METHOD OF MOMENTS

used to solve operator equation (linear):

$$LF = g \qquad \text{(deterministic problem)}$$

we are dealing with fcns in a Hilbert space and thus define an inner product (F, g) such that: if $\alpha, \beta \in \mathbb{R}$, $F, g, h \in \mathcal{H}$

$$\begin{cases} (F, g) = (g, F) \\ (\alpha F + \beta g, h) = \alpha(F, h) + \beta(g, h) \\ (F^*, F) = \|F\|^2 > 0 \quad \text{if } F \neq 0 \\ \qquad\qquad = 0 \quad \text{if } F = 0 \end{cases}$$

$*$ → complex conjugate

adjoint operator L^a : $(LF, g) = (F, L^a g)$

For all F in the domain of L

an operator is self-adjoint if $L^a = L$ and domain of L^a is that of L

i.e. $(LF, g) = (F, Lg)$

an operator is real if $Lf(x) \in \mathbb{R}$ when $f(x) \in \mathbb{R}$

an operator is positive definite if:

$$(f^*, Lf) > 0 \qquad \forall f \neq 0$$

if solution to $Lf = g$ exists and is unique for all g, then the inverse operator exists, L^{-1}, $\ni$

$$f = L^{-1} g$$

if g is known then this is the solution to original problem.

<u>example:</u> given $g(x)$ Find $f(x)$ $0 \le x \le 1$ $\ni$

(O.D.E) $$-\frac{d^2 f}{dx^2} = g(x) \quad (\text{B.C.'s})\ f(0) = f(1) = 0$$

$L = -\dfrac{d^2}{dx^2}$, <u>range</u> of L is space of all fcns g over $0 \le x \le 1$

, <u>domain</u> of L is the space of all fcns f over $0 \le x \le 1$ satisfying B.C's and having second derivatives in the range of L

Note: both differential operator and boundary conditions define the operator.

Inner product: $(f, g) = \int_0^1 f(x) g(x)\, dx$

but could also use: $(f, g) = \int_0^1 w(x) f(x) g(x)\, dx$

$w(x) > 0$ (weighting fcn)

the adjoint operator L^a depends on the inner product which is used.

$$(Lf, g) = \int_0^1 \left(-\frac{d^2 f}{dx^2}\right) g\, dx$$

using integration by parts: $\int u\, dv = uv - \int v\, du$

$$(Lf, g) = \int_0^1 g \left(-\frac{d^2 f}{dx^2}\right) dx = \int_0^1 \left(\frac{df}{dx}\right) \frac{dg}{dx}\, dx - \left[\frac{df}{dx} g\right]_0^1$$

$$= \int_0^1 f\left(-\frac{d^2 g}{dx}\right) dx + \left[f \frac{dg}{dx} - g \frac{df}{dx}\right]_0^1$$

last two terms are boundary terms and g on the boundary can be chosen so that these vanish. Then we have

$$(Lf, g) = (f, L^a g)$$

$L^a = -\frac{d^2}{dx} = L$ $\quad\therefore\quad$ L is self-adjoint

notice domain of L^a is same as domain of L.

L is a real operator and is positive definite:

$$(f^*, Lf) = \int_0^1 f^*\left(-\frac{d^2 f}{dx^2}\right) dx = \int_0^1 \frac{df^*}{dx}\frac{df}{dx}\,dx - \left[f^*\frac{df}{dx}\right]_0^1$$

$$= \int_0^1 \left|\frac{df}{dx}\right|^2 dx \quad > 0$$

inverse operator:

$$L^{-1} g = \int_0^1 G(x,x')\, g(x')\, dx'$$

$$G(x,x') = \begin{cases} x(1-x') & x < x' \\ (1-x)\,x' & x > x' \end{cases}$$

Note: since L is self-adjoint

$$(Lf_1, f_2) = (f_1, Lf_2) = (g_1, L^{-1}g_2) = (L^{-1}g_1, g_2)$$

and $\therefore$ L^{-1} is self-adjoint.

Approximations

now consider again : $Lf = g$. ①

expand f in a series $f_1, f_2, f_3 \ldots$ in $\mathcal{D}(L)$:

$$f = \sum_n \alpha_n f_n \qquad \alpha_n \in \mathbb{R} \qquad ②$$

we call f_n expansion fcns, or basis fcns

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if $\{f_n\}_1^\infty$ form a complete set of basis fcns then we can represent the exact solution.

if $\{f_n\}_1^N$ is finite then we can determine an approximate solution.

$$Lf = g = L\left(\sum_n \alpha_n f_n\right) = \sum_n \alpha_n Lf_n = g \qquad ③$$

we now define a set of weighting fcns or testing fcns $\{w_m\}_1^M$ in the range of L

and take inner product of ③ with each w_m:

$$\sum_n \alpha_n (w_m, Lf_n) = (w_m, g) \qquad m = 1, 2, \ldots \qquad ④$$

or in matrix form:

$$[l_{mn}][\alpha_n] = [g_m] \qquad ⑤$$

$$[l_{mn}] = \begin{bmatrix} (w_1, Lf_1) & (w_1, Lf_2) & \cdots \\ (w_2, Lf_1) & (w_2; Lf_2) & \cdots \\ \vdots & \vdots & \end{bmatrix}$$

$$\alpha_n = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_N \end{bmatrix} \qquad [g_m] = \begin{bmatrix} (w_1, g) \\ (w_2, g) \\ \vdots \\ (w_M, g) \end{bmatrix}$$

$$[\alpha_n] = [l_{mn}^{-1}][g_m]$$

define : $[f_n]^T = [f_1 \ \ f_2 \ \ f_3 \ \ \ldots]$

$$\therefore \ \boxed{f = [f_n]^T[\alpha_n] = [f_n]^T[l_{mn}^{-1}][g_m]}$$

this is in general approximate.

if $w_n = f_n \Rightarrow$ <u>Galerkin's method</u>.

$\{f_n\}$, $\{w_n\}$ must be appropriately chosen.

- f_n in the domain of L
- w_n in the domain of L^a
- linear combination of f_n represent f
- " " " w_n " f^a

Example: $\quad -\dfrac{d^2 f}{dx^2} = 1+4x^2 \qquad 0 \le x \le 1$

$f(0) = f(1) = 0$

exact solution (integrate twice):

$$\frac{df}{dx} = -\int (1+4x^2)\,dx = -x - \frac{4}{3}x^3 + A$$

$$f = \int \left(-x - \frac{4}{3}x^3 + A\right) dx = -\frac{1}{2}x^2 - \frac{1}{3}x^4 + Ax + B$$

$f(0) = B = 0$

$f(1) = -\frac{1}{2} - \frac{1}{3} + A \Rightarrow A = \frac{5}{6}$

$$\boxed{f(x) = \frac{5x}{6} - \frac{x^2}{2} - \frac{x^4}{3}}$$

moment method solution:

let $\quad f_n = x - x^{n+1} \qquad$ (basis fcns) $\quad n = 1, 2, \ldots N$

$$f = \sum_{n=1}^{N} a_n (x - x^{n+1})$$

- we have chosen $f_n \in \mathcal{D}(L) \rightarrow \quad L = -\dfrac{d^2}{dx^2}$, $\quad f(0) = f(1) = 0$

Apply Galerkin method: that is choose weighting fcns as $w_n = f_n = x - x^{n+1}$

$w_n \in \mathcal{D}(L^a) = \mathcal{D}(L)$ since L is self-adjoint.

$$\sum_n \alpha_n (\omega_m, LF_n) = (\omega_m, 1+4x^2)$$

$$(\omega_m, LF_n) = \int_0^1 \omega_m LF_n \, dx = \int_0^1 (x-x^{m+1}) \left(-\frac{d^2}{dx^2} (x-x^{n+1}) \right) dx$$

$$= \int_0^1 (x-x^{m+1}) \left((n+1) n \, x^{n-1} \right) dx$$

$$= \int_0^1 n(n+1) \, (x^n - x^{n+m}) \, dx = n(n+1) \left[\frac{x^{n+1}}{n+1} - \frac{x^{n+m+1}}{n+m+1} \right]_0^1$$

$$= n(n+1) \left[\frac{1}{n+1} - \frac{1}{n+m+1} \right] = n(n+1) \left[\frac{m}{(n+1)(n+m+1)} \right]$$

$$\boxed{(\omega_m, LF_n) = \frac{nm}{n+m+1}}$$

$$(\omega_m, 1+4x^2) = \int_0^1 (x-x^{m+1})(1+4x^2) \, dx$$

$$= \int_0^1 x - x^{m+1} + 4x^3 - 4x^{m+3} \, dx$$

$$= \left[\frac{x^2}{2} - \frac{x^{m+2}}{m+2} + \frac{4x^4}{4} - \frac{4x^{m+4}}{m+4} \right]_0^1$$

$$= \frac{1}{2} - \frac{1}{m+2} + 1 - \frac{4}{m+4} = \frac{3}{2} - \frac{1}{m+2} - \frac{4}{m+4}$$

$$= \frac{3(m+2)(m+4) - 2(m+4) - 8(m+2)}{2(m+2)(m+4)} = \frac{3m^2 + m8}{2(m+2)(m+4)}$$

$$\boxed{(w_m, 1+4x^2) = \frac{m(3m+8)}{2(m+2)(m+4)}}$$

For $N=1$:

$$\alpha_1\left(\frac{1}{3}\right) = \frac{11}{30} \qquad \alpha_1 = \frac{11}{10} \qquad f = \frac{11}{10}(x - x^2)$$

For $N=2$:

$$\begin{array}{c c} & \begin{array}{cc} n=1 & 2 \end{array} \\ \begin{array}{c} m=1 \\ 2 \end{array} & \begin{bmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{1}{2} & \frac{4}{5} \end{bmatrix} \end{array} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{array}{cc} \begin{bmatrix} \frac{11}{30} \\ \frac{7}{12} \end{bmatrix} & \begin{array}{c} m=1 \\ 2 \end{array} \end{array} \qquad \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{10} \\ \frac{2}{3} \end{bmatrix}$$

$$f = \frac{1}{10}(x - x^2) + \frac{2}{3}(x - x^3)$$

$$f = \frac{23}{30}x - \frac{x^2}{10} - \frac{2}{3}x^3$$

For $N=3$:

$$\begin{bmatrix} \frac{1}{3} & \frac{1}{2} & \frac{3}{5} \\ \frac{1}{2} & \frac{4}{5} & 1 \\ \frac{3}{5} & 1 & \frac{9}{7} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} \frac{11}{30} \\ \frac{7}{12} \\ \frac{51}{70} \end{bmatrix} \qquad \underline{\alpha} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{3} \end{bmatrix}$$

$$f = \frac{1}{2}(x - x^2) + \frac{1}{3}(x - x^4)$$

$$\boxed{f = \frac{5}{6}x - \frac{x^2}{2} - \frac{x^4}{3}}$$

exact! since f_n span the $\mathcal{D}(L)$

Note: $[\ell^{-1}]$ represents L^{-1}

example: solve the same problem

$$\begin{cases} -\dfrac{d^2f}{dx^2} = 1+4x^2 & 0 \le x \le 1 \\ f(0) = f(1) = 0 \end{cases}$$

by the Rayleigh - Ritz variational method on the quadratic functional for the problem. Use the same basis fcns.

$$I(f) = \int_0^1 \left(\frac{df}{dx}\right)^2 dx - 2\int_0^1 f(1+4x^2)\,dx$$

$$f = \sum_{n=1}^{N} \alpha_n f_n = [f_n]^T[\alpha_n] \qquad f_n = x - x^{n+1}$$

$$\frac{df}{dx} = \left[\frac{df_n}{dx}\right]^T[\alpha_n] \qquad f_n' = \frac{df_n}{dx} = -(n+1)x^n$$

$$I(f) = \int_0^1 \left([f_n']^T[\alpha_n]\right)^2 dx - 2\int_0^1 [f_n]^T[\alpha_n](1+4x^2)\,dx$$

$$\frac{\partial I}{\partial \alpha_m} = 2\int_0^1 \left([f_n']^T[\alpha_n]\right) f_m'\,dx - 2\int_0^1 f_m(1+4x^2)\,dx = 0$$

$$\int_0^1 \left([f_n']^T[\alpha_n]\right) f_m'\,dx = \int_0^1 f_m(1+4x^2)\,dx$$

$$\int_0^1 [f_n']^T[\alpha_n] f_m'\,dx = \int_0^1 [f_n'] f_m'\,dx\,[\alpha_n]$$

$$\int u\,dv = uv - \int v\,du \qquad \text{(partial integration)}$$

$$\therefore \int_0^1 [F_n'] F_m'\,dx = [F_n'] F_m \Big|_0^1 - \int_0^1 F_m [F_n'']\,dx$$

the first term is zero since $F_m(0) = F_m(1) = 0$

$$\therefore \int_0^1 [F_n'] F_m'\,dx\,[\alpha_n] = -\int_0^1 F_m [F_n'']\,dx\,[\alpha_n]$$

Note this is the same as the Galerkin method

$$-\int_0^1 F_m [F_n'']\,dx\,[\alpha_n] = \sum_n \alpha_n (F_m, LF_n)$$

$$\int_0^1 F_m (1+4x^2)\,dx = (F_m, g)$$

$\therefore$ Using Rayleigh-Ritz method for the minimization of $I(F)$ produces the same matrix equation as applying the Galerkin method.

$$\sum_n \alpha_n (F_m, LF_n) = (F_m, g)$$

Point Matching or Collocation

in the moment method we have

$$[l_{mn}] = [(w_m, Lf_n)]$$

$$[g_m] = (w_m, g)$$

$$(w_m, Lf_n) = \int_D w_m \, Lf_n \, dv$$

if we choose $w_m = \delta(\underline{x} - \underline{x}_m)$

then $\int_D w_m \, Lf_n \, dv = \int_D Lf_n \, \delta(\underline{x} - \underline{x}_m) \, dv$

$$= (Lf_n)\Big|_{\underline{x} = x_m}$$

thus: integration is side-stepped. This is equivalent to requiring that

$$\sum_{n=1}^{N} \alpha_n (Lf_n) = g$$

be sattisfied exactly at N discrete points in the domain, Hence the name point-matching.

<u>example:</u> $$-\frac{d^2u}{dx^2} = 1 + 4x^2 \qquad u(0) = u(1) = 0 \qquad (1)$$

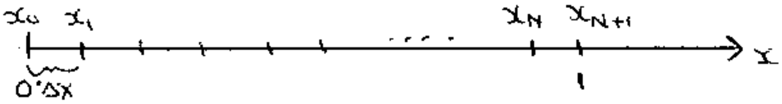
assume $$u \approx \sum_{n=1}^{N} \alpha_n u_n \qquad u_n \equiv x - x^{n+1} \quad (\text{basis}) \qquad (2)$$

$$Lu = \sum_{n=1}^{N} \alpha_n \left(-\frac{d^2}{dx^2}\left(x - x^{n+1}\right)\right) = 1 + 4x^2$$

$$\sum_{n=1}^{N} \alpha_n (n+1) n \, x^{n-1} = 1 + 4x^2 \qquad (3)$$

this represents 1 equation in N unknowns.

we now "match" this equation at N discrete points: (not on the boundary).



$$\Delta x = \frac{1}{N+1} \qquad x_m = \frac{1}{N+1} m$$

$$\therefore \quad \sum_{n=1}^{N} \alpha_n \left[n(n+1) \right] x_m^{n-1} = 1 + 4(x_m)^2 \qquad m = 1, 2, \ldots N$$

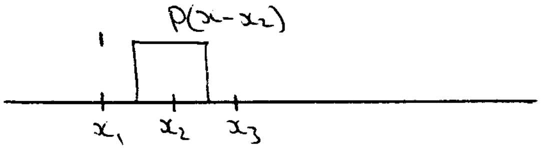
$$[l_{mn}][\alpha_n] = [g_m]$$

$$l_{mn} = n(n+1)\left(\frac{m}{N+1}\right)^{n-1} \qquad g_m = 1 + 4\left[\frac{m}{N+1}\right]^2$$

<u>exercise:</u> evaluate α_n for $n = 1, 2, 3$ and compare to the Galerkin method.

Method of Subsections

a) use "pulse" fcns for basis fcns.



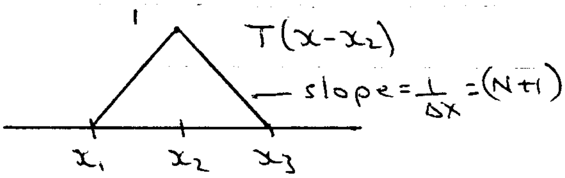
$$\Delta x = \frac{1}{N+1} \qquad x_n = \frac{n}{N+1}$$

$$P_n = P(x-x_n) = \begin{cases} 1 & |x-x_n| \le \frac{1}{2(N+1)} = \frac{\Delta x}{2} \\ 0 & |x-x_n| \ge \frac{1}{2(N+1)} = \frac{\Delta x}{2} \end{cases} \quad (1)$$

expanding:

$$u(x) = \sum_{n=1}^{N} \alpha_n P(x-x_n) \quad (2)$$

b) "Triangle" fcns for basis $\Rightarrow$ piecewise linear approximation

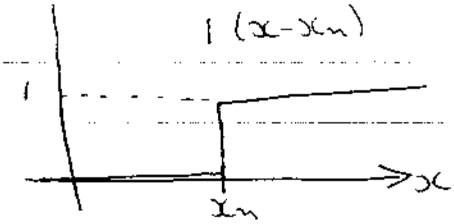


$$T_n = T(x-x_n) = \begin{cases} 1 - |x-x_n|(N+1) & |x-x_n| < \frac{1}{N+1} = \Delta x \\ 0 & |x-x_n| > \frac{1}{N+1} = \Delta x \end{cases} \quad (3)$$

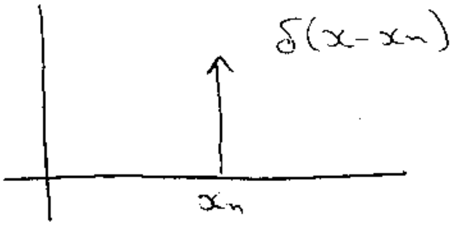
$$u(x) = \sum_{n=1}^{N} \alpha_n T(x-x_n) \quad (4)$$

if we are using the expansions (2) or (4) to approximate a function satisfying a differential equation then we will have to determine derivatives of P_n and T_n.

consider the unit step fcn:

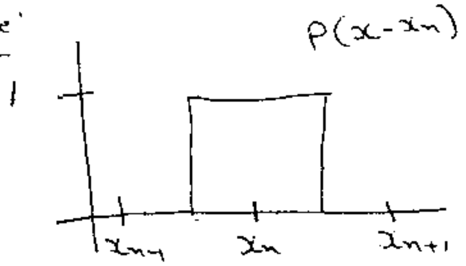


$$1(x-x_n) = \begin{cases} 1 & x \geq x_n \\ 0 & x \leq x_n \end{cases}$$

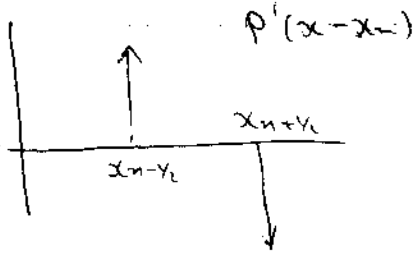


$$1'(x-x_n) = \delta(x-x_n) = \begin{cases} \infty & x = x \\ 0 & x \neq \end{cases}$$

For pulse:



$$P_n = P(x-x_n) = 1(x-x_{n-1/2}) - 1(x-x_{n}$$



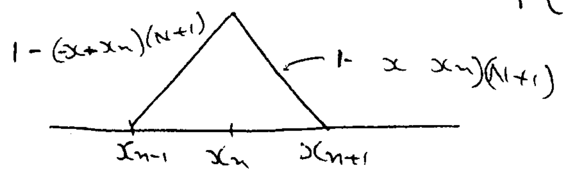
$$\frac{d}{dx} P(x-x_n) = \frac{d}{dx}\left[1(x-x_{n-1/2}) - 1(x-x_{n}\right.$$

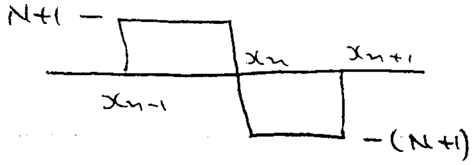
$$P'(x-x_n) = \delta(x-x_{n-1/2}) - \delta(x-x_{n+1/2})$$

Note: second derivative doesn't exist

Triangle fcn:

$$T(x-x_n) = \begin{cases} 1 - |x-x_n|(N+1) & |x-x_n| < \frac{1}{N+1} = \Delta x \\ 0 & |x-x_n| > \frac{1}{N+1} = \Delta x \end{cases}$$





$$T'(x-x_n) = (N+1)\left[1(x-x_{n-1}) - 1(x-x_n)\right]$$
$$-(N+1)\left[1(x-x_n) - 1(x-x_{n}\right.$$
$$= (N+1)\left[1(x-x_{n-1}) - 2\cdot 1(x-x_n) + 1(x-x_{n+}\right.$$

$$\frac{d^2T(x-x_n)}{dx} = (N+1)\left[\delta(x-x_{n-1}) - 2\delta(x-x_n) + \delta(x-x_{n+1})\right]$$

__example:__ $$-\frac{d^2u}{dx^2} = 1+4x^2 \qquad u(0)=u(1)=0$$

$$u = \sum_{n=1}^{N} \alpha_n T(x-x_n)$$

__Note:__ boundary conditions are automatically satisfied.

$$-\frac{d^2u}{dx^2} = \sum_{n=1}^{N} \alpha_n\left(-\frac{d^2}{dx^2}T(x-x_n)\right)$$
$$= \sum_{n=1}^{N} \alpha_n\left[-(N+1)\left[\delta(x-x_{n-1}) - 2\delta(x-x_n) + \delta(x-x_{n+1})\right]\right]$$

$$\sum_{n=1}^{N} \alpha_n (N+1)\left[-\delta(x-x_{n-1}) + 2\delta(x-x_n) - \delta(x-x_{n+1})\right] = 1+4x^2$$

we can now choose P_n as test fcns (or weighting Fcns)

$$\therefore \sum_{n=1}^{N} \alpha_n (N+1) \int_0^1 P(x-x_m)\left[-\delta(x-x_{n-1}) + 2\delta(x-x_n) - \delta(x-x_{n+1})\right] dx$$

$$= \int_0^1 P(x-x_m)(1+4x^2)\, dx$$

$$\therefore \quad l_{mn} = \begin{cases} 2(N+1) & \text{for } m=n \\ -(N+1) & \text{for } |m-n|=1 \\ 0 & \text{for } |m-n|>1 \end{cases}$$

$$g_m = \int_{x_{m-1/2}}^{x_{m+1/2}} (1+4x^2)dx = \left[x + \frac{4}{3}x^3\right]_{x_{m-1/2}}^{x_{m+1/2}}$$

$$= \frac{1}{N+1}(m+1/2) + \frac{4}{3}\frac{(m+1/2)^3}{(N+1)^3} - \frac{1}{N+1}(m-1/2) - \frac{4}{3}\frac{(m-1/2)^3}{(N+1)^3}$$

$$= \frac{1}{N+1}\left[1 + \frac{4}{3}\frac{(m+1/2)^3 - (m-1/2)^3}{(N+1)^2}\right]$$

$$= \frac{1}{N+1}\left[1 + \frac{4m^2 + \frac{1}{3}}{(N+1)^2}\right]$$

exercise: check for $N=1, 2, 3$.