RADIATION PROBLEMS

($e^{j\omega t}$ time variation)

$$\left.\begin{aligned} \bar{\nabla}\times\bar{E} &= -j\omega\bar{B} \\ \bar{\nabla}\times\bar{H} &= \bar{J} + j\omega\epsilon\bar{E} \\ \bar{\nabla}\cdot\bar{B} &= 0 \\ \bar{\nabla}\cdot\bar{D} &= \rho \end{aligned}\right\} \quad \begin{aligned} &(1) \\ &(2) \\ &(3) \\ &(4) \end{aligned}$$

$\bar{J}$ and ρ are the source.

$\bar{\nabla}\cdot\bar{B} = 0 \quad \therefore$ let $\bar{B} = \bar{\nabla}\times\bar{A}$ (5)

$\bar{A}$ - magnetic vector potential

$$\bar{\nabla}\times\bar{E} = -j\omega(\bar{\nabla}\times\bar{A}) \rightarrow \bar{\nabla}\times(\bar{E} + j\omega\bar{A}) = 0 \qquad (6)$$

$$\bar{E} + j\omega\bar{A} = -\bar{\nabla}\phi \qquad \bar{E} = -j\omega\bar{A} - \bar{\nabla}\phi \qquad (7)$$

$$\bar{\nabla}\times\bar{H} = \bar{J} + j\omega\epsilon(-j\omega\bar{A} - \bar{\nabla}\phi) \qquad (8)$$

$$\bar{\nabla}\times\left(\frac{\bar{\nabla}\times\bar{A}}{\mu}\right) = \bar{J} + \omega^2\epsilon\bar{A} - j\omega\epsilon\bar{\nabla}\phi \qquad (9)$$

$$\bar{\nabla}(\bar{\nabla}\cdot\bar{A}) - \bar{\nabla}^2\bar{A} = \mu\bar{J} + \omega^2\mu\epsilon\bar{A} - j\omega\mu\epsilon\bar{\nabla}\phi \qquad (10)$$

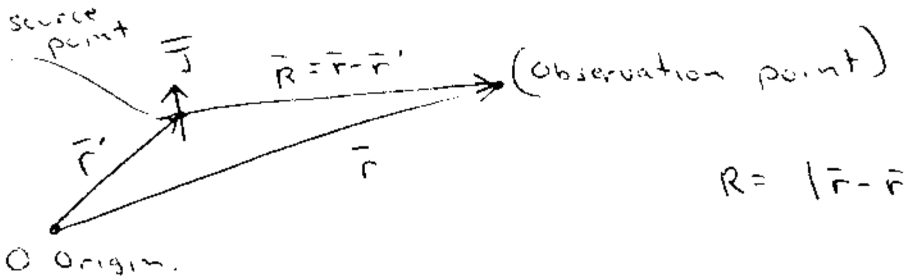
Lorentz condition: $\nabla(\bar{\nabla}\cdot\bar{A}) + j\omega\mu\epsilon\bar{\nabla}\phi = 0$ (11)

$$\therefore \quad \boxed{\bar{\nabla}^2\bar{A} + \omega^2\mu\epsilon\bar{A} = -\mu\bar{J}}$$ inhomogeneous equation (12)

$$\boxed{\bar{A} = \frac{\mu_0}{4\pi}\int_V \bar{J}\,\frac{e^{-jkR}}{R}\,dV}$$

(general solution) (13)

$k^2 = \omega^2\mu\epsilon$



$$R = |\bar{r} - \bar{r}'| \qquad (14)$$

if $\bar{\bar{J}}(\bar{r}') \;//\; z$-direction, $\bar{A} = \hat{z}\, A_z(x, y, z)$

(i.e. magnetic vector potential has only one component).

from (7)
$$E_z = -j\omega A_z - \frac{\partial \phi}{\partial z} \qquad (15)$$

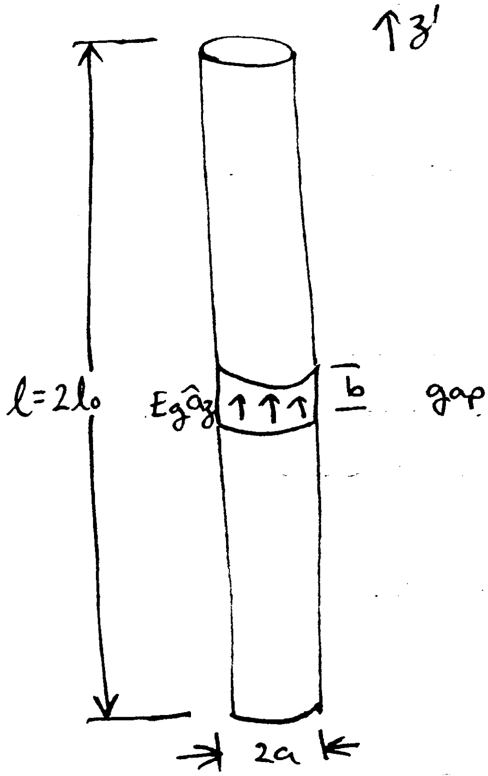
but from the Lorentz condition: (11)

$$\frac{\partial^2 A_z}{\partial z^2} + j\omega\mu\epsilon \frac{\partial \phi}{\partial z} = 0$$

$\therefore$ into (15):
$$E_z = -j\omega A_z + \frac{1}{j\omega\mu\epsilon} \frac{\partial^2 A_z}{\partial z^2} \qquad (16)$$

$$\boxed{E_z = \frac{1}{j\omega\mu\epsilon}\left[\frac{\partial^2 A_z}{\partial z^2} + k^2 A_z\right]} \qquad (17)$$

Antenna Impedance: Theoretical Considerations



- $I(z')$ is unknown

$I(0)$ – current at input (unknown)

$I(\pm l_0) = 0$, – current at ends.

– assume only z-directed current

tangential component of $\bar{E} = 0$ on the "perfect conductor"

we <u>impose</u> a gap electric field of $E_g \hat{a}_z$ $\quad (V_g = E_g b)$

$$\Rightarrow \quad E_T = 0 = E_z + E_g \qquad -\frac{b}{2} < z' < \frac{b}{2}$$

$$\therefore \quad E_z = -E_g$$

$$E_z = 0 \qquad \frac{b}{2} < |z'| < l_0$$

input impedance $= \quad Z_a = \dfrac{E_g b}{I(0)}$ $\quad \therefore$ need $I(0)$

$$A_z(x,y,z) = \frac{\mu_0}{4\pi} \int_{-l_0}^{l_0} \frac{e^{-jk_0R}}{R} I(z')\,dz'$$

observation point (x,y,z), (r,ϕ,z)

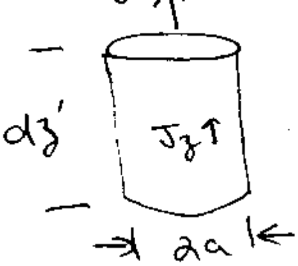
$r = (x^2+y^2)^{1/2}$

$$R = \left[r^2 + (z-z')^2\right]^{1/2}$$

$$\bar{E} = -j\omega \bar{A} + \frac{\bar{\nabla}\bar{\nabla}\cdot\bar{A}}{j\omega\varepsilon_0\mu_0} \Rightarrow E_z = -j\omega A_z + \frac{1}{j\omega\varepsilon_0\mu_0}\frac{\partial^2 A_z}{\partial z^2}$$

$$= \frac{1}{j\omega\varepsilon_0\mu_0}\left(\frac{\partial^2}{\partial z^2} + k_0^2\right) A_z$$

Note:



if perfectly conducting then $J_z = \dfrac{I}{2\pi a}$.

distance from a point on surface of conductor to observation point is

$$R = \left[(x-x')^2 + (y-y')^2 + (z-z')\right]^{1/2}$$
$$= \left[(x - a\cos\phi')^2 + (y - a\sin\phi')^2 + (z-z')\right]^{1/2}$$
$$= \left[x^2 + y^2 + a^2 - 2xa\cos\phi' - 2ya\sin\phi' + (z-z')\right]^{1/2}$$
$$= \left[r^2 + a^2 - 2xa\cos\phi' - 2ya\sin\phi' + (z-z')\right]^{1/2}$$

$$A_z = \frac{\mu_0}{4\pi} \int_{-l_0}^{l_0}\int_0^{2\pi} \frac{e^{-jk_0R}}{R} J_z \quad a\, d\phi'\, dz'$$

if the observation point is at the surface:

$$R = \left[a^2(\cos\phi - \cos\phi')^2 + a^2(\sin\phi - \sin\phi')^2 + (z-z')^2\right]^{1/2}$$
$$= \left[2a^2 - 2a^2\cos\phi\cos\phi' - 2a^2\sin\phi\sin\phi' + (z-z')^2\right]^{1/2}$$
$$= \left[2a^2\left(1 - \cos(\phi - \phi')\right) + (z-z')^2\right]^{1/2}$$

we assume that J_z is independent of ϕ', therefore if as ϕ' goes from 0 to 2π we take the average value of $1 - \cos(\phi - \phi') = \frac{1}{2}$

then we can use $R = \left[a^2 + (z-z')^2\right]^{1/2}$ and performing the ϕ' integration:

$$A_z = \frac{\mu_0}{4\pi} \int_{-l_0}^{l_0} \frac{e^{-jk_0R}}{R} I(z')\, dz'$$

$$\therefore \quad E_z(x,y,z) = \frac{1}{j\omega\varepsilon_o\mu_o}\left(\frac{\partial^2}{\partial z^2} + k_o^2\right)\frac{\mu_o}{4\pi}\int_{-l_o}^{l_o}\frac{e^{-jk_oR}}{R} I(z')\,dz'$$

on the surface of the antenna $\quad r^2 = a^2 = x^2 + y^2$

$$\begin{cases} \left(\dfrac{\partial^2}{\partial z^2} + k_o^2\right)\displaystyle\int_{-l_o}^{l_o} I(z')\frac{e^{-jk_oR}}{4\pi R}\,dz' = \begin{cases} -j\omega\varepsilon_o E_g & -\frac{b}{2} < z < \frac{b}{2} \\ 0 & \frac{b}{2} < |z| < l_o \end{cases} \\ R = \left[a^2 + (z-z')^2\right]^{1/2} \end{cases}$$

Pocklington's Integral Equation (1897)

- Numerical technique is required to solve for $I(z')$

we have: $$\left(\frac{d^2}{dz^2} + k_o^2\right) A_z = \begin{cases} -j\omega\varepsilon_o\mu_o E_g & -\frac{b}{2} < z < \frac{b}{2} \\ 0 & \frac{b}{2} < |z| < l \end{cases}$$

as the gap goes to zero, we can describe

$$E_g = V_g\,\delta(z) \qquad \begin{cases} \displaystyle\int_{-z}^{z}\delta(z')\,dz' = 1 \\ \delta(z') = 0,\ z' \neq 0 \end{cases}$$

Dirac delta Fcn

$\therefore$ we have:

$$\left(\frac{d^2}{dz^2} + k_0^2 \right) A_z = -j\omega\varepsilon_0\mu_0 V_g \,\delta(z)$$

For this equation to have a solution, A_z must be differentiable $\Rightarrow$ A_z must be continuous.

Since the second derivative produces a singularity, we cannot assume that the first derivative is continuous! In fact, integrating this equation from $z = 0-\epsilon$ to $z = 0+\epsilon$:

$$\int_{0-\epsilon}^{0+\epsilon} \left(\frac{d^2}{dz^2} + k_0^2 \right) A_z \, dz = \left. \frac{dA_z}{dz} \right|_{0-\epsilon}^{0+\epsilon} + \int_{0-\epsilon}^{0+\epsilon} k_0^2 A_z \, dz = -j\omega\varepsilon_0\mu_0 V_g$$

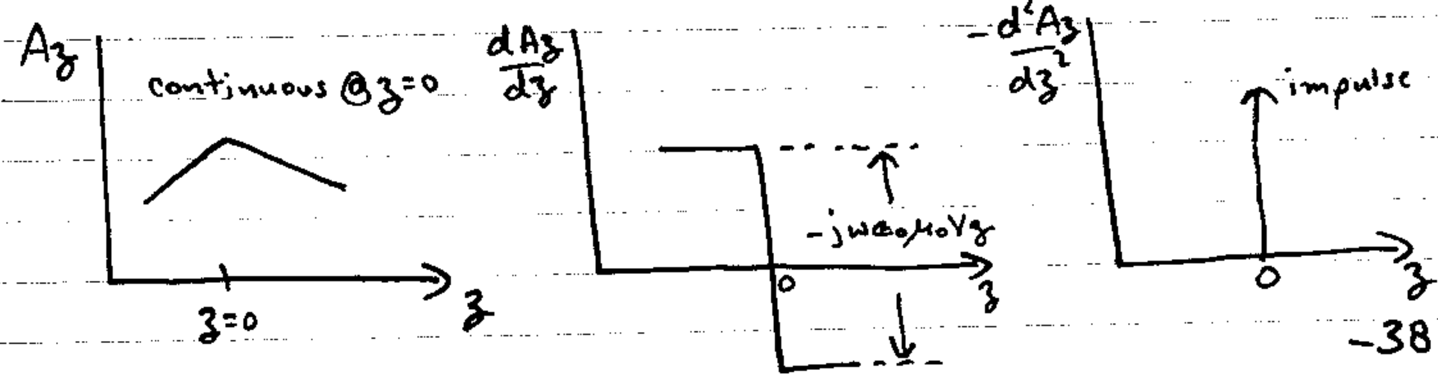
Since A_z must be bounded (otherwise the magnetic field will be unbounded), we have

$$\int_{0-\epsilon}^{0+\epsilon} k_0^2 A_z \, dz \Rightarrow 0 \quad \text{as} \quad \epsilon \Rightarrow 0$$

$\therefore$ we have a step-discontinuity in the first derivative:

$$\left. \frac{dA_z}{dz} \right|_{0-\epsilon}^{0+\epsilon} = -j\omega\varepsilon_0\mu_0 V_g$$

at any other location dA_z/dz is continuous.



the differential equation to solve in the two regions $0 < |z| < l_0$ is

$$\frac{d^2}{dz^2} A_z + k_0 A_z = 0 \qquad z \neq 0$$

general solution is
$$A_z = C_1 \cos k_0 z + C_2 \sin k_0 z \qquad z<0$$
$$= C_3 \cos k_0 z + C_4 \sin k_0 z \qquad z>0$$

continuity at $z=0 \Rightarrow C_1 = C_3$

$$\left.\frac{dA_z}{dz}\right|_{0-}^{0+} = k_0 C_4 - k_0 C_2 = -j\omega\varepsilon_0\mu_0 V_g.$$

A_z must be even fcn of z (due to symmetry).
$\Rightarrow \quad C_2 = -C_4$

$$\therefore \quad C_2 = \frac{j\omega\varepsilon_0\mu_0}{2k_0} V_g = \frac{j}{2} Y_0 \mu_0 V_g.$$

$$\therefore \quad A_z = -\frac{j}{2} Y_0 \mu_0 V_g \sin k_0 |z| + C_1 \cos k_0 z.$$

$$\therefore \left\{ \begin{array}{l} \dfrac{1}{4\pi} \displaystyle\int_{-l_0}^{l_0} I(z') \frac{e^{-jk_0R}}{R} dz' = -\frac{j}{2} Y_0 V_g \sin k_0 |z| + C \cos k_0 z \\ R = [a^2 + (z-z')^2]^{1/2} \qquad C = \dfrac{C_1}{\mu_0} \end{array} \right.$$

Hallén's Integral Equation (1938)

C to be found such that $I(z) = 0$ @ $z = \pm l_0$

Method of Moments

- a very general numerical technique for solving any operator equation of the form $L u(z) = f(z)$.

- the operator we're interested in is an integral equation, specifically Fredholm equation of the first kind:

$$\int_0^1 G(u,u')\, I(u')\, du' = f(u)$$

- $G(u,u')$ - known kernal, Green's function
- $f(u)$ - known function
- $I(u)$ - unknown function to be determined.

Step 1:

we expand $I(u)$ in terms of N basis functions:

$$I(u') = \sum_{n=1}^{N} I_n \Phi_n(u')$$

- I_n coefficients to be determined
- $\Phi_n(u)$ are the basis functions.

Step 2: substitute the expansion into the operator equation:

$$\sum_{n=1}^{N} I_n \int_0^1 G(u,u')\, \Phi_n(u')\, du' = f(u) = \sum_{n=1}^{N} I_n G_n(u)$$

$$G_n(u) = \int_0^1 G(u,u')\,\Phi_n(u')\,du'$$

"moments" of G(u,u') with respect to Φn(u')

so far we have 1 equation : $F(u) = \sum_{n=1}^{N} I_n G_n(u)$

and N unknowns: I_n , therefore we need to obtain N equations.

<u>Step 3:</u> Obtain N equations by "weighting" $F(u)=\sum_{n=1}^{N} I_n G_n$ by N weighting function ψ_m , $m=1,\dots,N$.

$$\int_0^1 F(u)\,\psi_m(u)\,du = \sum_{n=1}^{N} I_n \int_0^1 \psi_m(u)\,G_n(u)\,du$$

$\therefore$ if we set $G_{mn} = \int_0^1 \psi_m(u)\,G_n(u)\,du = \int_0^1\int_0^1 \psi_m(u)\,G(u,u')\,\Phi(u')\,du'du$

$$f_m = \int_0^1 F(u)\,\psi_m(u)\,du$$

we have the matrix equation:

$$\begin{bmatrix} G_{11} & \cdots & G_{1n} & \cdots & G_{1N} \\ \vdots & & \vdots & & \\ G_{m1} & \cdots & G_{mn} & \cdots & G_{mN} \\ \vdots & & \vdots & & \\ G_{N1} & \cdots & G_{Nn} & \cdots & G_{NN} \end{bmatrix} \begin{bmatrix} I_1 \\ \vdots \\ I_m \\ \vdots \\ I_N \end{bmatrix} = \begin{bmatrix} f_1 \\ \vdots \\ f_m \\ \vdots \\ f_N \end{bmatrix}$$

$$[G]\,\underline{I} = \underline{f}$$

which can be solved for $\underline{I}$

- if the $\psi_m(u)$ are chosen as $\delta(u-mh)$ the method is called <u>point-matching method.</u>

- if ψ_m are chosen the same as Φ_n it is called <u>Galerkin's method</u>.

Method of Least-Squares

- we start with the equation $f(u) = \sum_{n=1}^{N} I_n G_n(u)$

and try to minimize the norm of the Residual:

$$\min_{I_n} \left\| \sum_{n=1}^{N} I_n G_n(u) - f(u) \right\|^2 = \int_0^1 \underbrace{\left| \sum_{n=1}^{N} I_n G_n(u) - f(u) \right|^2}_{R^2} du$$

where $G_n(u) = \int_0^1 G(u,u')\, \Phi_n(u')\, du'$

this can be written as (minus the integral).

$$\left(\sum_{n=1}^{N} I_n G_n(u) - f(u) \right)\left(\sum_{n=1}^{N} I_n^* G_n^*(u) - f^*(u) \right) = F(I_{nr}, I_n$$

we want to minimize this with respect to the

currents $I_n = I_{nr} + j I_{ni}$, $I_n^* = I_{nr} - j I_{ni}$

$$\frac{\partial}{\partial I_{nr}} = \frac{\partial}{\partial I_n} \frac{\partial I_n}{\partial I_{nr}} = \frac{\partial}{\partial I_n} = \frac{\partial}{\partial I_n^*} \qquad \frac{\partial}{\partial I_{ri}} = \frac{\partial}{\partial I_n} \frac{\partial I_n}{\partial I_{ni}} = j \frac{\partial}{\partial I_n} = -j \frac{\partial}{\partial I_n^*}$$

$\therefore$ the stationary point is found by setting all the $\frac{\partial F}{\partial I_n^*} = 0$

$$\frac{\partial F}{\partial I_m^*} = 0 \Rightarrow \int_0^1 G_m^*(u) \left(\sum_{n=1}^{N} G_n(u) I_n - F(u) \right) du = 0$$

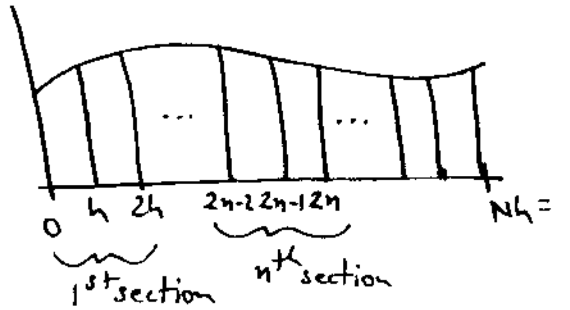
this implies that minimizing the R^2 is the same as using $G_m^*(u)$ as the weighting function.

$\therefore$ we have:

$$\left\{ \begin{array}{l} \displaystyle\sum_{n=1}^{N} G_{mn} I_n = F_m \\ \\ \displaystyle G_{mn} = \int_0^1 G_m^*(u)\, G_n(u)\, du \qquad F_m = \int_0^1 G_m^*(u)\, f(u)\, du \\ \displaystyle G_n(u) = \int_0^1 G(u,u')\, \Phi_n(u')\, du' \end{array} \right.$$

Numerical Integration (Simpson's Rule)

consider $\int_0^1 g(u)\,du$



we divide $0 \le u \le 1$ interval into an even number, N, of subdivisions of length h. so that $Nh=1$

over each subdivision of length $2h$ we approximate $g(u)$ by $A_1 + B_1 u + C_1 u^2$:

$$g_0 = g(0) = A_1$$
$$g_1 = g(h) = A_1 + B_1 h + C_1 h^2$$
$$g_2 = g(2h) = A_1 + B_1 2h + C_1 4h^2$$

let $g(0) = g_0$
$g(h) = g_1$
$g(2h) = g_2$
$\vdots$
$g(nh) = g_n$

$$\boxed{A_1 = g_0}$$

$$g_1 - g_0 = B_1 h + C_1 h^2$$
$$- \quad \frac{g_2 - g_0}{2} = B_1 h + C_1 2h^2$$

$$g_1 - \frac{g_2}{2} - \frac{g_0}{2} = -C_1 h^2 \quad \Rightarrow \quad \boxed{C_1 = \frac{g_0 + g_2 - 2g_1}{2h^2}}$$

$$g_1 - g_0 = B_1 h + \frac{g_0 + g_2 - 2g_1}{2} \quad \Rightarrow \quad \boxed{B_1 = \frac{4g_1 - 3g_0 - g_2}{2h}}$$

For the n^{th} subsection: $A_n + B_n[u - 2(n-1)h] + C_n[u - 2(n-1)h]^2$

with $\boxed{A_n = g_{2n-2}}$ $\boxed{B_n = \frac{4g_{2n-1} - 3g_{2n-2} - g_{2n}}{2h}}$

$$\boxed{C_n = \frac{g_{2n-2} + g_{2n} - 2g_{2n-1}}{2h^2}}$$

$$\int_0^{2h} g(u)\,du \simeq A_1 2h + \frac{1}{2} B_1 (2h)^2 + \frac{1}{3} C_1 (2h)^3$$

$$= \frac{h}{3}\left(g_0 + 4 g_1 + g_2\right)$$

$$\int_{2n-2}^{2n} g(u)\,du \simeq \frac{h}{3}\left(g_{2n-2} + 4 g_{2n-1} + g_{2n}\right)$$

$$\boxed{\int_0^1 g(u)\,du \simeq \frac{h}{3} \sum_{n=0}^{N} S_n g_n \qquad S_n = 1,4,2,4,2,4 \ldots 4,1}$$

exact for any cubic polynomial!

Simpson's Rule

Dipole Impedance: Numerical Solution

Consider Hallén's integral equation:

$$\int_{-l_0}^{l_0} \frac{e^{-jk_0R}}{R} I(z')\,dz' = -j2\pi Y_0 V_g \sin k_0|z| + 4\pi C \cos k_0 z$$

$$R = [(z-z')^2 + a^2]^{1/2}$$

now introduce the following normalized variables:

$$u = z/l_0 \qquad u' = z'/l_0 \qquad k_0 l_0 = \Theta \qquad a/l_0 = \alpha$$

$\Rightarrow$ integral goes from $\quad -1 < u' < 1 \qquad dz' = l_0\,du'$

$$R = l_0\,[(u-u')^2 + \alpha^2]^{1/2}$$

$$\therefore \quad \int_{-1}^{1} \frac{e^{-jk_0R}}{[(u-u')^2+\alpha^2]^{1/2}} I(u')\,du' = -j2\pi Y_0 V_g \sin\Theta|u| + 4\pi C \cos\Theta u$$

by symmetry the current $I(u')$ is an even fcn: $I(u') = I(-u$

$$\int_{-1}^{1} \frac{e^{-jk_0R}}{[(u-u')^2+\alpha^2]^{1/2}} I(u')\,du' = \int_{0}^{1} \frac{e^{-jk_0R}}{[(u-u')^2+\alpha^2]^{1/2}} I(u')\,du'$$

$$+ \int_{0}^{1} \frac{e^{-jk_0l_0[(u+u')+\alpha^2]^{1/2}}}{[(u+u')^2+\alpha^2]^{1/2}} I(u')\,du'$$

let $R_1 = [(u-u')^2 + \alpha^2]^{1/2}$, $R_2 = [(u+u')^2+\alpha^2]^{1/2}$

$$\therefore \int_0^1 \left(\frac{\cos\theta R_1 - j\sin\theta R_1}{R_1} + \frac{\cos\theta R_2 - j\sin\theta R_2}{R_2} \right) I(u')\,du'$$

$$= -j2\pi Y_0 V_g \sin\theta u + 4\pi C\cos\theta u$$

$$0 \le u \le 1$$

Note : α is a small number and at $u=u'$ or $u=-u'$

$R_1 = \alpha$ $\quad R_2 = \alpha$ $\quad \frac{\sin\theta\alpha}{\alpha} \approx \alpha$ but $\frac{\cos\theta\alpha}{\alpha}$ is large

$\therefore$ we want to "integrate out" the singularity analytically in the $\frac{\cos\theta R_1}{R_1}$ and $\frac{\cos\theta R_2}{R_2}$ terms.

$$\int_0^1 \left[\frac{\cos\theta R_1}{R_1} I(u') - \frac{\cos\theta\alpha}{R_1} I(u) \right] du'$$

$$+ \int_0^1 \left[\frac{\cos\theta R_2}{R_2} I(u') - \frac{\cos\theta\alpha}{R_2} I(u) \right] du'$$

$$+ \quad I(u)\cos\theta\alpha \underbrace{\int_0^1 \left[\frac{1}{R_1} + \frac{1}{R_2} \right] du'}$$

this is the singular part which we have to do analytically.

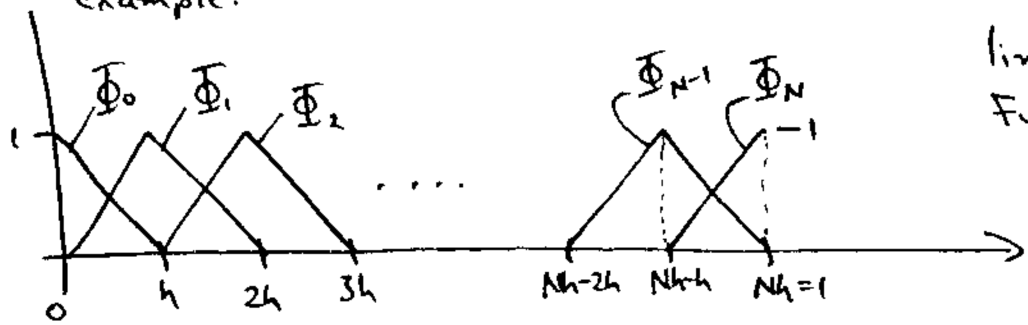
$$\int_0^1 \left(\frac{1}{R_1} + \frac{1}{R_2} \right) du' = \ln \frac{\left[1-u+\sqrt{\alpha^2+(1-u)^2}\right]\left[1+u+\sqrt{\alpha^2+(1+u)^2}\right]}{\alpha^2}$$

the integral equation is now of the form:

$$\int_0^1 G(u,u')\, I(u')\, du' = F(u)$$

consider any basis functions, $\Phi_n(u')$, which have support $2h$ and are equal to 1 at nh, that is $\Phi_n(nh)=1$, $\Phi_n(nh \pm h)=0$:

example:



linear basis functions.

$$\therefore\ I(u') = \sum_{n=0}^{N} I_n \Phi_n(u')$$

expansion in terms of the "sub-domain" basis functions.

putting this into the integral equation we have:

$$\int_0^1 G(u,u')\, I(u')\, du' = \int_0^1 G(u,u') \sum_{n=0}^{N} I_n \Phi_n(u')\, du'$$

$$= \sum_{n=0}^{N} I_n \int_0^1 G(u,u')\, \Phi_n(u')\, du' = F(u)$$

using Simpson's rule to evaluate the integral and using J divisions of the interval $0 \le u' \le 1$, $H = \frac{1}{J}$

we get:

$$\sum_{n=0}^{N} I_n \frac{H}{3} \sum_{j=0}^{J} G_j(u)\, S_j\, \Phi_n(jH) = F(u)$$

where $G_j(u) = G(u, jH)$, S_j are Simpson's coefficients.

denote: $$\psi_m(u) = \frac{H}{3} \sum_{j=0}^{J} G_j(u)\, S_j\, \Phi_m(jH)$$

For least square error, we should us $\psi_m^*(u)$

as testing (weighting) functions.

testing: $$\int_0^1 \psi_m^*(u) \sum_{n=0}^{N} I_n \psi_n(u)\, du = \int_0^1 \psi_m^*(u) F(u)\, du$$

again using Simpson's Rule for this integral, using.
$J'H' = 1$ (J' subdivisions of $0 \le u \le 1$ interval).

$$\sum_{i=0}^{J'} S_i\, \psi_m^*(iH') \left[\sum_{n=0}^{N} I_n \psi_n(iH') - F(iH') \right] = 0 \quad m = 0, 1, 2, \dots N$$

let $$[\Psi] = \begin{bmatrix} \psi_1(0) & \cdots & \psi_n(0) & \cdots & \psi_N(0) \\ \vdots & & & & \\ \psi_1(J'H') & \cdots & \psi_n(J'H') & \cdots & \psi_N(J'H') \end{bmatrix} \in \mathbb{C}^{J' \times N}$$

$$\underline{\underline{I}} = \begin{bmatrix} I_0 \\ \vdots \\ I_N \end{bmatrix} \qquad \underline{F} = \begin{bmatrix} F(0) \\ \vdots \\ F(J'H') \end{bmatrix}$$

$$[\Psi^*] = \begin{bmatrix} S_0\psi_1^*(0) \cdots & S_i\psi_1^*(iH') & \cdots S_{J'}\psi_1^*(J'H') \\ \vdots & & \\ S_0\psi_N^*(0) \cdots & S_i\psi_N^*(iH') & \cdots S_{J'}\psi_N^*(J'H') \end{bmatrix} \in \mathbb{C}^{N\times J'}$$

$\therefore$ the above equation can be written as:

$$[\Psi^*]\Big[[\Psi]\underline{I} - \underline{F}\Big] = 0$$

if $J' = N$ (i.e. choose N subdivisions for testing integral)
$(H' = h)$
$= \frac{1}{N}$

then $[\Psi^*] \in \mathbb{C}^{N\times N}$ $[\Psi] \in \mathbb{C}^{N\times N}$

and the solution to the above equation is just:

$$[\Psi]\underline{I} = \underline{F}$$

i.e. $\boxed{\sum_{n=0}^{N} I_n \psi_n(ih) = F(ih) \quad i = 0,1,2,\ldots N.}$

this is the same as "point matching", $\therefore$ using Simpson's rule for the weighting and usin N subdivisions makes the testing procedure independent of the testing Functions.

we thus have:

$$\sum_{n=0}^{N} I_n \frac{H}{3} \sum_{j=0}^{J} G_j(ih)\, S_j\, \phi_n(jH) = F(ih) \qquad i = 0, 1, 2, \ldots N$$

if we now choose $J = N$, i.e. $H = h$,
since $\phi_n(nh) = 1$, $\phi_n(nh \pm zh) = 0$ z any other integer other than 0.

we get:

$$\begin{cases} \dfrac{h}{3} \displaystyle\sum_{n=0}^{N} G_{mn} S_n I_n = F_m \qquad m = 0, 1, 2, \ldots N \\ G_{mn} = G(mh, nh) \qquad F_m = F(mh) \end{cases}$$

we could have got this by just evaluating the original integral equation using Simpsons rule and then using point matching.

See pages 38-67 of

R.E. Collin, Antennas and Radiowave Propagation, McGraw-Hill, 1985.

introducing a large error (see Probs. 2.21 and 2.22). Thus we will use Eq. (2.93) in spite of its limitations, since it is a numerically very simple and efficient procedure. The results obtained compare very favorably with those based on alternative numerical procedures or approximate analytical solutions. The attractive feature of this method is that the matrix elements are known—they are simply the values of the kernel function at the sample points multiplied by the Simpson weights. The method appears to converge quite rapidly with increasing N, and this is due to the fact that the dominant part of the kernel was extracted and integrated exactly.

The algebraic equations that will determine the current on the dipole antenna are obtained by using the final expression (2.93) derived above, after substituting Eq. (2.91) into Eq. (2.90) and adding the remaining terms from Eq. (2.89). It is found that the system of equations to be solved is

$$\frac{h}{3}\sum_{n=0}^{N} S_n\Bigg[\frac{(\cos\theta\sqrt{(n-m)^2h^2+\alpha^2}-j\sin\theta\sqrt{(n-m)^2h^2+\alpha^2})I_n - I_m\cos\theta\alpha}{\sqrt{(n-m)^2h^2+\alpha^2}}$$
$$+\frac{(\cos\theta\sqrt{(n+m)^2h^2+\alpha^2}-j\sin\theta\sqrt{(n+m)^2h^2+\alpha^2})I_n - I_m\cos\theta\alpha}{\sqrt{(n+m)^2h^2+\alpha^2}}\Bigg]$$
$$+ I_m\cos\theta\alpha\ln\frac{[1-mh+\sqrt{\alpha^2+(1-mh)^2}][1+mh+\sqrt{\alpha^2+(1+mh)^2}]}{\alpha^2}$$
$$= -j2\pi Y_0V_g\sin\theta mh + 4\pi C\cos\theta mh \qquad m=0,1,2,\ldots,N \tag{2.94}$$

Note that $I(l_0)=I_N$ must be equal to zero. The constant C may be found from the equation obtained for $m=0$. After the constant C has been eliminated, the equations that result can be expressed in the following matrix form:

$$\begin{bmatrix} R_{10} & R_{11} & R_{12}\cdots R_{1,N-1}\\ R_{20} & R_{21} & R_{22}\cdots R_{2,N-1}\\ \cdots & \cdots & \cdots\\ R_{N0} & R_{N1} & R_{N2}\cdots R_{N,N-1}\end{bmatrix}\begin{bmatrix} I_0\\ I_1\\ \vdots\\ I_{N-1}\end{bmatrix} = -j\frac{6\pi}{h}Y_0V_g\begin{bmatrix}\sin\theta h\\ \sin 2\theta h\\ \vdots\\ \sin N\theta h\end{bmatrix} \tag{2.95}$$

where the matrix elements R_{mn} are given by

$$R_{mn} = S_n\Bigg\{\frac{\cos\theta\sqrt{\alpha^2+(n-m)^2h^2}}{\sqrt{\alpha^2+(n-m)^2h^2}}+\frac{\cos\theta\sqrt{\alpha^2+(n+m)^2h^2}}{\sqrt{\alpha^2+(n+m)^2h^2}}$$
$$-2\cos\theta mh\frac{\cos\theta\sqrt{\alpha^2+n^2h^2}}{\sqrt{\alpha^2+n^2h^2}}\Bigg]$$
$$+\delta_{mn}\cos\theta\alpha\Bigg\{\frac{3}{h}\ln\frac{[1-nh+\sqrt{\alpha^2+(1-nh)^2}][1+nh+\sqrt{\alpha^2+(1+nh)^2}]}{\alpha^2}$$

$$-\sum_{j=0}^{N} S_j\Bigg[\frac{1}{\sqrt{\alpha^2+(j+n)^2h^2}}+\frac{1}{\sqrt{\alpha^2+(j-n)^2h^2}}\Bigg]\Bigg\}$$
$$-\delta_{0n}\cos\theta\alpha\cos\theta mh\Bigg[\frac{6}{h}\ln\frac{1+\sqrt{1+\alpha^2}}{\alpha}-\sum_{j=0}^{N}\frac{2S_j}{\sqrt{\alpha^2+(jh)^2}}\Bigg]$$
$$-jS_n\Bigg[\frac{\sin\theta\sqrt{\alpha^2+(n-m)^2h^2}}{\sqrt{\alpha^2+(n-m)^2h^2}}+\frac{\sin\theta\sqrt{\alpha^2+(n+m)^2h^2}}{\sqrt{\alpha^2+(n+m)^2h^2}}$$
$$-2\cos\theta mh\frac{\sin\theta\sqrt{\alpha^2+(nh)^2}}{\sqrt{\alpha^2+(nh)^2}}\Bigg] \tag{2.96}$$

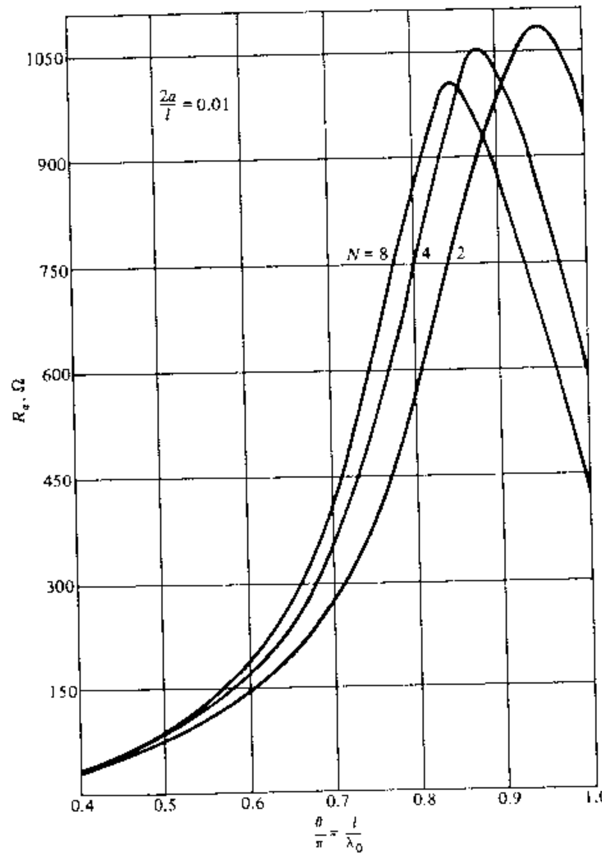


Figure 2.20 Computed value of radiation resistance for a dipole antenna using different values of N, $2a/l = 0.01$.

with $\delta_{mn} = 0$ for $n \neq m$ and $\delta_{nn} = 1$. The input impedance $Z_a = R_a + jX_a$ is given by V_g/I_0.

The simplest approximation that can be made is to choose $N = 2$, which corresponds to approximating the current on the antenna by a polynomial of degree 2. The numerical results for R_a and X_a with this approximation are shown in Figs. 2.20 and 2.21 for the case $a/l_0 = 0.01$. For comparison, the more accurate results obtained by using $N = 4$ and $N = 8$ are also shown.† The results for $N = 2$ predict the general behavior of Z_a and its dependence on the antenna radius quite well, with the exception that the curves are displaced to

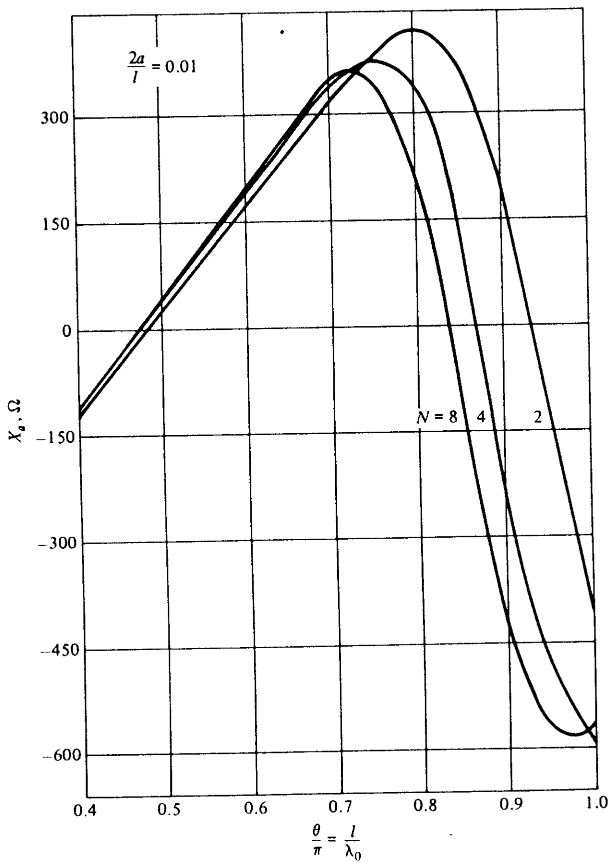


Figure 2.21 Computed value of input reactance for a dipole antenna using different values of N, $2a/l = 0.01$.

† The author is indebted to John Silvestro for the numerical computations.

the right relative to the more accurate results obtained using $N = 8$. In Figs. 2.22 and 2.23 results for R_a and X_a using $N = 2$, 8, and 12 are given for $\alpha = 0.001$. Figures 2.24 and 2.25 give corresponding results for $\alpha = 0.05$. These curves agree reasonably well with the measured values given in Figs. 2.13 and 2.14. In Figs. 2.26 and 2.27 the computed values of R_a and X_a using $N = 8$ and 12 for $\alpha = 0.0135$ are compared with the results of the King-Middleton improved second-order theory.† The King-Middleton theory has been shown to agree very well with measured data obtained by Mack‡. Since the numerical

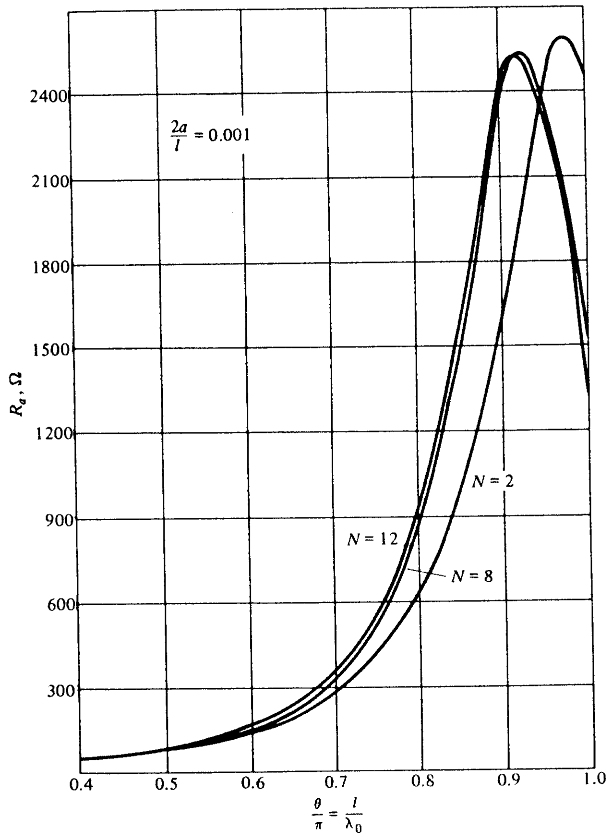


Figure 2.22 Computed value of radiation resistance for a dipole antenna with $2a/l = 0.001$.

† R. W. P. King, "Cylindrical Antennas and Arrays," Chap. 9 in R. E. Collin and F. J. Zucker (eds.), *Antenna Theory*, Pt. I, McGraw-Hill Book Company, New York, 1969.

‡ R. B. Mack, *A Study of Circular Arrays*, Cruft Laboratory Tech. Rept. Nos. 381–386, Harvard University, Cambridge, Mass., 1963.