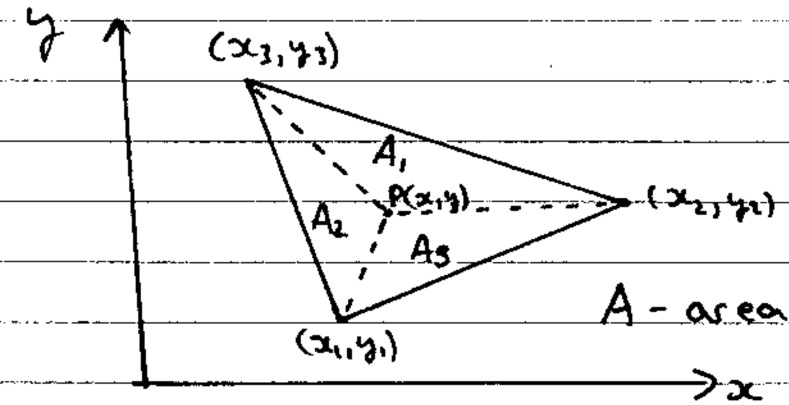
Higher - Order Elements

Triangular Elements using Area Coordinates

consider a point $p(x, y)$ inside a triangle:



when p is at vertex 1, (x_1, y_1), $A_1 = A, A_2 = 0, A_3 = 0$
when p is at vertex 2, (x_2, y_2), $A_1 = 0, A_2 = A, A_3 = 0$
when p is at vertex 3, (x_3, y_3), $A_1 = 0, A_2 = 0, A_3 = A$

for any location we define 3 coordinates L_1, L_2, L_3 given by:

$$L_1 = \frac{A_1}{A}, \quad L_2 = \frac{A_2}{A}, \quad L_3 = \frac{A_3}{A}$$

where A is the area of the triangle.

at any location we have $A = \sum_{i=1}^{3} A_i$

L_i is unity at vertex i and 0 at any other vertex.

$$L_1 + L_2 + L_3 = 1$$

$\therefore$ If we want linear basis functions then L_i can be used. (Note: $L_i(x,y)$)

In terms of the global coordinates (x_i, y_i), the area, A, is written as:

$$A = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$ (counter-clockwize listing of vertices)

$$L_1 = \frac{A_1}{A} = \frac{\frac{1}{2}\begin{vmatrix} 1 & x & y \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}}{A}$$

$$L_2 = \frac{A_2}{A} = \frac{\frac{1}{2}\begin{vmatrix} 1 & x & y \\ 1 & x_3 & y_3 \\ 1 & x_1 & y_1 \end{vmatrix}}{A}$$

$$L_3 = \frac{A_3}{A} = \frac{\frac{1}{2}\begin{vmatrix} 1 & x & y \\ 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \end{vmatrix}}{A}$$

$$\therefore \begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix} = \frac{1}{2A}\begin{pmatrix} x_2y_3 - x_3y_2 & y_2 - y_3 & x_3 - x_2 \\ x_3y_1 - x_1y_3 & y_3 - y_1 & x_1 - x_3 \\ x_1y_2 - x_2y_1 & y_1 - y_2 & x_2 - x_1 \end{pmatrix}\begin{pmatrix} 1 \\ x \\ y \end{pmatrix}$$

[Note: L_i is the same as ψ_i found before.]

when minimizing the functional in the finite element method we encounter integrals of the form:

$$I = \int_A \frac{\partial L_i}{\partial x} \frac{\partial L_j}{\partial x} \, dx \, dy$$

where we are integrating over the area of the triangle, A, (which is arbitrary).

In order to evaluate the partial derivatives, we use the chain rule:

$$\left.\frac{\partial L_i}{\partial x}\right|_y = \left.\frac{\partial L_i}{\partial L_1}\right|_{L_2} \left.\frac{\partial L_1}{\partial x}\right|_y + \left.\frac{\partial L_i}{\partial L_2}\right|_{L_1} \left.\frac{\partial L_2}{\partial x}\right|_y$$

Note: we only use 2 of the area coordinates, L_1 and L_2 since the third is dependent on it, $L_3(L_1, L_2) = 1 - L_1 - L_2$

$$\left.\frac{\partial L_1}{\partial x}\right|_y = \frac{y_2 - y_3}{2A} \qquad \left.\frac{\partial L_2}{\partial x}\right|_y = \frac{y_3 - y_1}{2A}$$

$$\therefore \quad \left.\frac{\partial L_i}{\partial x}\right|_y = \left.\frac{\partial L_i}{\partial L_1}\right|_{L_2} \left(\frac{y_2 - y_3}{2A}\right) + \left.\frac{\partial L_i}{\partial L_2}\right|_{L_1} \left(\frac{y_3 - y_1}{2A}\right)$$

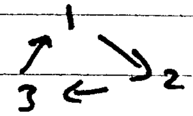
$$\frac{\partial L_1}{\partial x} = \frac{1}{2A} \left[(y_2 - y_3) + 0\,(y_3 - y_1) \right] = \frac{1}{2A}(y_2 - y_3)$$

$$\frac{\partial L_2}{\partial x} = \frac{1}{2A} \left[0\,(y_2 - y_3) + (y_3 - y_1) \right] = \frac{1}{2A}(y_3 - y_1)$$

$$\frac{\partial L_3}{\partial x} = \frac{1}{2A} \left[-(y_2 - y_3) - (y_3 - y_1) \right] = \frac{1}{2A}(y_1 - y_2)$$

$$\therefore \boxed{\frac{\partial L_i}{\partial x} = \frac{1}{2A}\left(y_{i+1} - y_{i+2}\right)}.$$

with cyclic indices $1 \leq i \leq 3$



similarly:

$$\boxed{\frac{\partial L_i}{\partial y} = \frac{1}{2A}\left(x_{i+2} - x_{i+1}\right)}$$

$\therefore$ we have the following integrals:

$$\begin{cases} I = \displaystyle\int_A \frac{\partial L_i}{\partial x}\frac{\partial L_j}{\partial x}\,dx\,dy = \frac{1}{4A}\left(y_{i+1} - y_{i+2}\right)\left(y_{j+1} - y_{j+2}\right) \\ I' = \displaystyle\int_A \frac{\partial L_i}{\partial y}\frac{\partial L_j}{\partial y}\,dx\,dy = \frac{1}{4A}\left(x_{i+2} - x_{i+1}\right)\left(x_{j+2} - x_{j+1}\right) \end{cases}$$

Another common integral is:

$$I = \int_A L_i L_j \,dx\,dy$$

we can use the following relationship for integrating powers of L_i:

$$\boxed{\int_A L_1^{m_1} L_2^{m_2} L_3^{m_3}\,dx\,dy = 2A \frac{m_1!\,m_2!\,m_3!}{(m_1 + m_2 + m_3 + 2)!}}$$

$$\therefore \quad \text{for} \quad I = \int_A L_i L_j \, dx\, dy$$

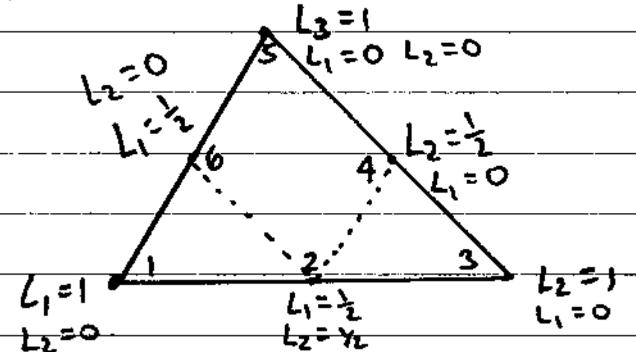
if $i \neq j \Rightarrow m_1 = m_2 = 1,\ m_3 = 0$ and we get

$$\left.\int_A L_i L_j \, dx\, dy = 2A \, \frac{1!\,1!\,0!}{4!} = \frac{A}{12} \right\}$$

if $i = j \Rightarrow m_1 = 2 \quad m_2 = m_3 = 0$

$$\left.\int_A L_i^2 \, dx\, dy = 2A \, \frac{2!}{4!} = \frac{A}{6} \right\}$$

Quadratic Triangular Elements



$$\phi_i = a + bL_1 + cL_2 + dL_1^2 + eL_2^2 + fL_1L_2$$

(quadratic polynomial)

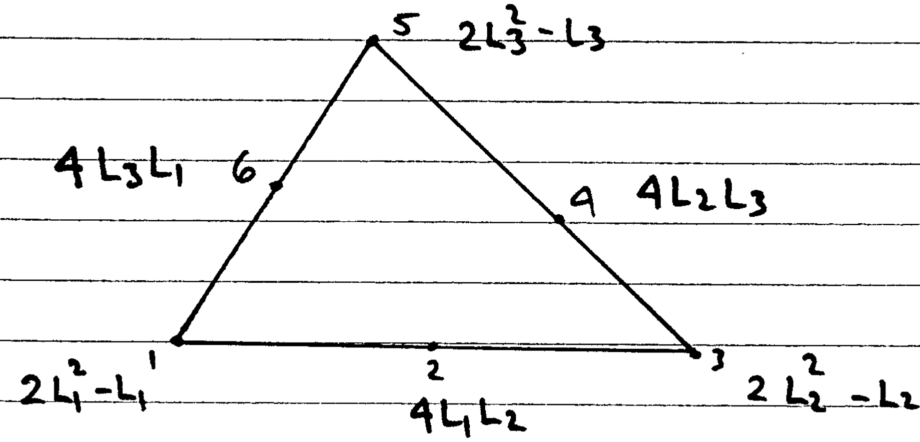
each basis function has 6 unknowns, we impose 6 constraints. For example, @ vertex 1:

$$\begin{array}{c} \text{Vertex} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \quad \begin{array}{c} \\ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{array} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & \frac{1}{2} & 0 & \frac{1}{4} & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & \frac{1}{2} & 0 & \frac{1}{4} & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix}$$

we get: $[a\ b\ c\ d\ e\ f]^T = [0\ -1\ 0\ 2\ 0\ 0]^T$

$\therefore\ \phi_1 = 2L_1^2 - L_1$

similarly: $\phi_2 = 4L_1L_2$
$\phi_3 = 2L_2^2 - L_2$
$\phi_4 = 4L_2L_3$
$\phi_5 = 2L_3^2 - L_3$
$\phi_6 = 4L_3L_1$



Since the basis functions are written in terms of area coordinates, our previous integration formula can still be used:

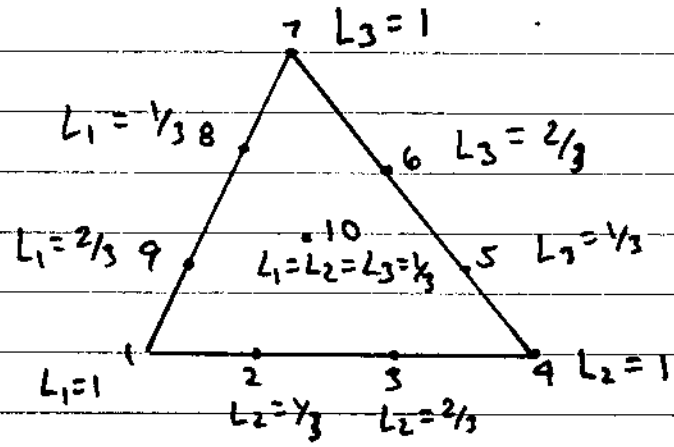
$$I = \int_A \frac{\partial \phi_i}{\partial x} \frac{\partial \phi_j}{\partial x}\, dx\, dy$$

$$\left.\frac{\partial \phi_i}{\partial x}\right|_y = \left.\frac{\partial \phi_i}{\partial L_1}\right|_{L_2} \left.\frac{\partial L_1}{\partial x}\right|_y + \left.\frac{\partial \phi_i}{\partial L_2}\right|_{L_1} \left.\frac{\partial L_2}{\partial x}\right|_y$$

ex: $\dfrac{\partial \phi_2}{\partial L_1} = 4L_2$

Cubic Triangular Elements

C^0 element



$$\phi_i = a + bL_1 + cL_2 + dL_1^2 + eL_2^2 + fL_1L_2 + gL_1^2L_2 + hL_1L_2^2 + iL_1^3 + jL_2^3$$

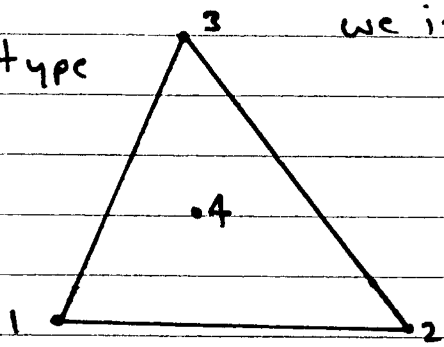
TABLE 3.2. Linear, Quadratic, and Cubic Basis Functions for Triangular Elements

	Functions		
Basis	Linear	Quadratic	Cubic
ϕ_1	L_1	$2L_1^2 - L_1$	$\frac{1}{2}L_1(3L_1-1)(3L_1-2)$
ϕ_2	L_2	$4L_1L_2$	$\frac{9}{2}L_1L_2(3L_1-1)$
ϕ_3	L_3	$2L_2^2 - L_2$	$\frac{9}{2}L_1L_2(3L_2-1)$
ϕ_4	—	$4L_2L_3$	$\frac{1}{2}L_2(3L_2-1)(3L_2-2)$
ϕ_5	—	$2L_3^2 - L_3$	$\frac{9}{2}L_2L_3(3L_2-1)$
ϕ_6	—	$4L_3L_1$	$\frac{9}{2}L_2L_3(3L_3-1)$
ϕ_7	—	—	$\frac{1}{2}L_3(3L_3-1)(3L_3-2)$
ϕ_8	—	—	$\frac{9}{3}L_3L_1(3L_3-1)$
ϕ_9	—	—	$\frac{9}{2}L_3L_1(3L_1-1)$
ϕ_{10}	—	—	$27L_1L_2L_3$

in these elements continuity of the function value at the vertices, between triangles, is imposed naturally. (this is why they are called C^0 elements).

If we want to impose continuity of the first derivatives ∂_x and ∂_y we use fewer nodes:

C'-type



we impose:

$$\phi|_i,\ \frac{\partial \phi}{\partial x}\Big|_i,\ \frac{\partial \phi}{\partial y}\Big|_i \qquad i=1,2,3,4$$

if we want to expand $u(x,y)$ inside the triangle, we write:

$$u(x,y) = \sum_{j=1}^{4} u_j \phi_{00j} + \sum_{j=1}^{3}\Big[u'_{xj}\, \phi_{10j} \frac{\partial x}{\partial L_1} + u'_{xj}\, \phi_{01j} \frac{\partial x}{\partial L_2} + u'_{yj}\, \phi_{10j} \frac{\partial y}{\partial L_1} + u'_{yj}\, \phi_{01j} \frac{\partial y}{\partial L_2} \Big]$$

TABLE 3.3. Cubic Basis Functions for Elements with Continuous $\partial\hat{u}/\partial x$ and $\partial\hat{u}/\partial y$ at Nodal Locations

	Basis Function	Functional Form
1	ϕ_1 ϕ_{001}	$L_1^2(L_1+3L_2+3L_3)-7L_1L_2L_3$
	ϕ_2 ϕ_{101}	$L_1^2(a_3L_2-a_2L_3)+(a_2-a_3)L_1L_2L_3$
	ϕ_3 ϕ_{011}	$L_1^2(b_2L_3-b_3L_2)+(b_3-b_2)L_2L_1L_3$
2	ϕ_4 ϕ_{002}	$L_2^2(L_2+3L_3+3L_1)-7L_1L_2L_3$
	ϕ_5 ϕ_{102}	$L_2^2(a_1L_3-a_3L_1)+(a_3-a_1)L_1L_2L_3$
	ϕ_6 ϕ_{012}	$L_2^2(b_3L_1-b_1L_3)+(b_1-b_3)L_1L_2L_3$
3	ϕ_7 ϕ_{003}	$L_3^2(L_3+3L_1+3L_2)+7L_1L_2L_3$
	ϕ_8 ϕ_{103}	$L_3^2(a_2L_1-a_1L_2)+(a_1-a_2)L_1L_2L_3$
	ϕ_9 ϕ_{013}	$L_3^2(b_1L_2-b_2L_1)+(b_2-b_1)L_1L_2L_3$
4	ϕ_{10} ϕ_{004}	$27L_1L_2L_3$

$a_i = x_k - x_j \qquad b_i = y_j - y_k$

After Felippa, 1966.