Von Neumann Method for Stability Analysis

let $\bar{u}_i^n$ be the exact solution to a difference eq. and
u_i^n the actual computed solution

the error, due to round-off (and errors in the initial data), can be represented as:

$$u_i^n = \bar{u}_i^n + \varepsilon_i^n \qquad (1)$$

i.e. ε_i^n is the error at time level n and mesh point i.

Consider the one-way wave equation

$$\partial_t u + a\,\partial_x u = 0 \qquad (2)$$

and the Forward-time centered-space finite difference scheme:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -\frac{a}{2\Delta x}\left(u_{i+1}^{n+1} - u_{i-1}^{n+1}\right) \qquad (3)$$

plugging (1) into (2) we have:

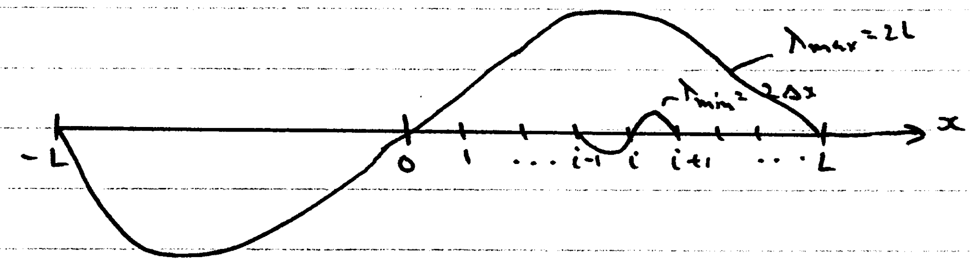
$$\frac{\bar{u}_i^{n+1} - \bar{u}_i^n}{\Delta t} + \frac{\varepsilon_i^{n+1} - \varepsilon_i^n}{\Delta t} = -\frac{a}{2\Delta x}\left(\bar{u}_{i+1}^n - \bar{u}_{i-1}^n\right) - \frac{a}{2\Delta x}\left(\varepsilon_{i+1}^n - \varepsilon_i^n \qquad (4)\right.$$

now $\bar{u}_i^n$ satisfies (3) exactly (by definition) therefore we find that

$$\frac{\varepsilon_i^{n+1} - \varepsilon_i^n}{\Delta t} = -\frac{a}{2\Delta x}\left(\varepsilon_{i+1}^n - \varepsilon_{i-1}^n\right) \qquad (5)$$

$\Rightarrow$ that the numerical error ε_i^n evolves over time in the same way as the numerical solution u_i^n.

now consider the Fourier series on a finite region $(0, L)$



Note that the Fourier representation on $(0, L)$ will "continue" the representation periodically over $(-\infty, \infty)$.

the fundamental freq. $f_0 = \frac{1}{\lambda_{max}} = \frac{1}{2L} \Rightarrow k_{min} = k_0 = \frac{2\pi}{\lambda_{max}} = \frac{\pi}{L}$

the max. freq. $f_{max} = \frac{1}{\lambda_{min}} = \frac{1}{2\Delta x} \Rightarrow k_{max} = \frac{\pi}{\Delta x}$

given that we have N intervals $\Rightarrow \Delta x = \frac{L}{N}$

and $x_i = i\Delta x \quad i = 0, 1, \ldots N$ are the mesh pts.

and the harmonics on the mesh are

$$k_j = j\,k_{min} = j\frac{\pi}{L} = j\frac{\pi}{N\Delta x} \qquad j = 0, 1, \ldots N. \qquad (6)$$

$$K_N = \frac{\pi}{\Delta x} = K_{max}$$

now any finite mesh function such as ε_i^n can be represented in a Fourier series:

$$\varepsilon_i^n = \sum_{j=-N}^{N} E_j^n e^{I k_j \cdot i \Delta x} = \sum_{j=-N}^{N} E_j^n e^{I i j \pi / N} \qquad (7)$$

where $I = \sqrt{-1}$ and E_j^n is the amplitude of the j^{th} harmonic.

$E_0^n \Rightarrow$ constant function or D.C. component

$k_j \Delta x \triangleq \phi_j = \frac{j\pi}{N}$ sometime called "phase angle"

note the phase takes on values $(-\pi, \pi)$ in increments of $\frac{\pi}{N}$

$$\varepsilon_i^n = \sum_{j=-N}^{N} E_j^n e^{I i \phi_j} \qquad (8)$$

$\phi_0 = 0 \Rightarrow$ low freq.

$\phi_N = \pi \Rightarrow$ high freq. $\Rightarrow \phi_N = \frac{N\pi}{N} = \pi \Rightarrow k_j \Delta x = \frac{2\pi}{\lambda_{j=N}} \Delta x$

$\Rightarrow \lambda_{j=N} = 2\Delta x = \lambda_{min}$.

each harmonic in (8) or (7) must satisfy the difference equation (5)

$\therefore$ if we consider a single harmonic $E_j^n e^{Ii\phi_j}$ (i^{th} location) its time evolution is determined by the numerical scheme.

plugging into (5) we have: (dropping the j subscript)

$$\frac{(E^{n+1} - E^n)}{\Delta t} e^{Ii\phi} + \frac{a}{2\Delta x}\left(E^n e^{I(i+1)\phi} - E^n e^{I(i-1)\phi}\right) = 0 \quad (9)$$

dividing by $e^{Ii\phi}$ and multiplying by Δt

$$(E^{n+1} - E^n) + \frac{\sigma}{2} E^n \left(e^{I\phi} - e^{-I\phi}\right) = 0 \quad (10) \qquad \sigma = \frac{a\Delta t}{\Delta x}$$

- we define the amplification factor, G,

$$G \triangleq \frac{E^{n+1}}{E^n} \quad (11)$$

and the scheme is stable if $|G| \leq 1 \quad \forall \phi$ (12)

as can be seen, G is a function of time step, Δt, freq., k_j, and mesh size Δx.

From (10):

$$\frac{E^{n+1}}{E^n} - 1 + \frac{\sigma}{2}\left(e^{I\phi} - e^{-I\phi}\right) = 0$$

or

$$G - 1 + \frac{\sigma}{2} 2I \sin\phi = 0$$

- $$\boxed{G = 1 - I\sigma \sin\phi} \quad (13)$$

amp. factor for F-C one-way wave eq.

$$|G|^2 = 1 + \sigma^2 \sin^2\phi \quad > 1 \qquad (14)$$

$\therefore$ the F-C scheme is unconditionally unstable.

<u>example</u> First-order upwind for $\partial_t u + a\,\partial_x u = 0$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -\frac{a}{\Delta x}\left(u_i^n - u_{i-1}^n\right) \qquad (15)$$

inserting a single harmonic $E^n e^{Ii\phi}$

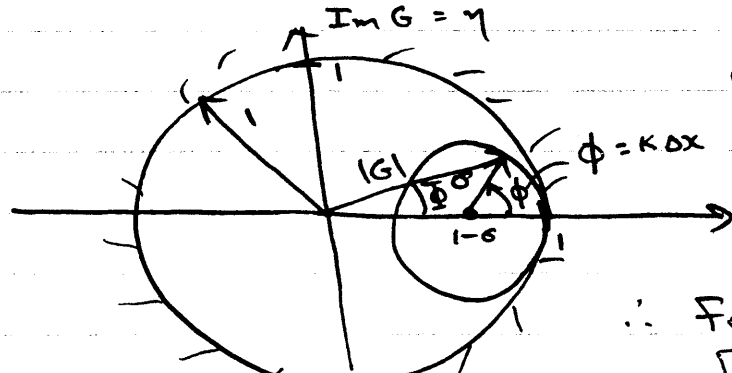
$$\left(E^{n+1} - E^n\right) e^{Ii\phi} + \sigma E^n \left(e^{Ii\phi} - e^{I(i-1)\phi}\right) = 0$$

dividing by $E^n e^{Ii\phi}$

$$\begin{aligned} G &= 1 - \sigma + \sigma e^{-I\phi} \\ &= 1 - 2\sigma \sin^2\frac{\phi}{2} - I\sigma\sin\phi \qquad (16) \\ &= \xi + I\eta \end{aligned}$$

$$\begin{cases} \xi = 1 - 2\sigma\sin^2\frac{\phi}{2} = (1-\sigma) + \sigma\cos\phi \\ \eta = -\sigma\sin\phi \end{cases}$$

this is the parametric equations for a circle centred at $(\xi = 1-\sigma, \eta = 0)$ and radius $= \sigma$



$G = |G| \angle \Phi = \xi + I\eta$

$\therefore$ For stability

$$\boxed{0 < \sigma \le 1} \qquad (17)$$

CFL condition

this is the Courant-Friedrichs-Lewy condition.

$\sigma = \dfrac{a\,\Delta t}{\Delta x}$ is called the <u>Courant number.</u>

Another way of stating the Von Neumann method for stability is to say that no harmonic of the difference solution u_i^n should be allowed to increase without bound.

expanding: $u_i^n = \sum_{m=-N}^{N} v_m^n e^{I i K_m \Delta x} = \sum_{m=-N}^{N} v_m^n e^{I i \phi}$

where v_m^n is the amplitude of the m^{th} harmonic of u_i^n. We then define

$$G \triangleq \frac{v^{n+1}}{v^n} = G(\phi, \Delta t, \Delta x)$$

and this will result in the same thing as $G \triangleq \dfrac{\varepsilon^{n+1}}{\varepsilon^n}$

since v_i^n and ε_i^n satisfy the same difference equation.

<u>Example</u>: Leap-Frog Scheme

$$\frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} = -\frac{a}{2\Delta x}\left(u_{i+1}^n - u_{i-1}^n\right)$$

inserting $v^n e^{I i \phi} = u_i^n$

$$\frac{\left(v^{n+1} - v^{n-1}\right)}{2\Delta t} e^{I i \phi} = -\frac{a}{2\Delta x} v^n \left[e^{I(i+1)\phi} - e^{I(i-1)\phi}\right]$$

Dividing by $v^n e^{I i \phi}$ we get:

$$\frac{v^{n+1}}{v^n} - \frac{v^{n-1}}{v^n} = -\sigma \left[e^{I\phi} - e^{-I\phi} \right] = -\sigma 2I \sin \phi$$

$$G - \frac{1}{G} = -\sigma 2I \sin\phi$$

or $G^2 + G\sigma 2I \sin\phi - 1 = 0$

$$\therefore \boxed{G = -I\sigma \sin\phi \pm \sqrt{1 - \sigma^2 \sin^2\phi}}$$ amplification Factor for L-F scheme.

For $\sigma = 1$ $|G| = 1$ $\Rightarrow$ "neutrally stable"

For $\sigma < 1$: $|G|^2 = \sigma^2 \sin^2\phi + 1 - \sigma^2 \sin^2\phi = 1$

$\Rightarrow$ "neutrally stable"

$\Rightarrow$ no dissipation.

if $\sigma > 1$: $G = I \left[-\sigma \sin\phi \pm \sqrt{\sigma^2 \sin^2\phi - 1} \right]$

take $\phi = \frac{\pi}{2}$ $|G| = \sigma + \sqrt{\sigma^2 - 1} > 0$

$\Rightarrow$ unstable

Example : 2-D hyperbolic system

$$\partial_t u + A\,\partial_x u + B\,\partial_y u = 0$$

the Lax-Friedrichs scheme is :

$$u^{n+1}_{i,j} = \frac{1}{4}\left(u^n_{i,j+1} + u^n_{i+1,j} + u^n_{i-1,j} + u^n_{i,j-1}\right)$$

$$- \frac{\Delta t}{2\Delta x} A \left(u^n_{i+1,j} - u^n_{i-1,j}\right) - \frac{\Delta t}{2\Delta y} B \left(u^n_{i,j+1} - u^n_{i,j-1}\right)$$

the two-dimensional Fourier decomposition is:

$$u^n_{i,j} = \sum_{k_x, k_y} v^n e^{I k_x i \Delta x} e^{I k_y j \Delta y}$$

$$= \sum_{\phi_x, \phi_y} v^n e^{iI\phi_x} e^{jI\phi_y}$$

or one component can be expressed as:

$$v^n e^{iI\phi_x} e^{jI\phi_y}$$

plugging this in to the L-F scheme and dividing by $v^n e^{iI\phi_x} e^{jI\phi_y}$ we get

unit matrix

$$G = \frac{v^{n+1}}{v^n} = \frac{1}{4}\left(e^{I\phi_y} + e^{I\phi_x} + e^{-I\phi_x} + e^{-I\phi_y}\right) \underline{I}$$

$$- \frac{\Delta t}{2\Delta x} A \left[e^{I\phi_x} - e^{-I\phi_x}\right] - \frac{\Delta t}{2\Delta y} B \left(e^{I\phi_y} - e^{-I\phi_y}\right)$$

$$\boxed{G = \frac{1}{2}\left(\cos\phi_x + \cos\phi_y\right) - \frac{\Delta t}{\Delta x} A I \sin\phi_x - \frac{\Delta t}{\Delta y} B I \sin\phi_y.}$$