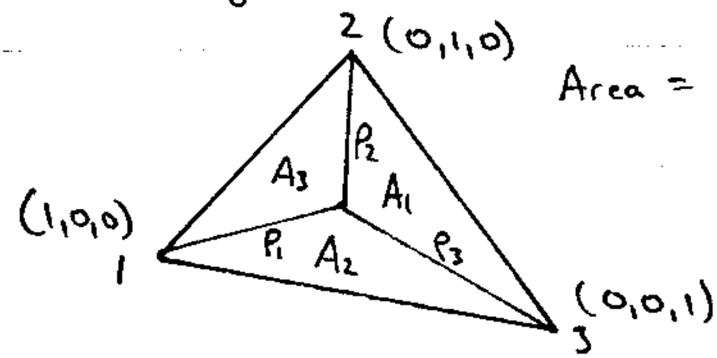
$$\bar{A}_i^{pq} = \frac{\mu}{4\pi} \int_{T^q} \left(\frac{l_i}{2A^q}\right) \bar{\rho}_i \, \frac{e^{-jkR^p}}{R^p} \, ds' \tag{23}$$

and

$$\phi_i^{pq} = -\frac{1}{4\pi j\omega\varepsilon} \int_{T^q} \underbrace{\left(\frac{l_i}{A^q}\right)}_{\nabla_s \cdot \bar{J}} \frac{e^{-jkR^p}}{R^p} \, ds' \tag{24}$$

$$R^p = |\bar{r}^{cp} - \bar{r}'|$$

Using <u>normalized area coordinates</u>:



(ξ, η, ζ)

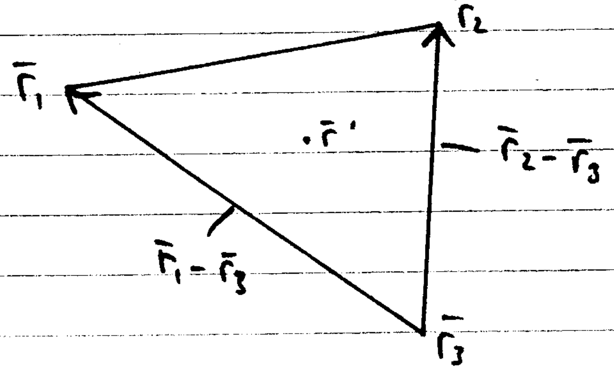
$$\left\{ \begin{array}{l} \xi = \dfrac{A_1}{A^q} \quad \eta = \dfrac{A_2}{A^q} \quad \zeta = \dfrac{A_3}{A^q} \\ \\ \xi + \eta + \zeta = 1 \qquad 0 \le \xi, \eta, \zeta \le 1 \end{array} \right. \tag{25}$$

a point $\bar{r}'$ inside the triangle, T^q, is represented as:

$$\bar{r}' = \xi \bar{r}_1 + \eta \bar{r}_2 + \zeta \bar{r}_3 \qquad (26)$$

$\bar{r}_i$ is the position vector to vertex i.

That $\bar{r}'$ can be represented in this way is shown as follows:



any point in the plane where the above triangle sits can be represented as a linear combination of $\bar{r}_1 - \bar{r}_3$ and $\bar{r}_2 - \bar{r}_3$, say

$$\bar{r}' = \xi(\bar{r}_1 - \bar{r}_3) + \eta(\bar{r}_2 - \bar{r}_3) + \bar{r}_3$$

$\therefore$ when $\xi = \eta = 0 \quad \bar{r}' = \bar{r}_3$

The above can be written as:

$$\bar{r}' = \xi \bar{r}_1 + \eta \bar{r}_2 + (1 - \xi - \eta)\bar{r}_3$$

and letting $\zeta = 1 - \xi - \eta$ we get (26)

Integration of a function over a triangle:

$$\int_{T^q} g(\bar{r}')\,ds = I$$

can be performed in Local coordinates

$$I = 2A^q \int_0^1 \int_0^{1-\eta} g(\xi \bar{r}_1 + \eta \bar{r}_2 + (1-\xi-\eta)\bar{r}_3)\,d\xi\,d\eta$$

letting: $g'(\xi,\eta) = g(\xi \bar{r}_1 + \eta \bar{r}_2 + (1-\xi-\eta)\bar{r}_3)$

the integral is written as:

$$I = 2A^q \int_0^1 \int_0^{1-\eta} g'(\xi,\eta)\,d\xi\,d\eta$$

and this can be approximated using quadrature rules:

$$I \approx 2A^q \sum_{j=1}^{J} g'(L_{1j}, L_{2j})\,w_j$$

(L_{1j}, L_{2j}) is the j^{th} sample point where the function $g'(\xi,\eta)$ is to be evaluated

w_j is the weighting which is given to the function value at the j^{th} sample point.

TABLE 3.1. Quadrature Formulas for Triangular Finite Elements

n	Polynomial Degree	i	L_{1i}	L_{2i}	L_{3i}	w_i
1	Linear	1	0.333 333	0.333 333	0.333 333	0.500 000
3	Quadratic	1	0.500 000	0.500 000	0.000 000	0.166 667
		2	0.000 000	0.500 000	0.500 000	0.166 667
		3	0.500 000	0.000 000	0.500 000	0.166 667
4	Cubic	1	0.333 333	0.333 333	0.333 333	−0.281 250
		2	0.600 000	0.200 000	0.200 000	0.260 417
		3	0.200 000	0.600 000	0.200 000	0.260 417
		4	0.200 000	0.200 000	0.600 000	0.260 417
7	Quintic	1	0.333 333	0.333 333	0.333 333	0.112 500
		2	0.797 427	0.101 287	0.101 287	0.062 970
		3	0.101 287	0.797 427	0.101 287	0.062 970
		4	0.101 287	0.101 287	0.797 427	0.062 970
		5	0.059 716	0.470 142	0.470 142	0.066 197
		6	0.470 142	0.059 716	0.470 142	0.066 197
		7	0.470 142	0.470 142	0.059 716	0.066 197

From Brebbia and Connor, 1973.

Now in order to evaluate (23) and (24), i.e. $\bar{A}_i^{pq}$ and ϕ_i^{pq}, we write:

$$\left(\bar{\rho}_i = \pm(\bar{r}' - \bar{r}_i) \quad i = 1,2,3\right)$$

$$\begin{cases} \bar{A}_i^{pq} = \dfrac{\mu}{4\pi} \displaystyle\int_{T^q} \left(\frac{l_i}{2A^q}\right) \bar{\rho}_i \, \frac{e^{-jkR^p}}{R^p} \, ds' \\ \quad = \pm \dfrac{\mu l_i}{4\pi} \left(\bar{r}_1 I_\xi^{pq} + \bar{r}_2 I_\eta^{pq} + \bar{r}_3 I_\zeta^{pq} - \bar{r}_i I^{pq} \right) \end{cases}$$

and

$$\begin{cases} \phi_i^{pq} = \dfrac{1}{4\pi\varepsilon j\omega} \displaystyle\int_{T^q} \left(\frac{l_i}{A^q}\right) \frac{e^{-jkR^p}}{R^p} \, ds' \\ \quad = \mp \dfrac{l_i}{j2\pi\omega\varepsilon} I^{pq} \end{cases}$$

$$
\left\{
\begin{aligned}
I^{pq} &= \int_0^1 \int_0^{1-\eta} \frac{e^{-jkR^p}}{R^p} \, d\xi \, d\eta \\
I_\xi^{pq} &= \int_0^1 \int_0^{1-\eta} \xi \frac{e^{-jkR^p}}{R^p} \, d\xi \, d\eta \\
I_\eta^{pq} &= \int_0^1 \int_0^{1-\eta} \eta \frac{e^{-jkR^p}}{R^p} \, d\xi \, d\eta \\
I_\zeta^{pq} &= I^{pq} - I_\xi^{pq} - I_\eta^{pq}
\end{aligned}
\right.
$$

$$
\begin{aligned}
R^p &= \left| \bar{r}^{cp} - \bar{r}' \right| \\
&= \left| \bar{r}^{cp} - \xi \bar{r}_1 - \eta \bar{r}_2 - \zeta \bar{r}_3 \right| \\
&= \left| \bar{r}^{cp} - \xi \bar{r}_1 - \eta \bar{r}_2 - (1-\xi-\eta) \bar{r}_3 \right|
\end{aligned}
$$

in order to evaluate the above integrals, the singularity should first be subtracted out and evaluated analytically.

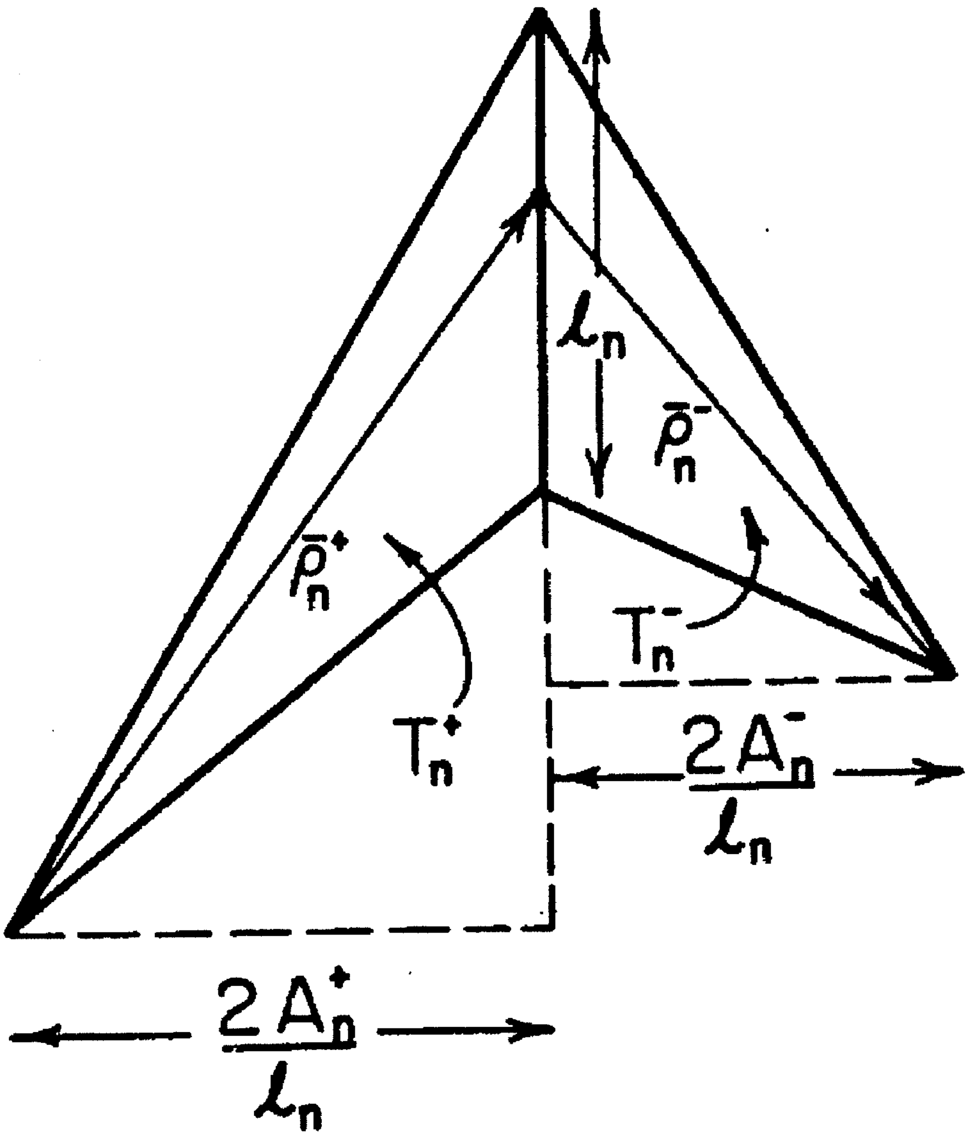


Fig. 3. Geometry for construction of component of basis function normal to edge.

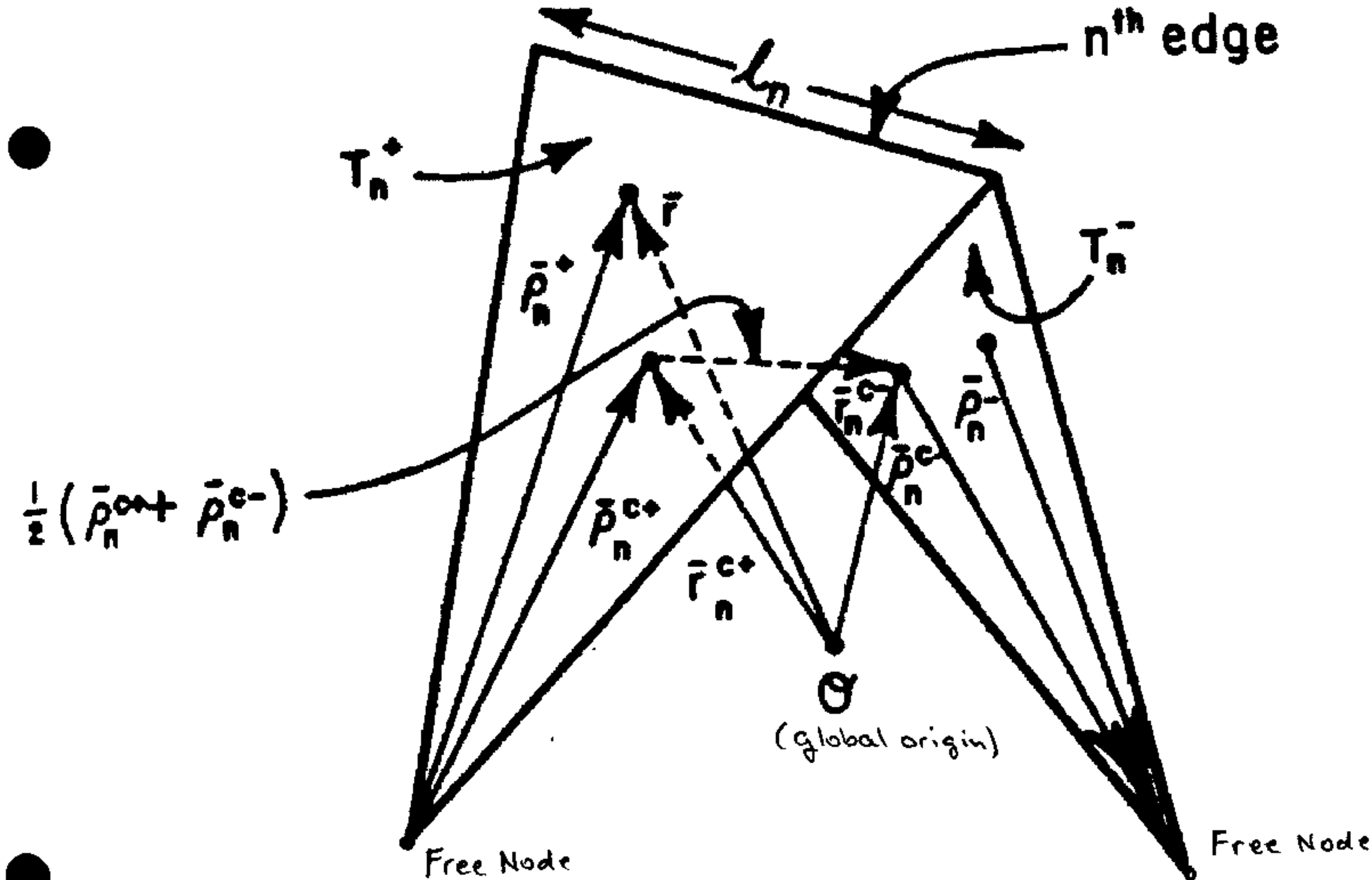


Fig. 2. Triangle pair and geometrical parameters associated with interior edge.

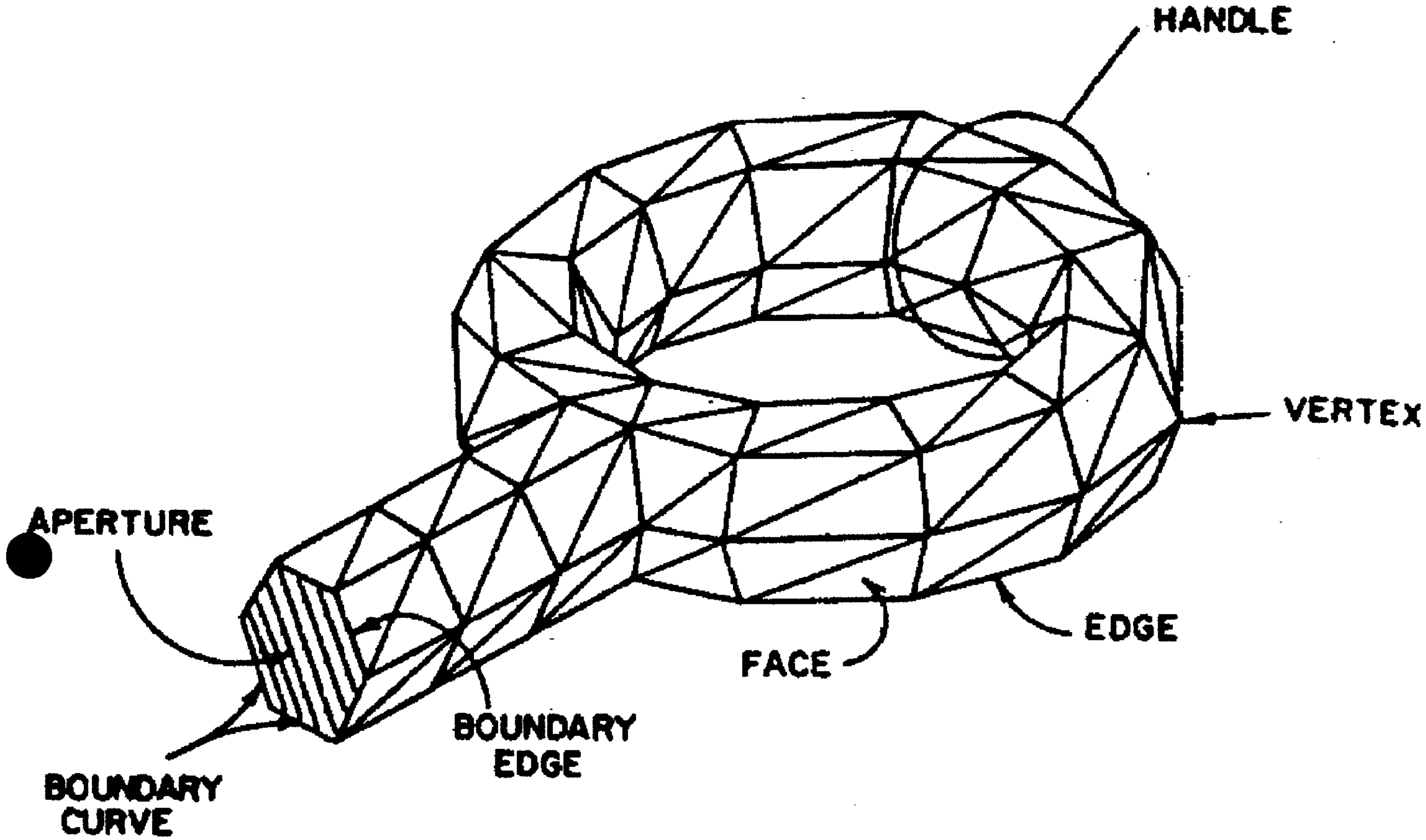


Fig. 1. Arbitrary surface modeled by triangular patches.

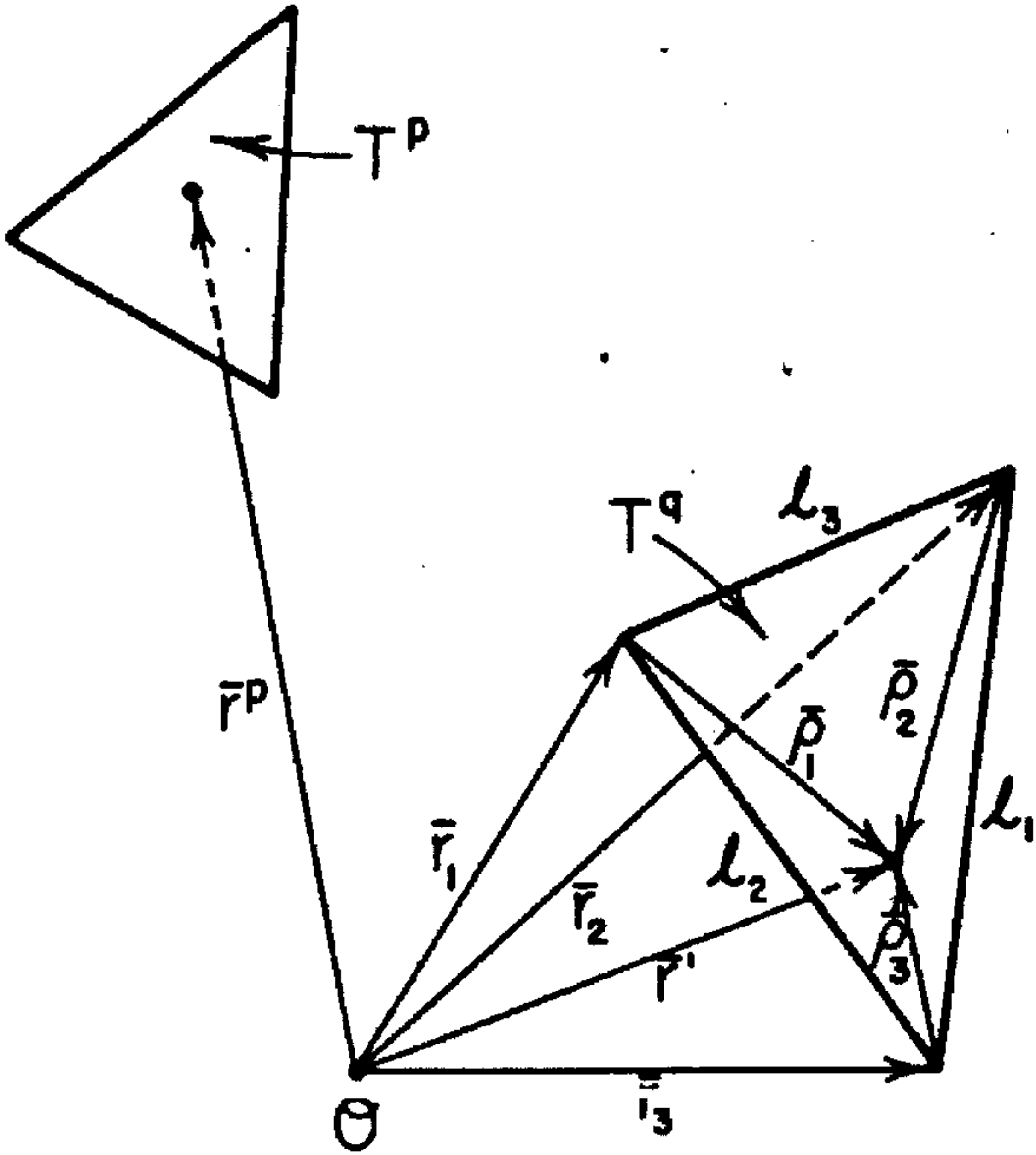


Fig. 4. Local coordinates and edges for source triangle T^q with observation point in triangle T^p.