TWO QUESTIONS ABOUT PROPERTIES OF LARGE GRAPHS: ON GENERALIZED TURÁN NUMBERS AND THE CHROMATIC NUMBER OF RANDOM LIFTS

by

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TWO QUESTIONS ABOUT PROPERTIES OF LARGE GRAPHS: ON GENERALIZED TURÁN NUMBERS AND THE CHROMATIC NUMBER OF RANDOM LIFTS

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In this thesis, I consider questions and results from two fields: extremal graph theory and random graph theory.

First, I consider a generalization of the classic theorem of Turán that states that the maximum number of edges in a K_{r+1}-free graph is uniquely obtained by a balanced complete r-partite graph. Following Alon and Shikhelman, instead of counting edges while forbidding K_{r+1}, I count copies of a target graph T while forbidding copies of F. I investigate the behavior of the extremal graphs and their graphon limits in the case where T and F are stars or cliques and provide partial results in more general cases. In particular, I prove the following:

Theorem. *For any integers $r \geq 2$ and $t \geq 2$ and sufficiently large n, any S_t-extremal K_{r+1}-free graph on n vertices is complete r-partite. The complete r-partite graphon that maximizes S_t density is either the balanced r-partite Turán graphon or has $r-1$ parts of size α and one of size β where α is the maximum of a specified function.*

Furthermore, there is a function $t^* = t^*(r)$, which satisfies $r < t^* \leq r + \sqrt{2r}$, such that when $t < t^*$ the optimal graphon is the Turán graphon and when $t \geq t^*$ it is of the imbalanced form.

Second, using the small subgraph conditioning method and several other techniques, I obtain an upper bound for the chromatic number of random lifts of a general

d-regular graph. Specifically, I prove

Theorem. *Let k be an integer and define*

$$\ell_k = \frac{2(k-1)^3}{k(k-2)} \log(k-1).$$

If $d < \ell_k$ and G is a d-regular graph, then asymptotically almost surely a random n-lift of H is k-colorable.

In the case where the host graph G is complete, I will also show the following:

Theorem. *Let $k \geq 2$ and define*

$$u_k = \frac{2 \log k}{\log k - \log(k-1)} < (2k-1) \log k.$$

For any $d \geq u_k$, asymptotically almost surely a random n-lift of K_{d+1} is not k-colorable.

Together, these results prove that asymptotically almost surely the chromatic number of a random n-lift of K_{d+1} takes one of at most two values. This result complements results on the chromatic number of regular graphs chosen uniformly with the configuration model.

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On the Generalized Turán Theorem

1.1 Background

1.1.1 The Generalized Turán Theorem

In 1941, Turán introduced in [42] the celebrated extremal problem that now bears his name: how many edges can a (simple) graph contain if it has no clique containing $r+1$ vertices? Turán proved that the answer, denoted $\mathrm{ex}(n, K_{r+1})$, is asymptotically $(1-\frac{1}{r})\binom{n}{2}$ and that the unique extremal graph is the complete r-partite graph on n vertices with parts of size $\lceil \frac{n}{r} \rceil$ or $\lfloor \frac{n}{r} \rfloor$ (so no pair of parts differs in size by more than one). We call this graph the *Turán graph* and denote it $T_r(n)$.

The first extensions to Turán's theorem considered forbidding graphs other than cliques. For any graph F, we say a graph is F*-free* if it contains no (not necessarily induced) subgraph isomorphic to F. Then $\mathrm{ex}(n, F)$ is the maximal number of edges in an F-free graph. The general question is settled asymptotically by the Erdős-Stone Theorem which states

$$\mathrm{ex}(n, F) = \left(1 - \frac{1}{\chi(F)-1} + o(1)\right)\binom{n}{2}$$

for non-complete graphs F. We note that the result was actually first proved by

Erdős and Simonovits in [16], but their proof relies heavily on the earlier result in [17] on which Stone was a coauthor.

To fully generalize the problem, one may consider counting subgraphs other than edges. We denote by $\mathrm{ex}(n, T, F)$ the maximal number of subgraphs T in an F-free graph on n vertices. (Here T is the "target" graph while F is "forbidden.") Several specific cases were investigated (see, for example, [9, 26]) before 2015 when Alon and Shikhelman introduced a systematic study in [4] in which they determine, among other results, that for forbidden graphs F with $\chi(F) = k + 1 > r$,

$$\mathrm{ex}(n, K_r, F) = (1 + o(1))\binom{k}{r}\left(\frac{n}{k}\right)^r.$$

A more precise result can be found in [35]. Since then, the area has been widely studied; see [14, 23, 28, 32, 34] for an (incomplete) sampling of authors and results.

In this thesis we consider results regarding $\mathrm{ex}(n, T, K_{r+1})$, counting graphs T in graphs without cliques of size $r + 1$. In particular, we investigate for which T the extremal graphs are complete multipartite. We then consider the specific cases where T and F are cliques or stars, extending a result of [14] regarding S_t-extremal complete r-partite graphons.

1.1.2 Notation and Definitions

If A is a set and k is an integer, we use $\binom{A}{k}$ to represent the set of subsets of A containing k distinct elements. We represent the set of integers $\{1, 2, \ldots, n\}$ by $[n]$.

A *graph* $G = (V, E)$ is an ordered pair consisting of a finite set of *vertices* V and *edges* $E \subseteq \binom{V}{2}$. Note that loops and multiedges are not permitted. Given a graph G, let $V(G)$ denote the vertex set of G and $E(G)$ denote the edge set of G. We use $n(G)$ to denote $|V(G)|$ and $e(G)$ to denote $|E(G)|$. If $\{u, v\} \in E(G)$, we say u is *adjacent* to v in G and denote the adjacency by $u \sim_G v$. When the graph is clear from context, we may just write $u \sim v$. We often refer to edges as $e = uv$, rather than use set notation. The *neighborhood* of v, denoted $N(v)$, is the set of vertices u such that $uv \in E(G)$. The *degree* of a vertex v is the number of edges containing v, or equivalently, the size of the neighborhood of v. We use $\Delta(G)$ and $\delta(G)$ to denote the maximum and minimum degree in G, respectively.

We denote the *complete graph* on r vertices, in which $|V| = r$ and $E = \binom{V}{2}$, by K_r. A *cycle* C_k is a graph on vertex set $[k]$ such that vertex i is adjacent only to vertices $i+1$ and $i-1$, with operations taken modulo k. A *path graph* P_ℓ is a graph on vertex set $[\ell+1]$ in which vertex 1 is adjacent only to vertex 2, vertex $\ell+1$ is adjacent only to vertex ℓ, and vertex i is adjacent only to vertices $i+1$ and $i-1$ for $2 \leq i \leq \ell$. In particular, P_ℓ contains ℓ edges and $\ell+1$ vertices.

A subset of vertices $I \subseteq V(G)$ is *independent* if $u \not\sim v$ for every $u, v \in I$. A graph is *complete r-partite* if there is a partition of V into $V_1, \ldots, V_r$ (where one or more parts may be empty), such that each V_i is an independent set and $v_i \sim v_j$ for each $v_i \in V_i, v_j \in V_j$ such that $i \neq j$. We use the term *bipartite* instead of 2-partite. We may refer to a graph as *complete multipartite* without specifying r. The *star graph* S_t is a complete bipartite graph where one part has size t and the other has size one. In general, we denote the complete bipartite graph with parts of size p and q by $K_{p,q}$.

A *graph homomorphism* from H to G is a function $f : V(H) \to V(G)$ such that

$$u \sim_H v \implies f(u) \sim_G f(v).$$

Note that the implication is not bidirectional; when $u \not\sim_H v$, we put no restriction on the adjacency of $f(u)$ and $f(v)$ in G. If f is an injective function we say that H is a *subgraph* of G, which we denote $H \subseteq G$. If f is an injection and also

$$u \sim_H v \iff f(u) \sim_F f(v)$$

we say H is an *induced* subgraph of G. Informally, H is a subgraph of G if a copy of H, possibly with extra edges, can be found in G, and if that copy has no extra edges it is induced. Given any $U \subseteq V$, the subgraph induced by U, denoted $G[U]$, is the graph with vertex set U and edge set $\binom{U}{2} \cap E(G)$; note that $G[U]$ is indeed an induced subgraph of G. If G is an induced subgraph of H and H is an induced subgraph of G, or equivalently if there is a bijective function $f : V(H) \to V(G)$ such that

$$u \sim_H v \iff f(u) \sim_F f(v),$$

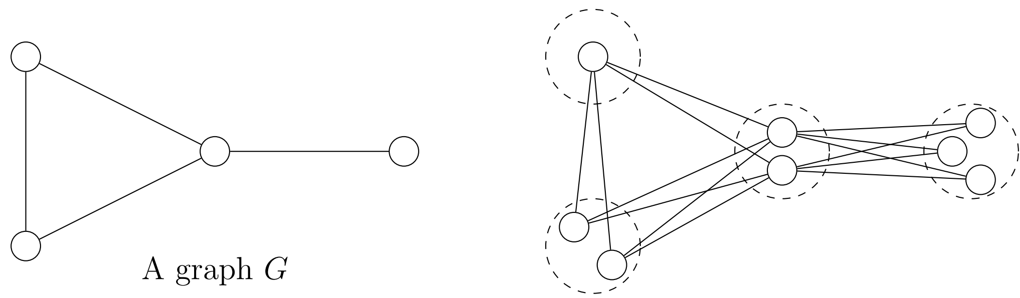
then we say G and H are isomorphic. A subgraph $K \subseteq G$ that is isomorphic to a complete graph (that is, a subgraph containing every possible edge) is called a *clique*. For graphs T and G, the function $n_T(G)$ counts the number of (not necessarily induced) subgraphs of G that are isomorphic to T. When counting cliques, cycles, paths, or stars, we use $k_r(G), c_k(G), p_\ell(G)$ or $s_t(G)$, respectively, so as to avoid double subscripts.

If G and H are graphs, we denote the disjoint union of G and H by $G \cup H$. More precisely, $G \cup H$ is a graph with vertex set $V(G) \sqcup V(H)$ and edge set $E(G) \cup E(H)$.

The chromatic number of G, denoted $\chi(G)$, is the smallest k such that there exists a function $f : V \to [k]$ satisfying

$$u \sim v \implies f(u) \neq f(v).$$

Given a graph G, a *blowup* of G is a graph in which every vertex v of G is replaced by an independent set I_v and $x \in I_v$ is adjacent to $y \in I_u$ if and only if $v \sim_G u$. We refer to I_v as the *fiber* of v. Blowups have been used in a variety of extremal contexts; see, for example, [4, 19, 36, 37]. Figure 1.1 contains an example of a blowup.



A graph G

A sample blowup of G

Figure 1.1: An example of a blowup

1.1.3 Graph Limits and Graphons

In our discussion of the generalized Turán theorem, we will encounter the concept of graphons. On one hand, the topic of graphons is deep unto itself; the main resource used to prepare this background, [33], is a several-hundred page text introducing the theory. On the other hand, our use of graphons requires little more than a cursory understanding of their foundations. Indeed, if the reader thinks of graphons as graphs with vertex set the size of the continuum, very little of this thesis will make less sense. Nevertheless, for the sake of completeness, we provide a brief, semi-formal introduction to graphons in this section.

We begin with the concept of graph densities. If G and H are graphs, we define the *density of H in G*, denoted, $d(H, G)$, to be the probability that a random injective function $f : V(H) \to V(G)$ is a graph homomorphism. For example, if H is a single vertex, $d(H, G) = 1$ for any graph G as every function from a single vertex to $V(G)$

is a graph isomorphism. If H is a single edge, then

$$d(H,G) = \frac{2e(G)}{n(G)(n(G)-1)} = \frac{e(G)}{\binom{n(G)}{2}}$$

is often called the edge density of G: every edge $v_1v_2 \in E(G)$ contributes two homomorphisms. As a final example, if G is a complete graph and $n(H) \leq n(G)$ then

$$d(H,G) = 1 \tag{1.1}$$

as every injection $f : V(H) \to V(G)$ is a graph homomorphism.

As an aside, this is only one of several definitions of graph density and, in a fuller treatment of the topic, would be called *injective homomorphism density*. Given certain modest conditions, all of these different notions of density produce the same theory of graphons, so focusing solely on injective homomorphism density will be sufficient to develop the necessary understanding of graphons and thus we henceforth drop the "injective homomorphism" modifier.

Suppose $(G_j)_{j=1}^{\infty}$ is a sequence of graphs such that $n(G_j) \to \infty$. Then for each finite graph H we get a sequence $d(H, G_j)$ of densities. We say the sequence of graphs (G_j) converges if, for every finite graph H, the sequence of real numbers $d(H, G_j)$ converges. For example, the result in (1.1) proves that the sequence $(K_n)_{n=1}^{\infty}$ converges to one as for any graph H we see $d(H, K_n) = 1$ for every $n \geq n(H)$.

The theory of graphons, at least partially, was developed to find limit objects corresponding to convergent graph sequences. Formally, a graphon is a bounded symmetric measurable function $W : [0,1]^2 \to \mathbb{R}$ such that for every $(x,y) \in [0,1]^2$,

$$0 \leq W(x,y) \leq 1.$$

Much less formally, we often think of graphons as graphs with vertex set $[0, 1]$ with "edge density" $\int_{[0,1]^2} W(x, y)\ dxdy$. Indeed, if $x, y \in [0, 1]$ then one can imagine an edge from x to y exists probabilistically with probability $W(x, y)$. Graphons are often visualized by coloring a square with various shades of gray where darker shades represent larger probabilities (see Figure 1.2).

An empty graphon — A complete graphon — A typical graphon

Figure 1.2: Visualizations of graphons

Using this notion, we extend our notion of density to graphons. Define

$$d(H, W) = \int_{[0,1]^{n(H)}} \prod_{ij \in E(H)} W(x_i, x_j)\ \mathrm{d}x_1 \cdots \mathrm{d}x_{n(H)} \tag{1.2}$$

At first glance this formula may look imposing, but it is the natural equivalent of an injective graph homomorphism: given $n(H)$ "vertices" $x_1, \ldots, x_{n(H)} \in [0, 1]$, we multiply the probability x_i is adjacent to x_j for each edge $x_i x_j \in H$.

We will only consider graphons in which $W(x_i, x_j)$ is either zero or one, in which this interpretation is even simpler: each $x \in [0, 1]$ is adjacent to those $y \in [0, 1]$ for which $W(x, y) = 1$ so that the graphon acts as a sort of "$[0, 1] \times [0, 1]$" adjacency matrix, and the term

$$\prod_{x_i x_j \in E(H)} W(x_i, x_j)$$

is one if and only if $x_1, \ldots, x_{n(H)}$ have the adjacencies required to form a subgraph isomorphic H and zero otherwise.

The most important result for our applications of graphons is that a sequence of graphs $(G_j)_{j=1}^{\infty}$ converges if and only if there is a graphon W such that for every finite graph H,

$$\lim_{j\to\infty} d(H, G_j) = d(H, W).$$

In this case $d(H, W)$ can be thought of as the probability that a random function $x : V(H) \to [0, 1]$ is a homomorphism. Moreover, this convergence is unique in a weak sense. If $\varphi : [0, 1] \to [0, 1]$ is measure preserving, then we may define W^{φ} by

$$W^{\varphi}(x, y) = W(\varphi(x), \varphi(y)).$$

In our infinite graph analogy this corresponds to relabeling vertices. With a bit of measure theory one can show $d(H, W) = d(H, W^{\varphi})$ for all graphs H and thus the limit graphon is not strictly unique. However, if for all H

$$d(H, W) = \lim_{j\to\infty} d(H, G_j) = d(H, W')$$

one can show $W' = W^{\varphi}$ for some measure preserving φ and thus the limit graphon is unique up to this "relabeling."

Graphons have been used in many applications in mathematics [22, 30, 39, 43] as well as fields such as statistics [44, 48], machine learning [1], and economics [11]. We will focus on their applications to extremal graph theory. Specifically, any question regarding an extremal number $\mathrm{ex}(n, T, F)$ can be reformulated in terms of finding the graphon with $d(F, W) = 0$ that maximizes $d(T, W)$. Often, the continuous nature of graphons leads to analysis less cluttered by concerns such as choosing distinct vertices or divisibility of vertices into parts. For example, when counting subgraphs of $T_r(n)$, calculations often include falling factorials as vertices selected in a given part often must be distinct. In a graphon, however, such concerns are unnecessary: analytically, sets which select the same vertex more than once have measure zero.

Furthermore, Turán graphs often require careful handling of ceiling and floor functions when r does not divide n (indeed, the reader will find such calculations in this thesis), but, without too much trouble, one can show that for fixed r the sequence $(T_r(n))_{n=1}^{\infty}$ converges to

$$W(x,y) = \begin{cases} 0, & x,y \in [\frac{i-1}{r}, \frac{i}{r}] \text{ for some } 1 \leq i \leq r \\ 1, & \text{otherwise.} \end{cases}$$

Figure 1.3 contains visualizations of the Turán graphons for $r = 3$ (left) and $r = 4$ (right).

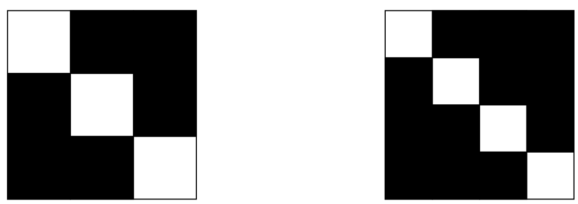


Figure 1.3: Turán graphons

1.2 Counting Arbitrary Graphs and Forbidding Cliques

In this section we investigate what can be said about $\text{ex}(n, T, K_{r+1})$ for general graphs T. In particular, we're interested in determining which T have a complete multipartite extremal graph. We begin with an easy observation.

Proposition 1.1. *If F is a subgraph of T then* $\mathrm{ex}(n, T, F) = 0$ *and every F-free graph on n vertices is extremal.*

Proof. If $n_T(G) > 0$, then G must contain an F subgraph, so G is not F-free. $\square$

With this in mind, we will henceforth only study $\mathrm{ex}(n, T, F)$ where $F \not\subseteq T$. We say T is *r-partite extremal* if T is K_{r+1}-free and there is some N such that for all $n \geq N$ there is some complete r-partite graph G on n vertices such that

$$n_T(G) = \mathrm{ex}(n, T, K_{r+1}).$$

Furthermore, we say T is *strictly r-partite extremal* if there is some N such that for all $n \geq N$ any graph G with

$$n_T(G) = \mathrm{ex}(n, T, K_{r+1})$$

is complete r-partite.

First we note that r-partite extremal graphs have chromatic number at most r:

Proposition 1.2. *If T is a K_{r+1}-free graph such that $\chi(T) > r$, then T is not r-partite extremal.*

Proof. Let C be a complete r-partite graph. Then $\chi(C) \leq r < \chi(T)$, so C contains no copies of T. However, for any $n \geq |V(T)|$, if I is an independent set of $n - |V(T)|$ vertices then $T \cup I$ is a K_{r+1}-free graph on n vertices with one copy of T. Thus we conclude $\mathrm{ex}(n, T, K_{r+1}) > 0$ and C is not optimal. $\square$

We now consider less trivial results regarding r-partite extremal graphs. The following result was proved by Györi, Pach, and Simonovits in [27]:

Theorem 1.3 (Györi, Pach, Simonovits [27]). *Let T be a complete multipartite graph with s parts. Let $r \geq s$ and $n \geq \max(|V(T)| + 1, r)$. Then any K_{r+1}-free graph G on n vertices with*

$$n_T(G) = \text{ex}(n, T, K_{r+1})$$

must be complete r-partite. In other words, complete s-partite graphs are strictly r-partite extremal.

We give a brief summary of their proof. If u and v are non-adjacent vertices in G, the *Zykov symmetrization* $Z_{u \to v}(G)$ turns v into a clone of u by removing edges vx such that $u \not\sim x$ and adding edges vy such that $u \sim y$. The authors prove that for any complete multipartite T, any K_{r+1}-free graph G, and any non-adjacent u and v in $V(G)$, one of $Z_{u \to v}(G)$ or $Z_{v \to u}(G)$ must contain at least as many copies of T as G does. They then prove that as long as G is not complete r-partite, one can iteratively apply Zykov symmetrizations so that at each step either the number of T strictly increases or it stays constant but the number of pairs of non-adjacent vertices with non-identical neighborhoods strictly decreases. As the number of copies of T is bounded above by $\text{ex}(n, T, K_{r+1})$, this process only stops by reaching a complete r-partite graph and therefore T is r-partite extremal. To show strictness, one can carefully analyze the step in which the last non-adjacent pair of vertices with non-identical neighborhoods are symmetrized and conclude it cannot have had $\text{ex}(n, T, K_{r+1})$ copies of T.

Let $\mathcal{C}_r$ be the family of complete r-partite graphs, $\mathcal{B}_r$ denote the family of r-partite extremal graphs, $\mathcal{A}_r$ denote the family of graphs with chromatic number at most r and $\mathcal{K}_r$ be the family of K_{r+1}-free graphs. Then Proposition 1.2 and Theorem 1.3, together with the fact that any graph containing a K_{r+1} has chromatic number greater than r, imply

$$\mathcal{C}_r \subseteq \mathcal{B}_r \subseteq \mathcal{A}_r \subseteq \mathcal{K}_r.$$

It is well known that $\mathcal{A}_r$ is a strict subset of $\mathcal{K}_r$ (take, for example, an odd cycle and a dominating set of the appropriate size to get a K_{r+1}-free graph with chromatic number $r+1$). It is natural to ask if the other inclusions are strict.

In the case $r = 2$, Györi, Pach, and Simonovits show in [27] that $\mathcal{B}_r$ is a strict subset of $\mathcal{A}_r$ with the following example:

Proposition 1.4. *There exist infinitely many bipartite graphs T such that for infinitely many n, no complete bipartite graph has $\mathrm{ex}(n, T, K_3)$ copies of T and thus $\mathcal{B}_2 \subsetneq \mathcal{A}_2$.*

Proof. We build the graph T as follows: fix an integer a, to be determined later, and start with two stars S_{a-2} with central vertices c_1 and c_2. Add vertices x and y and edges c_1x, xy, and yc_2. The resulting graph has $2a$ vertices.

Consider the number of copies of T in any complete bipartite graph $K_{p,q}$. The structure of T forces the two large independent sets to be embedded in different sides of the partition; indeed, from each part we select the leaves of one star, the center of the other star, and a vertex for the central edge of the path connecting the stars. Thus the number of T in $K_{p,q}$ is

$$\binom{p}{a-2, 1, 1, p-a}\binom{q}{a-2, 1, 1, q-a}.$$

As $p + q = n$, by a straightforward but unedifying argument the maximum occurs at the balanced bipartite graph which contains

$$\binom{\frac{n}{2}}{a-2,1,1,\frac{n}{2}-a}^2 = \left(\binom{\frac{n}{2}}{a-2}\binom{n}{2}\right)^2 + O(n^{2a-1})$$
$$= \frac{n^{2a-4}}{2^{2a-4}(a-2)!^2} \cdot \frac{n^4}{16} + O(n^{2a-1})$$
$$= \frac{n^{2a}}{2^{2a}(a-2)!^2} + O(n^{2a-1})$$

many copies of T.

Now consider instead the number of copies of T in G, a blowup of C_5 with four smaller parts of size $\frac{n}{2a}$ and one larger part of size $(a-2)\frac{n}{a}$. (For ease of computation we assume a divides n. Ultimately we will see the number of copies of T in G beats the number of copies in $K_{p,q}$ by an asymptotic factor and thus introducing floors and ceilings will only complicate the calculation.) We get a lower bound for the number of copies of T in G by counting only those copies where both sets of $a-2$ leaves of T are contained in the large fiber and each vertex in the path is in a unique smaller part. Thus there are at least

$$\binom{(a-2)\frac{n}{a}}{a-2,a-2,(a-2)\frac{n}{a}-(2a-4)}\left(\frac{n}{2a}\right)^4 \geq \frac{((a-2)\frac{n}{a}-(2a-4))^{2a-4}}{(a-2)!^2} \cdot \frac{n^4}{16a^4}$$
$$= \frac{n^{2a}}{(a-2)!^2} \cdot \frac{(1-\frac{2}{a}-\frac{2(a-2)}{n})^{2a-4}}{16a^4}$$

copies of T in G. We may require $n \geq a(a-2)$ so that $\frac{2(a-2)}{n} \leq \frac{2}{a}$ and $a > 16$ so that

$a - 4 > \frac{1.5a}{2}$, giving

$$\frac{(1 - \frac{2}{a} - \frac{2(a-2)}{n})^{2a-4}}{16a^4} \geq \frac{(1 - \frac{4}{a})^{2a-4}}{16a^4}$$
$$= \frac{(a-4)^{2a-4}}{2^4 a^{2a}}$$
$$> \frac{(1.5)^{2a-4}}{a^4} \cdot \frac{1}{2^{2a}}.$$

By comparing derivatives, one can show $(1.5)^{2a-4} > a^4$ when $a > 16$. Thus asymptotically, and so for large enough a, G contains more copies of T than any $K_{p,q}$. □

We now prove $\mathcal{C}_2 \subsetneq \mathcal{B}_2$, completely resolving the question of subset containment in the $r = 2$ case. Recall that $p_3(G)$ is number of subgraphs of G isomorphic to P_3, the path graph with three edges.

Theorem 1.5. *For all $n \geq 4$, the balanced complete bipartite graph $T_2(n)$ uniquely satisfies $p_3(T_2(n)) = \text{ex}(n, P_3, K_3)$. Note that $\chi(P_3) = 2$ and that P_3 is not a complete bipartite graph, and thus $\mathcal{C}_2 \subsetneq \mathcal{B}_2$.*

Proof. First we count $p_3(T_2(n))$. Choose any edge $e = xy$ in $T_2(n)$. We count the number of P_3 whose edges have the form v_1x, xy, yv_2; that is, P_3 containing e as the "central" edge. As $T_2(n)$ is bipartite, we know $N(x)$ and $N(y)$ are disjoint. Thus choosing any neighbor of x other than y for v_1 and any neighbor of y other than x for v_2 gives such a P_3, and all P_3 with e as the central edge can be formed in this way. Let $V(T_2(n)) = L \sqcup R$ be a partition of the vertices into independent sets. Then for each $x \in L$, $d(x) = |R|$ and similarly for each $y \in R$, $d(y) = |L|$. We conclude there are $(|R|-1)(|L|-1)$ such P_3. This is true for each edge in $T_2(n)$, so noting that each P_3 has a unique central edge we see $p_3(T_2(n)) = e(T_2(n))(|L|-1)(|R|-1)$. We will consider the ratio

$$\frac{p_3(T_2(n))}{e(T_2(n))} = (|L|-1)(|R|-1) = \begin{cases} (\frac{n}{2}-1)^2, & n \text{ even} \\ (\frac{n+1}{2}-1)(\frac{n-1}{2}-1), & n \text{ odd} \end{cases}$$

This ratio measures the average, across the edges of $T_2(n)$, of the number of P_3 in which that edge acts as the central edge.

Assume, for sake of contradiction, that $G \neq T_2(n)$ is a triangle-free graph such that $p_3(G) \geq p_3(T_2(n))$. We know from Turán's theorem that $e(G) < e(T_2(n))$, so we conclude

$$\frac{p_3(G)}{e(G)} > \frac{p_3(T_2(n))}{e(T_2(n))}.$$

Let $e = xy$ be an edge in G that is the central edge in strictly more than $\frac{p_3(T_2(n))}{e(T_2(n))}$ many P_3. As G is triangle-free, we again know $N(x)$ and $N(y)$ are disjoint and thus, using the same reasoning as in the $T_2(n)$ case, e is the center of $(d(x)-1)(d(y)-1)$ many P_3. Furthermore, $d(x)+d(y) \leq n$ as every vertex (including x and y) is adjacent to at most one of x and y. Thus $d(y) \leq n - d(x)$ and

$$(d(x)-1)(d(y)-1) \leq (d(x)-1)(n-d(x)-1).$$

Note that the function $f(z) = -(z-1)(z-(n-1))$ is a parabola opening downward with maximum at $z = \frac{n}{2}$. When n is even, we see

$$\begin{aligned}\left(\frac{n}{2}-1\right)^2 &= \frac{p_3(T_2(n))}{e(T_2(n))} \\ &< \frac{p_3(G)}{e(G)} \\ &\leq (d(x)-1)(d(y)-1) \\ &\leq \left(\frac{n}{2}-1\right)^2,\end{aligned}$$

a contradiction. When n is odd, we cannot set $d(x) = \frac{n}{2}$ as the degree must be an integer. As $(d(x)-1)(d(y)-1)$ is increasing in both $d(x)$ and $d(y)$, the maximum value, even when restricted to integral choices of $d(x)$ and $d(y)$, must occur when $d(x)+d(y) = n$. Setting $d(x) = \frac{n}{2}+\varepsilon$, this forces $d(y) = \frac{n}{2}-\varepsilon$. Consider that

$$\left(\frac{n}{2}-1\right)^2 - \left(\frac{n}{2}+\varepsilon-1\right)\left(\frac{n}{2}-\varepsilon-1\right) = \varepsilon^2.$$

Thus the further from $\frac{n}{2}$ we set $d(x)$, the further $(d(x)-1)(d(y)-1)$ is from the maximum. In order to minimize this distance, and thus maximize the output given the integral constraint, we should choose the ε of smallest magnitude that makes $\frac{n}{2}+\varepsilon$ an integer. When n is odd, this value is $\pm\frac{1}{2}$. Thus we see, for n odd,

$$\begin{aligned}\left(\frac{n+1}{2}-1\right)\left(\frac{n-1}{2}-1\right) &= \frac{p_3(T_2(n))}{e(T_2(n))} \\ &< \frac{p_3(G)}{e(G)} \\ &\leq (d(x)-1)(d(y)-1) \\ &\leq -(d(x)-1)(d(x)-(n-1)) \\ &\leq \left(\frac{n+1}{2}-1\right)\left(\frac{n-1}{2}-1\right),\end{aligned}$$

a similar contradiction.

We conclude no such G exists and therefore $T_2(n)$ is the unique P_3-extremal K_3-free graph on n vertices. □

Note that Theorem 1.5 is a consequence of a result in [27], though the proof given above introduces the new idea of measuring the ratio $\frac{p_3(G)}{e(G)}$.

A natural question resulting from Proposition 1.4 is what graphs G that are not complete multipartite graphs satisfy

$$n_T(G) = \mathrm{ex}(n, T, K_{r+1})$$

for some graph T. The example from that proposition suggests blowups of C_5 could be another family of extremal graphs. In the following theorem, we prove that the only blowups of cycles which can be extremal are those of C_5 or C_4, where we note that the set of blowups of C_4 is exactly the set of complete bipartite graphs that are not stars.

Theorem 1.6. *Let T be a graph with at least one edge. Let G be a blowup of C_k for some $k > 5$ with $n_T(G) > 0$. Then there exists H, a blowup of either C_4 or C_5, with $n_T(H) > n_T(G)$.*

Proof. We will generate a sequence of graphs $G = G_0, G_1, \ldots, G_n = H$ on the same vertex set such that each G_i is a blowup of a cycle of length k_i with $k_i > k_{i+1}$, $k_n = 4$ or 5, and $n_T(G_i) \leq n_T(G_{i+1})$ for $0 \leq i < n$. Then we will argue there must be some i such that $n_T(G_i) < n_T(G_{i+1})$.

We generate G_{i+1} from G_i using the following algorithm: first choose an arbitrary fiber of G_i and label it V_1. Then choose an arbitrary neighboring fiber to give the label V_2. Continue following the cycle to label fibers V_3 through V_{k_i}. Then check for the following condition: there is a pair of adjacent fibers V_j and V_{j+1} (with addition modulo k_i) such that no copy of T in G_i contains an edge between V_j and V_{j+1}. If the condition is met, use Method 1 below, and otherwise use Method 2.

Method 1: If $k_i \leq 4$, terminate the sequence. Otherwise, we generate G_{i+1} by removing all edges between V_j and V_{j+1} and adding all edges between V_{j-1} and V_{j+1} and between V_j and V_{j+2}. Essentially, we "collapse" V_j and V_{j+1} into one vertex class. Then G_{i+1} is a blowup of a cycle on $k_i - 1$ vertices and, since no copy of T included the deleted edges, $n_T(G_i) \leq n_T(G_{i+1})$.

Method 2: If $k_i \leq 5$, terminate the sequence. Otherwise, we may assume that between every pair of fibers V_j and V_{j+1} there is an edge contained in some copy of T in G_i. We generate G_{i+1} from G_i by adding edges from V_1 to V_4, from V_2 to V_5, and from V_3 to V_6. Then vertices in $V_2 \cup V_4$ share a common neighborhood (namely, $V_1 \cup V_3 \cup V_5$), as do the vertices in $V_3 \cup V_5$. Therefore G_{i+1} is a blowup of a C_{k_i-2}; in this case we've "folded" V_2 into V_4 and V_3 into V_5. In this method we only add edges to G_{i+1} and so $n_T(G_{i+1}) \geq n_T(G_i)$.

Note that, assuming the condition is not met, the new vertex classes created by Method 2 will contain edges spanned by some copy of T. Thus once the condition is no longer met, we will continue using Method 2 until the process terminates. As Method 1 reduces k_i by 1 and Method 2 reduces k_i by 2, and as Method 2 will always apply if Method 1 does not, we conclude the algorithm terminates if and only if $k_n \in \{4, 5\}$.

Now we must argue that for some step in this process we have $n_T(G_i) < n_T(G_{i+1})$. First we argue that if we ever use Method 2 then n_T strictly increases. Recall that Method 2 only occurs when there is, for every pair V_j, V_{j+1}, a copy of T containing an edge between the two fibers. Choose a copy of T containing an edge $e = uw$ with $u \in V_3$ and $w \in V_4$. Choose $\hat{u} \in V_5$ and $\hat{w} \in V_2$. In G_{i+1} we have $N(u) = N(\hat{u})$

and $N(w) = N(\hat{w})$, so there is a copy T' of T in G_{i+1} where the roles of u and $\hat{u}$ are switched, as are the roles of w and $\hat{w}$. (Note that T' may or may not include $\hat{u}$ or $\hat{w}$. This does not affect our argument. By "switching roles," we mean $\hat{u}$ is adjacent to $N_T(u)$ in T' and, if $\hat{u} \in T$, then u is adjacent to $N_T(\hat{u})$ and otherwise T' does not include u, and v and $\hat{v}$ are treated analogously.) Then T' is not present in G_i as $\hat{u}$ is adjacent to $\hat{w}$ in T' but these vertices are not adjacent in G_i. We conclude $n_T(G_i) < n_T(G_{i+1})$.

Thus it is sufficient to argue that $n_T(G_i) < n_T(G_{i+1})$ for some i in the case that we only apply Method 1. If the process terminates after only using Method 1, then one of two cases has occurred:

Case A The algorithm applied Method 1 and then the condition to apply Method 1 was no longer met, but Method 2 was not applied because $k_i \leq 5$.

Case B The algorithm applied Method 1 and the condition to apply Method 1 was still met, but the graph produced was a blowup of C_4.

First consider Case A. In the last step of the algorithm, we start with a graph G_i in which there are fibers V_j and V_{j+1} such that no copy of T contains an edge between them, but once collapsed to form G_{i+1}, every pair of consecutive fibers has an edge contained in some copy of T. Give the name V to the fiber in G_{i+1} formed by the collapse of V_j with V_{j+1} in G_i. As G_{i+1} contains a copy of T between V and a neighboring fiber, fix such a copy of T and assume vu is an edge in T, where $v \in V$. If that copy of T does not exist in G_i, then we have $n_T(G_i) < n_T(G_{i+1})$, as required. Otherwise, in G_i, we have $v \in V_j$ or V_{j+1}, say V_j (as if $v \in V_{j+1}$ the argument is analogous.) We know T does not contain an edge from v to any vertex in V_{j+1}, as we assumed no copy of T contained an edge spanning V_j and V_{j+1}. Thus $u \in V_{j-1}$. Also, we choose $\hat{v}$ in V_{j+1} and consider the copy of T in G_{i+1} with the roles of v and

$\hat{v}$ switched. We can create this copy of T as $N(v) = N(\hat{v})$ in G_{i+1}, but as this copy of T contains an edge from $\hat{v} \in V_{j+1}$ to $u \in V_{j-1}$, which does not exist in G_i, this copy of T also does not exist in G_i and so $n_T(G_i) < n_T(G_{i+1})$.

Now consider Case B. We show this case cannot occur. The idea behind the argument is that blowups of C_4 are also complete bipartite graphs as non-adjacent vertex classes have identical neighborhoods. Thus the edges that are supposedly unused are indistinguishable from those that are included in a copy of T. We now formalize this notion. Suppose V_1 and V_2 are the vertex classes for which no copy of T spans an edge. Fix a copy of T and an edge $e = uw$ in T, as well as $v_1 \in V_1$ and $v_2 \in V_2$. We claim there is a way to replace u and w with v_1 and v_2 in a way that preserves a copy of T. To remove redundant cases, assume $u \in V_i$ and $w \in V_j$ with $i < j$ and reverse the names of u and w if not. This leaves three subcases:

Subcase (i) If $u \in V_1$, then $w \in V_4$: we've assumed no copy of T contains an edge spanning V_1 and V_2, but $uw \in T$, and vertices in V_1 are not adjacent to vertices in V_3.

Subcase (ii) If $u \in V_2$, then $w \in V_3$ as vertices in V_2 are not adjacent to those in V_4.

Subcase (iii) Finally, if $u \in V_3$ then $w \in V_4$.

In Subcase (i), u and v_1 are both in V_1 and thus have the same neighborhood, and as $N(V_2) = V_1 \cup V_3 = N(V_4)$, $N(v_2) = N(w)$. Thus swapping the roles of u with v_1 and w with v_2 produces a copy of T containing v_1v_2, an edge spanning V_1 and V_2. In Subcase (ii), u and v_2 are both in V_2 and $N(V_3) = V_2 \cup V_4 = N(V_1)$ so v_1 and w also share neighborhoods, again allowing us to switch their roles. Finally, in Subcase (iii), we switch the roles of v_1 with u and v_2 with w. In every case, we're able to identify a copy of T containing an edge that spans V_1 and V_2, so Case B cannot occur. $\square$

1.3 The Generalized Turán Theorem for Stars and Cliques

In this section, we consider specific cases of the generalized Turán theorem where T and F are stars or cliques. We start by reviewing the cases that were considered by other authors before turning to the remaining novel case.

1.3.1 Previously Studied Cases

The first case of this type to be considered was $\mathrm{ex}(n, K_t, K_{r+1})$, or counting small cliques while forbidding large cliques, by Zykov in [49]. In a result that has been often rediscovered, he proved that the Turán graph is optimal; that is,

$$\mathrm{ex}(n, K_t, K_{r+1}) = k_t(T_r(n)).$$

As $K_a \subseteq K_b$ whenever $a \leq b$, we have

$$\mathrm{ex}(n, K_t, K_{r+1}) = 0 = k_t(T_r(n))$$

if $t > r$, and so every K_{r+1}-free graph on n vertices is equally extremal. Otherwise, as in Turán's original theorem, the Turán graph is the unique extremal graph. It was in the proof of this theorem that Zykov introduced the concept of Zykov symmetrization, mentioned in Section 1.2. In fact, as complete graphs are complete multipartite (with each vertex acting as a different part), Theorem 1.3 can be considered an extension of Zykov's original result.

Next we consider the two cases when the forbidden graph is a star. Setting

$F = S_{r+1}$ is equivalent to mandating $\Delta(G) \leq r$. As $\Delta(K_t) = t - 1$, when $t > r + 1$ we have $k_t(G) = 0$ for any graph G satisfying $\Delta(G) \leq r$ and thus $\text{ex}(n, K_t, S_{r+1}) = 0$. Otherwise, Engbers and Galvin in [15], and independently Wood in [45], showed that for fixed $\Delta(G)$ one maximizes $k_t(G)$ by taking disjoint unions of larger K_{r+1} cliques. (Note that $\Delta(K_{r+1}) = r$.) When $r + 1$ divides n, then, we have

$$\text{ex}(n, K_t, S_{r+1}) = k_t\left(\frac{n}{r+1} K_{r+1}\right),$$

where aG is the union of a disjoint copies of G. The extremal graph is unique. Engbers and Galvin also consider the case when $r + 1$ does not divide n. Take a and b such that $n = a(r + 1) + b$ with $a, b \geq 0$ and $b < r + 1$. Then

$$\text{ex}(n, K_t, S_{r+1}) = k_t(aK_{r+1} \cup K_b).$$

In other words, after building as many disjoint K_{r+1} as possible, the remaining vertices should form a smaller clique. When $b \geq t$, this is the unique extremal graph. However, when $b < t$, the clique K_b contains no copies of K_t. In this case, somewhat surprisingly, there is no way to "make use of" these excess vertices. If G_b is any graph on b vertices, we have

$$k_t(aK_{r+1} \cup G_b) = k_t(aK_{r+1} \cup K_b) = \text{ex}(n, K_t, S_{r+1}).$$

As there are multiple non-isomorphic choices for G_b for any $b > 1$, the extremal graph is not unique for $1 < b < t$.

Engbers and Galvin's proof is a clever application of double counting. They count pairs (v, K) where K is a K_t containing v. If we use $k_t(v)$ to denote the number of cliques of size t containing v, then the number of such pairs is

$$tk_t(G) = \sum_{v \in V(G)} k_t(v) = \sum_{v \in V(G)} k_{t-1}(G[N(v)]).$$

As $\Delta(G) \leq r$, at best the neighborhood of v forms a clique with r vertices, producing $\binom{r}{t-1}$ many cliques of size $t-1$. Thus

$$tk_t(G) \leq n\binom{r}{t-1}$$

which, after some calculation, is equivalent to $k_t(G) \leq \frac{n}{r+1}\binom{r+1}{t}$, the number of K_t in $\frac{n}{r+1}$ disjoint copies of K_{r+1}. Uniqueness of the extremal graph and the case where $r+1$ does not divide n follow with some additional work.

Finally, we consider the case where $F = S_{r+1}$ and $T = S_t$, counting small stars while forbidding large stars. We have

$$s_t(G) = \sum_{v \in V(G)} \binom{d(v)}{t}$$

as each subset of $N(v)$ of size t is a distinct copy of S_t with v as the center. Thus in order to maximize $s_t(G)$, we should make each vertex of as large degree as possible. By requiring G to have no copies of S_{r+1}, we restrict each $\Delta(G) \leq r$. Therefore any r-regular graph will be extremal whenever such a graph exists.

When $n \leq r$, any graph on n vertices has degree at most $r-1$ and thus has no copies of S_{r+1}. Therefore, while we cannot construct an r-regular graph, the complete graph will maximize all subgraph counts, and in particular $s_t(G)$. When $n > r$ and nr is even, we can construct an r-regular graph as follows: label the vertices $0, 1, \ldots, n-1$. If r is even, for each vertex i add edges from i to $i+k$, for $1 \leq k \leq \frac{r}{2}$, with addition modulo n. If r is odd, for each vertex i add edges from i to $i+k$, for $1 \leq k \leq \frac{r-1}{2}$, with addition modulo n, and, since n must be even, also add edges between pairs i, j such that $j = i + \frac{n}{2}$. Otherwise, $n > r$ and n and r are both odd. In this case, no r-regular graph can exist as the sum of the degrees of the vertices, which should be twice the number of edges, would be an odd number. Thus in order to maximize $s_t(G)$, we seek a graph with $n-1$ vertices of degree r and one vertex of degree $r-1$. We can construct such a graph by labeling the vertices $0, 1, \ldots, n-1$, adding edges from vertex i to vertex $i+k$ for each $1 \leq k \leq \frac{r-1}{2}$ and adding edges from j to $j + \frac{n-1}{2}$ for $1 \leq j \leq \frac{n-1}{2}$. Note that this adds one additional edge to each vertex except the vertex with label 0. Thus we conclude

$$\mathrm{ex}(n, S_t, S_{r+1}) = \begin{cases} n\binom{r}{t} & n > r,\ nr \text{ even} \\ (n-1)\binom{r}{t} + \binom{r-1}{t} & n > r,\ nr \text{ odd} \\ 0 & n \leq r \end{cases}$$

and, as in general there are multiple non-isomorphic r-regular graphs, the extremal graphs are not unique.

1.3.2 The Novel Case: Counting Stars and Forbidding Cliques

Finally, we consider the case of $T = S_t$ and $F = K_{r+1}$ which will prove considerably more complex.

As in other cases, we must take special care when considering graphs on a small number of vertices. If $n \leq r$, the complete graph on n vertices contains no K_{r+1} and maximizes the count of any subgraph. For $n < t+1$, no graph on G vertices contains any copies of S_t and thus any K_{r+1}-free graph is extremal. We call a vertex *dominating* if it is adjacent to every other vertex. Copies of S_t in a graph G on $n = (t+1)$ vertices are in one-to-one correspondence with dominating vertices. Because r dominating vertices, together with any other vertex, span a K_{r+1}, with the unique exception of K_r (for which $n = r$ and so we've already covered) there are at most $\min(n, r-1)$ dominating vertices in a K_{r+1}-free graph. Thus the extremal graph when $n = t+1$ is an independent set of size $n - \min(n, r-1)$ and a set of dominating vertices of size $\min(n, r-1)$. We now consider only $n \geq \max(t+2, r+1)$.

Recall from Theorem 1.3 in Section 1.2 that for this range of n, complete multipartite graphs, including stars, are r-partite extremal for all r. Thus in order to determine $\mathrm{ex}(n, S_t, K_{r+1})$, one must merely decide the sizes of each part of an extremal complete multipartite graph. First we do so for $t = 2$.

Theorem 1.7. *For $n \geq 3$ and $r \geq 2$, the Turán graph $T_r(n)$ uniquely maximizes the number of S_2 among graphs with no K_{r+1}. That is,*

$$\mathrm{ex}(n, S_2, K_{r+1}) = s_2(T_r(n)).$$

Proof. As noted above, any extremal graph is complete r-partite. Let G be a complete multipartite graph on r parts with sizes $v_1, v_2, \ldots, v_r$. Let G' be a complete multipartite graph on r parts with sizes $v_1 + 1, v_2 - 1, \ldots, v_r$. We wish to calculate the change in the number of S_2 between G and G'. (We will actually calculate twice the change, for ease of calculation.) Note that the number of S_2 with center at vertex v is $\binom{d(v)}{2}$. Thus

$$s_2(G) = \sum_{i=1}^{r} v_i \binom{n - v_i}{2}$$

as each part as v_i vertices of degree $n - v_i$. As the terms of the sum only depend on

v_i and because G' has parts of the same size as G for $i \geq 3$ we have

$$\begin{aligned} 2(s_2(G') - s_2(G)) &= 2\left(\left[(v_1+1)\binom{n-(v_1+1)}{2} + (v_2-1)\binom{n-(v_2-1)}{2}\right] \quad - \right. \\ &\qquad \left. - \quad \left[v_1\binom{n-v_1}{2} + v_2\binom{n-v_2}{2}\right]\right) \\ &= (n-v_1-1)\big((v_1+1)(n-v_1-2) - v_1(n-v_1)\big) \quad + \\ &\qquad + \quad (n-v_2)\big((v_2-1)(n-v_2+1) - v_2(n-v_2-1)\big) \\ &= (n-v_1-1)(n-3v_1-2) + (n-v_2)(3v_2-n-1) \\ &= (v_2-v_1-1)(4n-3(v_1+v_2)-2) \end{aligned}$$

Suppose $v_2 \geq v_1 + 2$. Then $v_2 - v_1 - 1 > 0$. Furthermore, as $n \geq 3$ by assumption and

$$n = \sum_{i=1}^{r} v_i \geq v_1 + v_2$$

we have

$$4n - 3(v_1+v_2) - 2 \geq 4n - 3n - 2 = n - 2 > 0.$$

Thus if G was not $T_r(n)$, we can increase $s_2(G)$ by rebalancing parts. We may continue this process, increasing $s_2(G)$ at every step, until we reach $T_r(n)$. □

In general, it is difficult to determine the sizes of the parts of an extremal complete multipartite graph. Instead, we consider the corresponding graphon version of the problem. We extend the notion of a complete r-partite graph to graphons by generalizing the Turán graphons described in Section 1.1.3: partition $[0,1]$ into intervals $U_1, \ldots, U_r$ and let $W(x,y) = 1$ unless x and y are both in U_i for some i. For example, Figure 1.4 contains the visualization of a four-partite graphon where

$$U_1 = [0, \tfrac{1}{4}], \quad U_2 = (\tfrac{1}{4}, \tfrac{3}{8}], \quad U_3 = (\tfrac{3}{8}, \tfrac{2}{3}], \quad \text{and} \quad U_4 = (\tfrac{2}{3}, 1].$$

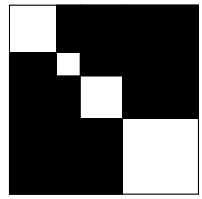


Figure 1.4: A typical four-partite graphon

We want to calculate the density of S_t in our complete r-partite graphon. Suppose $U_1, \ldots, U_r$ partition $[0,1]$ and have sizes $|U_i| = \rho_i$. Then if $x \in U_i$, we have

$$\int_{[0,1]} W(x,y)\ \mathrm{d}y = 1 - \rho_i$$

and thus if, without loss of generality, x_1 is the center of our star

$$\begin{aligned}
\rho(S_t, W) &= \int_{[0,1]^{t+1}} \prod_{j=2}^{t+1} W(x_1, x_j)\ \mathrm{d}x_1 \cdots \mathrm{d}x_{n(H)} \\
&= \int_{[0,1]} \left(\int_{[0,1]^t} \prod_{j=2}^{t+1} W(x_1, x_j)\ \mathrm{d}x_2 \cdots \mathrm{d}x_{n(H)} \right) \mathrm{d}x_1 \\
&= \int_{[0,1]} (1 - \rho_i)^t\ \mathrm{d}x_1 \\
&= \sum_{i=1}^{r} \rho_i (1 - \rho_i)^t
\end{aligned}$$

We have reduced the question of determining the extremal graphon to determining what sizes $\rho_1, \ldots, \rho_r$ maximize

$$F(\rho_1, \ldots, \rho_r) = \sum_{i=1}^{r} \rho_i (1 - \rho_i)^t$$

subject to the constraints

$$\sum_{i=1}^{r} \rho_i = 1 \qquad \text{and} \qquad \forall i, \rho_i \geq 0.$$

We will prove the following theorem, first presented in [14] for special cases of r and t, for all $r \geq 2$ and $t \geq 2$.

Theorem 1.8. *Let $r, t \geq 2$ be integers. Define $\Omega \subseteq [0,1]^r$ to be the set of points $(\rho_1, \rho_2, \ldots, \rho_r)$ satisfying*

$$\sum_{i=1}^{r} \rho_i = 1.$$

Define $f : [0,1] \to \mathbb{R}$ and $F : \Omega \to \mathbb{R}$ by

$$f(\rho) = \rho(1-\rho)^t \qquad \textit{and} \qquad F(\rho_1, \rho_2, \ldots, \rho_r) = \sum_{i=1}^{r} f(\rho_i).$$

Then F is maximized at an interior point of Ω. There are at most two possibilities for this critical point. One is the Turán solution $(\frac{1}{r}, \frac{1}{r}, \ldots, \frac{1}{r})$. The only other possibility is the skew solution $(\alpha, \alpha, \ldots, \alpha, \beta)$, with $r-1$ parts of size α and one of size β, having $f'(\alpha)$ largest. If any unbalanced solution exists, then this skew solution exists.

We start by noting that the result in [14] proves Theorem 1.8 for the cases

$$(r,t) \text{ for } r \geq 6 \text{ and } t \geq \begin{cases} 3 & \text{if } r \geq 9, \\ 4 & \text{if } r = 8, \\ 5 & \text{if } r = 7, \\ 37 & \text{if } r = 6 \end{cases} \tag{1.3}$$

Our strategy for extending this result to all $r \geq 2$ and $t \geq 2$ will depend on partitioning pairs (r, t) into four classes, one of which is completely contained in the pairs above, and providing an argument for each case. First, though, we follow the proof of [14] to establish several facts.

Note that interior critical points of F satisfy

$$\nabla F(\rho_1, \ldots, \rho_r) = (f'(\rho_1), \ldots, f'(\rho_r)) = \lambda(1, \ldots, 1).$$

Set

$$g(\rho) = f'(\rho) = (1-\rho)^{t-1}(1-(t+1)\rho)$$
$$h(\rho) = f''(\rho) = t(1-\rho)^{t-2}((t+1)\rho - 2)$$

Note that g has two critical points: a minimum at $\rho = 2/(t+1)$ and a second critical point at the boundary point $\rho = 1$. As $g(0) = 1$ and $g(1) = 0$, we conclude g is positive and decreasing until $\rho = 1/(t+1)$, where g has a zero. Then g is negative and decreasing until $\rho = 2/(t+1)$, after which it is negative and increasing until $\rho = 1$. Set $\phi_{\min} = g(2/(t+1))$. For each $\phi \in (0, 1] \cup \{\phi_{\min}\}$ there is a unique ρ such that $g(\rho) = \phi$. If $(\rho_1, \ldots, \rho_r)$ is a critical point such that

$$g(\rho_1) = \cdots = g(\rho_r) = \phi \in (0, 1] \cup \{\phi_{\min}\}$$

then, as the ρ_i sum to 1, it must be the point $(\frac{1}{r}, \ldots, \frac{1}{r})$, which we refer to as the Turán solution. Otherwise, for each $\phi \in (\phi_{\min}, 0]$ there are two ρ that satisfy $g(\rho) = \phi$, one satisfying $1/(t+1) \leq \rho < 2/(t+1)$ and another satisfying $2/(t+1) < \rho \leq 1$. Critical points in this case, without loss of generality, have the form $(\alpha, \ldots, \alpha, \beta, \ldots, \beta)$ where

$$\frac{1}{t+1} \leq \alpha < \frac{2}{t+1} \qquad \text{and} \qquad \frac{2}{t+1} < \beta \leq 1. \tag{1.4}$$

We call such solutions (a, b)-skew, where there are a copies of α and b copies of β.

We will need the following result:

Lemma 1.9 ([14] Lemma 3.6)**.** *If $(\alpha, \ldots, \alpha, \beta, \ldots, \beta)$ is an (a, b)-skew critical point with $g(\alpha) = g(\beta) = \phi$ and $a < r - 1$, then there is an $(a+1, b-1)$-skew critical point with parameters $\hat{\alpha}$ and $\hat{\beta}$ such that $g(\hat{\alpha}) = g(\hat{\beta}) = \hat{\phi} > \phi$ and*

$$\begin{aligned} F(\alpha, \ldots, \alpha, \beta, \ldots, \beta) &= af(\alpha) + bf(\beta) \\ &< (a+1)f(\hat{\alpha}) + (b-1)f(\hat{\beta}) = F(\hat{\alpha}, \ldots, \hat{\alpha}, \hat{\beta}, \ldots, \hat{\beta}). \end{aligned}$$

The proof of Lemma 1.9 is an application of the Mean Value Theorem. It contains several calculations of various derivatives which add little to the discussion, so we omit it here and refer readers to the original paper.

We are now prepared to prove Theorem 1.8.

Proof of Theorem 1.8. First we show that F is not maximized on the boundary of the domain. Suppose, without loss of generality, that $\rho_1 = 0$ and $\rho_r \neq 0$. Let $\hat{\rho_1} = \hat{\rho_r} = \frac{\rho_r}{2}$. Each term of the sum defining F, other than the first and last, remains unchanged. Originally, the first term was 0 and the last was $\rho_r(1-\rho_r)^t$. Now each term is $\frac{\rho_r}{2}(1-\frac{\rho_r}{2})^t$, giving a sum of $\rho_r(1-\frac{\rho_r}{2})^t > \rho_r(1-\rho_r)^t$.

As the domain of F is closed and bounded and F is continuous, it must achieve its maximum and thus that maximum must occur at an interior point. As proved above, such points may be Turán or (a, b)-skew.

If there are no skew critical points then the only interior critical point is the Turán solution. In this case, F must attain its maximum here.

Otherwise, there exists at least one (a, b)-skew critical point. Repeatedly applying Lemma 1.9, we see that the critical point at which F attains its maximum is either the Turán solution or an $(r-1, 1)$-skew solution. Therefore we may set $\beta = 1-(r-1)\alpha$ and, slightly abusing notation, express F as a function of one variable:

$$F(\alpha) = (r-1)f(\alpha) + f(1-(r-1)\alpha)$$

We will show that among skew solutions $F(\alpha)$ is maximized either by $\frac{1}{r}$ or that α which satisfies $g(\alpha) = g(1-(r-1)\alpha) = \phi$ and, if there are multiple such α, the one with ϕ largest.

We consider four cases:

(A) $\{(r, t) : t \leq r\}$

(B) $\{(r, t) : r \geq 7, t \geq r+1\}$

(C) $\{(r, t) : 2 \leq r \leq 6, t \geq 3r-1\}$

(D) The remaining pairs: $\{(r, t) : 2 \leq r \leq 6, r+1 \leq t \leq 3r-2\}$.

The pairs in case B are a subset of those from (1.3) and thus are covered by in [14]. For each of the finitely many case D pairs, one can simply calculate the maximum value of F and will find that in each case the maximum occurs at one of the two points. It thus falls to us to prove Theorem 1.8 in cases A and C.

We claim that for pairs in case A, no skew solution exists and thus the conditions on skew solutions in Theorem 1.8 hold vacuously.

Recall from (1.4) that $\alpha > 1/(t+1)$ and that

$$\beta = 1 - (r-1)\alpha > \frac{2}{t+1} \implies \alpha < \frac{t-1}{(r-1)(t+1)}.$$

We conclude

$$\alpha \in \left(\frac{1}{t+1}, \frac{t-1}{(r-1)(t+1)} \right).$$

Of course, if this interval is empty then we have no choices for α and, due to Lemma 1.9, no skew solution exists. Thus in order for a skew solution to exist we must have

$$\frac{t-1}{(r-1)(t+1)} > \frac{1}{t+1} \implies t-1 > r-1 \implies t > r.$$

We conclude for $t \leq r$ no skew solution exists.

Now consider pairs in case C. We use a weaker condition than that of (1.4): if a skew solution exists, $\alpha \in [0, \frac{2}{t+1})$. We claim that for $r \geq 2$ and $t \geq 3r-1$, F has exactly one critical point in this range.

Set

$$G(\alpha) = F'(\alpha) = (r-1)g(\alpha) - (r-1)g(1-(r-1)\alpha).$$

First we claim $G(0) > 0$ and $G(\frac{2}{t+1}) < 0$.

We have

$$\begin{aligned}
G(0) &= (r-1)f'(0) - (r-1)f'(1-(r-1)\cdot 0) \\
&= (r-1)[f'(0) - f'(1)] \\
&= (r-1)[(-1)(1-0)^{t-1}((t+1)\cdot 0 - 1) - (-1)(1-1)^{t-1}((t+1)\cdot 1 - 1)] \\
&= (r-1)[1-0] \\
&> 0
\end{aligned}$$

and, with Lemma A.2,

$$\begin{aligned}
G\left(\frac{2}{t+1}\right) &= (r-1)f'\left(\frac{2}{t+1}\right) - (r-1)f'\left(1-\frac{2(r-1)}{t+1}\right) \\
&= (r-1)\left[-\left(1-\frac{2}{t+1}\right)^{t-1} + \left(\frac{2(r-1)}{t+1}\right)^{t-1}(t-2(r-1))\right] \\
&\leq (r-1)\left[-\left(\frac{r-1}{r}+\frac{1}{3r}\right)^{t-1} + \left(\frac{2}{3}\cdot\frac{(r-1)}{r}\right)^{t-1}(t-2(r-1))\right] \\
&< (r-1)\left[-\left(\frac{r-1}{r}\right)^{t-1} + \left(\frac{2}{3}\cdot\frac{(r-1)}{r}\right)^{t-1} t\right] \\
&= \tfrac{3}{2}(r-1)\left(\frac{r-1}{r}\right)^{t-1}\left[t\cdot\left(\frac{2}{3}\right)^{t} - \frac{2}{3}\right].
\end{aligned}$$

Each term but the last is positive. As $r \geq 2$ and $t \geq 3r-1$, we may take $t \geq 5$ and thus Lemma A.3 implies the last term is negative. We conclude $G(\frac{2}{t+1}) < 0$ as claimed.

As G is continuous, by the Intermediate Value Theorem it has at least one root in $[0, \frac{2}{t+1})$. As $G = F'$, a root of G indicates a critical point of F.

To prove F has at most one critical point, we claim G has at most one zero in $[0, \frac{2}{t+1})$ by showing G is concave up on that interval. We have

$$G''(\alpha) = F^{(3)}(\alpha) = (r-1)f^{(3)}(\alpha) - (r-1)^3 f^{(3)}(1-(r-1)\alpha).$$

Now

$$f^{(3)}(\alpha) = t(t-1)(1-\alpha)^{t-3}(3-(t+1)\alpha) > 0,$$

as $(t+1)\alpha < 2$ and $(1-\alpha) > 0$ for $\alpha \in [0, \frac{2}{t+1})$, and

$$
\begin{aligned}
f^{(3)}(1-(r-1)\alpha) &= (-1)^3 \cdot t \cdot (t-1) \cdot (1-(1-(r-1)\alpha))^{t-3} \quad \times \\
&\quad \times \quad ((t+1)(1-(r-1)\alpha)-3) \\
&= t(t-1)((r-1)\alpha)^{t-3}(t-2-(r-1)(t+1)\alpha) \\
&> t(t-1)((r-1)\alpha)^{t-3}(3r-1-2-2(r-1)) \\
&= r-1 \\
&> 0.
\end{aligned}
$$

Thus G is concave up on $[0, \frac{2}{t+1})$ and is positive at one endpoint but negative at the other, so it has at most one zero. We conclude F has at most one critical point on $[0, \frac{2}{t+1})$ and thus at most one skew solution of type $(\alpha, \alpha, \ldots, \alpha, \beta)$ exists. If that solution exists it trivially has $f'(\alpha)$ largest.

We have demonstrated Theorem 1.8 holds in all four cases and the proof is complete. □

Next we partially determine for which r and t a Turán solution is optimal and for which a skew solution optimizes F.

Theorem 1.10. *There is a function $t^* = t^*(r)$ satisfying $r < t^* \leq r + \sqrt{2r}$ such that for $t < t^*$, F is maximized at the Turán solution and when $t \geq t^*$, F is maximized at a skew solution.*

Note when considering pairs in case A in the proof of Theorem 1.8 we proved the lower bound of Theorem 1.10. We work towards the upper bound through the following lemmas.

Lemma 1.11. *For fixed* r*, define*

$$q(t) = \left(\frac{r-1}{t+1}\right)^{1/t}\left(1 - \frac{1}{t+1}\right).$$

For $t > r-2$*,* $q(t)$ *is increasing in* t*.*

Proof. We analyze the logarithmic derivative of q:

$$\begin{aligned}
\frac{d}{dt}[\ln q(t)] &= \frac{d}{dt}\left[\ln\left(\frac{r-1}{t+1}\right)^{1/t}\left(1-\frac{1}{t+1}\right)\right] \\
&= \frac{d}{dt}\left[\frac{1}{t}\cdot\ln\left(\frac{r-1}{t+1}\right) + \ln\left(1-\frac{1}{t+1}\right)\right] \\
&= -\frac{1}{t^2}\cdot\ln\left(\frac{r-1}{t+1}\right) - \frac{1}{t}\cdot\frac{t+1}{r-1}\cdot\frac{r-1}{(t+1)^2} + \frac{1}{1-\frac{1}{t+1}}\cdot\frac{1}{(t+1)^2} \\
&= \frac{\ln(t+1)-\ln(r-1)}{t^2} - \frac{1}{t(t+1)} + \frac{1}{t(t+1)} \\
&= \frac{\ln(t+1)-\ln(r-1)}{t^2}.
\end{aligned}$$

Thus

$$\frac{dq}{dt} = q(t)\cdot\frac{d}{dt}[\ln q(t)] = \left(\left(\frac{r-1}{t+1}\right)^{1/t}\left(1-\frac{1}{t+1}\right)\right)\cdot\left(\frac{\ln(t+1)-\ln(r-1)}{t^2}\right).$$

We have $t+1 > r-1$ whenever $t > r-2$, which makes the derivative positive and shows q is increasing in t. □

Lemma 1.12. *For* $t > r-2$*, the skew solution with the largest target value, say with* $\alpha = \alpha_{\text{opt}}$*, is strictly larger than* $[q(t)]^t$*. That is,*

$$F(\alpha_{\text{opt}}) > [q(t)]^t.$$

Proof. Because α_{opt}, like all choices for α, satisfies (1.4), the optimal skew solution,

α_{opt}, has objective function at least as large as $F(\frac{1}{t+1})$. For $t > r-2$ we have $\frac{r-1}{t+1} < 1$ and so

$$\begin{aligned} F(\alpha_{\text{opt}}) \geq F(\tfrac{1}{t+1}) &= \frac{r-1}{t+1}\left(1-\frac{1}{t+1}\right)^t + \left(1-\frac{r-1}{t+1}\right)\left(\frac{r-1}{t+1}\right)^t \\ &> \frac{r-1}{t+1}\left(1-\frac{1}{t+1}\right)^t \\ &= [q(t)]^t. \end{aligned}$$

□

Lemma 1.13. *Define*

$$\Psi(r,x) = r\left(1+\frac{x}{r}\right)^{r+x}.$$

For fixed $r \geq 3$, if $\eta \geq 0$ is a root of $\Psi(r,x) - \Psi(r-1,x)$ then

$$[q(r-1+\eta)]^t = \left(1-\frac{1}{r}\right)^t = F(\tfrac{1}{r}).$$

Proof. First note that as $1-(r-1)\cdot\frac{1}{r} = \frac{1}{r}$, we have

$$F(\tfrac{1}{r}) = r\cdot\frac{1}{r}\left(1-\frac{1}{r}\right)^t = \left(1-\frac{1}{r}\right)^t.$$

Then

$$
\begin{aligned}
q(r-1+\eta) &= \left(\frac{r-1}{(r-1+\eta)+1}\right)^{\frac{1}{r-1+\eta}} \left(1-\frac{1}{(r-1+\eta)+1}\right) \\
&= \left(\frac{r-1}{r+\eta}\right)^{\frac{1}{r-1+\eta}} \left(\frac{r-1+\eta}{r+\eta}\right) \\
&= \left(\frac{(r-1)(r-1+\eta)^{r-1+\eta}}{(r+\eta)^{r+\eta}}\right)^{\frac{1}{r-1+\eta}} \\
&= \left(\frac{(r-1)(r-1+\eta)^{r-1+\eta}}{r^{r-1+\eta}\cdot r(1+\frac{\eta}{r})^{r+\eta}}\right)^{\frac{1}{r-1+\eta}} \\
&= \left(\frac{(r-1)(r-1+\eta)^{r-1+\eta}}{r^{r-1+\eta}\cdot (r-1)(1+\frac{\eta}{(r-1)})^{r-1+\eta}}\right)^{\frac{1}{r-1+\eta}} && \text{Definition of } \eta \\
&= \frac{(r-1+\eta)}{r(1+\frac{\eta}{r-1})} \\
&= \frac{r-1}{r} \\
&= 1-\frac{1}{r}
\end{aligned}
$$

Raising both sides to the t power gives the result. □

Note that if an η as described in Lemma 1.13 exists, it is unique. If not, there would be roots η_1 and η_2 satisfying $0 < \eta_1 < \eta_2$ and by Lemmas 1.11 and 1.13 we'd have

$$F(\tfrac{1}{r}) = q(r-1+\eta_1) < q(r-1+\eta_2) = F(\tfrac{1}{r}).$$

Theorem 1.14. *Suppose $x(r) \geq 0$ satisfies $\Psi(r, x(r)) - \Psi(r-1, x(r)) < 0$ for $r \geq 3$. Then for $t \geq r-1+x(r)$, the complete r-partite graphon that maximizes F is skew.*

Proof. Note that

$$\Psi(r, 0) - \Psi(r-1, 0) = r \cdot 1 - (r-1) \cdot 1 = 1 > 0.$$

As it is clear $\Psi(r,x)-\Psi(r-1,x)$ is continuous for $r\geq 3$, if

$$\Psi(r,\ell(r))-\Psi(r-1,\ell(r))<0$$

the Intermediate Value Theorem assures there is some $0<\eta<\ell(r)$ satisfying

$$\Psi(r,\eta)-\Psi(r-1,\eta)=0.$$

We have

$$\begin{aligned} F(\alpha_{\text{opt}}) &> [q(t)]^t && \text{by Lemma 1.12} \\ &\geq [q(r-1+x(r))]^t && \text{by Lemma 1.11} \\ &\geq [q(r-1+\eta)]^t && \text{again by Lemma 1.11} \\ &= F(\tfrac{1}{r}) && \text{by Lemma 1.13} \end{aligned}$$

□

We are now ready to prove the upper bound in Theorem 1.10.

Theorem 1.15. *For $t>r+\sqrt{2r}$, F is maximized by a skew solution.*

Proof. We will apply Theorem 1.14 with $x=1+\sqrt{2r}$.

By the Mean Value Theorem, there is $c\in[r-1,r]$ such that

$$\Psi(r,x)-\Psi(r-1,x)=\frac{1}{r-(r-1)}\cdot\frac{\partial\Psi}{\partial r}(c,x)=\frac{\partial\Psi}{\partial r}(c,x).$$

We have

$$\frac{\partial\Psi}{\partial r}(c,x)=\left(1+\frac{x}{c}\right)^{x+c}\left(-x+c\ln\left(1+\frac{x}{c}\right)+1\right).$$

For $r > 1$ and $x > 0$, each $c \in [r-1, r]$ satisfies $\left(1 + \frac{x}{c}\right)^{x+c} > 0$. Thus the sign of $\frac{\partial \Psi}{\partial r}(c, x)$ depends on the $-x + c \ln\left(1 + \frac{x}{c}\right) + 1$ term. Setting $x = 1 + \sqrt{2r}$, for $r \geq 26$ Lemma A.4 implies that term, and thus $\frac{\partial \Psi}{\partial r}(c, x)$, is negative for any $c \in [r-1, r]$. Thus by Theorem 1.14, F is maximized by a skew solution for $r \geq 26$ and

$$t \geq r - 1 + 1 + \sqrt{2r} = r + \sqrt{2r}.$$

Finally, we note that in order to extend the result to $2 \leq r \leq 25$, one need only evaluate

$$\Psi(r, 1 + \sqrt{2r}) - \Psi(r-1, 1 + \sqrt{2r})$$

and confirm the result is negative. In each case it is. □

We have now shown that F is maximized by the Turán solution for $t \leq r$ and by the skew solution with $f'(\alpha)$ largest for $t > r + \sqrt{2r}$. All that remains to prove Theorem 1.10 is to prove the existence of the function t^* which we do in the following theorem.

Theorem 1.16. *For any $r \geq 2$, there is $t^* = t^*(r)$ such that for any integer t, F is maximized by the Turán solution when $t < t^*$ and by a skew solution for $t \geq t^*$.*

Proof. Theorem 1.15 establishes there are pairs (r, t) for which a skew solution is optimal. Therefore for each r there is a smallest integer τ, $r < \tau \leq r + \sqrt{2r}$ for which a skew solution is optimal.

Let $\alpha \in (\frac{1}{\tau+1}, \frac{1}{r})$ such that $F_\tau(\alpha) > F_\tau(\frac{1}{r})$. We will prove $F_{\tau+1}(\alpha) > F_{\tau+1}(\frac{1}{r})$ and thus as $\alpha \in (\frac{1}{\tau+2}, \frac{1}{r}) \supseteq (\frac{1}{\tau+1}, \frac{1}{r})$,

$$\max_{\alpha' \in (\frac{1}{\tau+2}, \frac{1}{r})} F_{\tau+1}(\alpha') \geq F_{\tau+1}(\alpha) > F_{\tau+1}(\tfrac{1}{r}).$$

Recall from the proof of Lemma 1.13 that $F_\tau(\frac{1}{r}) = (1-\frac{1}{r})^\tau$. Thus we have

$$\begin{aligned}
F_\tau(\alpha) > F_\tau(\tfrac{1}{r}) &\implies F_\tau(\alpha) > \left(1-\frac{1}{r}\right)^\tau \\
&\implies F_\tau(\alpha)\cdot(1-\tfrac{1}{r}) > \left(1-\frac{1}{r}\right)^{\tau+1} \\
&\implies F_\tau(\alpha)\cdot(1-\tfrac{1}{r}) > F_{\tau+1}(\tfrac{1}{r})
\end{aligned}$$

Next note that

$$\begin{aligned}
F_\tau(\alpha)\cdot(1-\tfrac{1}{r}) = &\left(1-\frac{1}{r}\right)\left((r-1)\alpha(1-\alpha)^\tau + (1-(r-1)\alpha)((r-1)\alpha)^\tau\right) \\
=&\left(\left(1-\frac{1}{r}\right)-(1-\alpha)+(1-\alpha)\right)(r-1)\alpha(1-\alpha)^\tau \quad + \\
&+ \quad \left(\frac{r-1}{r}-(r-1)\alpha+(r-1)\alpha\right) \quad \times \\
&\times \quad (1-(r-1)\alpha)((r-1)\alpha)^\tau \\
=&\left(\alpha-\frac{1}{r}\right)(r-1)\alpha(1-\alpha)^\tau + (r-1)\alpha(1-\alpha)^{\tau+1} \quad + \\
&+ \quad (r-1)\left(\frac{1}{r}-\alpha\right)(1-(r-1)\alpha)((r-1)\alpha)^\tau \quad + \\
&+ \quad (1-(r-1)\alpha)((r-1)\alpha)^{\tau+1} \\
=&F_{\tau+1}(\alpha)-(r-1)\left(\frac{1}{r}-\alpha\right)(f_\tau(\alpha)-f_\tau(1-(r-1)\alpha))
\end{aligned}$$

As $\alpha < \frac{1}{r}$, the sign of this second term depends entirely on $f_\tau(\alpha) - f_\tau(1-(r-1)\alpha)$. Recall that

$$f_t{}'(\rho) = (1-\rho)^{t-1}(1-(t+1)\rho)$$

and note that $f_t{}' < 0$ on $\frac{1}{t+1} < \rho < 1$. Thus f is decreasing on this range, and as $\frac{1}{\tau+1} < \alpha$ and

$$\alpha < \frac{1}{r} \implies r\alpha < 1 \implies \alpha < 1-(r-1)\alpha < 1,$$

we have $f_\tau(\alpha) > f_\tau(1-(r-1)\alpha)$. Therefore we conclude

$$F_{\tau+1}(\alpha) > F_{\tau+1}(\alpha) - (r-1)\left(\frac{1}{r} - \alpha\right)(f_\tau(\alpha) - f_\tau(1-(r-1)\alpha)) > F_{\tau+1}(\tfrac{1}{r})$$

as claimed.

Finally, by induction we may apply the same reasoning to any $t > \tau$ to see that if α is the parameter of a skew solution maximizing F_t, then $F_{t+1}(\alpha) > F_{t+1}(\frac{1}{r})$ and so the pair $(r, t+1)$ also has a skew solution. Defining $t^*(r)$ to be the τ corresponding to r completes the proof. □

As a concluding remark, we note that Bollobás and Nikiforov considered the question of maximizing the degree power

$$\sum_{v \in V(G)} d(v)^t$$

in K_{r+1}-free graphs. They found, for sufficiently large n, very similar bounds for when the Turán graph maximizes the degree power in K_{r+1}-free graphs:

Theorem 1.17 (Bollobás, Nikiforov [10])**.** *For $0 < t < r$, the sum*

$$\sum_{v \in V(G)} d(v)^t$$

is maximized, among K_{r+1}-free graphs, by the Turán graph $T_r(n)$. Furthermore, for $t > r + \sqrt{2r}$ and n sufficiently large, the Turán graph is not optimal.

As the graphon extensions of counting stars and degree power are identical, Theorem 1.10 can be viewed as fairly simple extension of the above theorem to graphons. We note that our theorem also identifies the extremal graphon in the case where

Turán is not optimal and proves that, at least in the graphon case, there is a unique value of t at which the extremal case changes from Turán to skew. Finding a better bound for t^* remains an open question.

Chapter 2

Random Lifts of Regular Graphs

2.1 Background

2.1.1 Notation and Definitions

Many of the terms in this chapter, particularly regarding graphs and their chromatic number, were defined in Section 1.1.2. We say a graph is d-regular if every $v \in V(G)$ has degree exactly d. Given a labeling of $V(G)$ by $[n(G)]$, an *adjacency matrix* for G is an $n(G) \times n(G)$ square matrix with entry 1 in row i and column j if $v_i \sim v_j$ and 0 otherwise.

In this chapter we will often consider the rate of growth of a function. The statement $f(n) = O(g(n))$ means that there are fixed positive constants C and N such that for all $n \geq N$,

$$f(n) \leq C \cdot g(n).$$

In other words, the growth of f is bounded by some constant multiple of g. Conversely, the statement $f(n) = \Omega(g(n))$ means there are constants c and N such that for all $n \geq N$,

$$c \cdot g(n) \leq f(n),$$

or that f grows at least as fast as a constant multiple of g. If $f = O(g(n))$ and

$f = \Omega(g(n))$, or equivalently if there are fixed positive constants c, C, and N such that

$$c \cdot g(n) \leq f(n) \leq C \cdot g(n)$$

for $n \geq N$, then we say $f(n) = \Theta(g(n))$, indicating f grows at the same rate as g, up to a constant factor. If

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$$

then we make the stronger statement $f(n) \sim g(n)$, or f and g are asymptotically equal. Finally, the statement $f(n) = o(g(n))$ means

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$$

or that f grows at a rate strictly smaller than g.

We will also use several tools from probability theory. A *probability space* (Ω, Σ, p) is a set Ω, a σ-algebra Σ over Ω, and a function $p : \Sigma \to [0, 1]$ such that $p(\Omega) = 1$ and for countably many pairwise disjoint $\{S_i\}_{i=1}^{\infty} \subseteq \Sigma$,

$$p(\bigcup_{i=1}^{\infty} S_i) = \sum_{i=1}^{\infty} p(S_i).$$

The elements of Σ are called *events*. The function p is called the *probability distribution for* Ω and for each $S \in \Sigma$, $\Pr[S] = p(S)$ is the probability that S occurs. When Σ and p are clear, we often refer to the probability space as just Ω. A *random variable* on (Ω, Σ, p) is a p-measurable function $X : \Omega \to \mathbb{R}$. If Ω is countable and $\Sigma = 2^{\Omega}$ then X is a *discrete random variable*. The *expected value* of a discrete random variable X, denoted $\mathbf{E}[X]$, is defined as

$$\mathbf{E}[X] = \sum_{x \in X} x \cdot \Pr[X = x].$$

If Ω is a set of graphs, we call the probability space (Ω, Σ, p) a *graph model* and outcomes of this space *random graphs*.

Let $\{\Omega_n\}_{n=1}^{\infty}$ be a sequence of probability spaces. We say a sequence of events $\{Q_n\}_{n=1}^{\infty}$ occurs *asymptotically almost surely* if the probability of Q_n occurring in Ω_n tends to 1 as n tends to infinity. As a special case, if Ω_n is a sequence of graph models and $G_n \in \Omega_n$, we may say G has some property asymptotically almost surely, by which we mean the probability that G_n has that property tends to one as n tends to infinity.

We also introduce notation for several vector expressions. If $\boldsymbol{u}$ and $\boldsymbol{v}$ are vectors of the same length, we use $\langle \boldsymbol{u}, \boldsymbol{v} \rangle$ to denote the inner product of $\boldsymbol{u}$ and $\boldsymbol{v}$. If $\boldsymbol{u} = (u_1, \ldots, u_k)$ contains integers then

$$\boldsymbol{u}! = \prod_{i=1}^{k} u_i!$$

and if the entries of $\boldsymbol{u}$ sum to n then

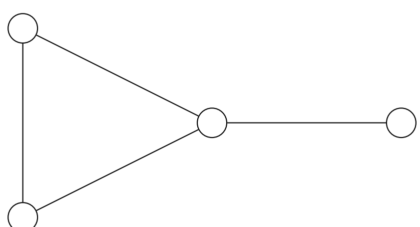
$$\binom{n}{\boldsymbol{u}} = \frac{n!}{\boldsymbol{u}!} = \frac{n!}{u_1! \cdots u_k!}$$

is a multinomial coefficient.

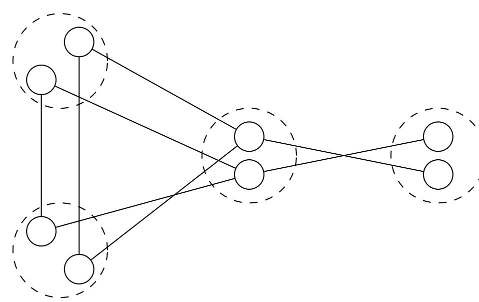
Logarithms in this section are of natural base unless otherwise indicated.

2.1.2 Covering Maps and Lifts

If L and G are graphs, we say $\Pi : V(L) \to V(G)$ is a *covering map* if Π is a surjective graph homomorphism such that if $uv \in E(G)$ then for each $x \in \Pi^{-1}(u)$ there is a unique $y \in \Pi^{-1}(v)$ such that $xy \in L$. If such a Π exists, then $|\Pi^{-1}(u)| = |\Pi^{-1}(v)|$, and in general if u is in the same connected component of G as some $v' \in V(G)$ then preimages of u and v' will have the same size. If $|\Pi^{-1}(v)| = n$ for all $v \in V(G)$ (which may not occur if G has multiple connected components) then we call L an *n-lift* of G. Note that an n-lift is similar to a blowup except instead of replacing vertices by complete bipartite graphs, they are replaced by a perfect matching which, unlike in a blowup, forces each fiber in a connected component to have the same size. Figure 2.1 contains an example of a lift.



A graph G



A sample 2-lift of G

Figure 2.1: An example of a lift

Suppose L is an n-lift of G for some $n \geq 2$. What can be said about $\chi(L)$? Given any proper coloring $\chi : V(G) \to [k]$, we can define $\tilde{\chi} : V(L) \to [k]$ by $\tilde{\chi}(v) = \chi(f(v))$. Then if $u \sim_L v$, $f(u) \sim_G f(v)$ and thus

$$\tilde{\chi}(u) = \chi(f(u)) \neq \chi(f(v)) = \tilde{\chi}(v).$$

We conclude $\chi(L) \leq \chi(G)$. However, this inequality may be far from tight. For example, let $G = K_{d+1}$ and n be an arbitrary even number. For each $u \in V(G)$, assign each $x \in f^{-1}(u)$ a label in $[n]$. Then for each $uv \in E(G)$ and each $x \in f^{-1}(u)$ with an odd label, add an edge to the vertex in $f^{-1}(v)$ with label one larger. (By symmetry, an edge will also be added between $x \in f^{-1}(u)$ with an even label to the vertex in $f^{-1}(v)$ with label one smaller.) This process gives a lift L that is bipartite (with vertices with even labels in one part and those with odd labels in the other) and so has chromatic number two regardless of d.

As we cannot improve the bounds on the chromatic number of general n-lifts, we might instead ask what can be said about a typical n-lift. Let $\mathcal{L}_n(G)$ be the random graph model where an n-lift of G is selected uniformly at random. This model of random lifts was introduced in a series of papers by Amit, Linial, Matoušek and Rozenman ([5, 6, 7, 31]). One can sample $L_n \in \mathcal{L}_n(G)$ by replacing each vertex of G with a set of n vertices and then, for each edge independently of the others, selecting a perfect matching uniformly at random. In [7], Amit, Linial, and Matoušek proved that, for any fixed graph G there is a function f_Δ, depending only on $\Delta(G)$ and that tends toward zero with $\Delta(G)$, such that asymptotically almost surely

$$\chi(L) \leq (1 + f_\Delta)\frac{\Delta(G)}{\log \Delta(G)}.$$

Moreover, they showed that, for a nonempty G, asymptotically almost surely

$$\chi(L) \geq \sqrt{\chi(G)/3 \log \chi(G)}.$$

The authors asked if there was a tighter lower bound of the form

$$\chi(L) = \Omega(\Delta(G)/\log \Delta(G))$$

to match the growth rate of the upper bound and showed that this was the case when $G = K_{d+1}$, proving that asymptotically almost surely $\chi(L) = \Theta(d/\log d)$. They conjectured that not only does such a bound exist for every graph, but for any graph G there is a single value k_G such that, asymptotically almost surely, $\chi(L) = k_G$. As $2 \leq \chi(L) \leq \chi(G)$, this conjecture is true for bipartite graphs G and in the same paper the authors prove it holds when $\chi(G) = 3$, but in general it remains open, even for cases such as $G = K_5$; in fact, the 2012 result of Farzad and Theis [18] that the conjecture holds for K_5 with an edge missing requires a complex, three-phase algorithm.

2.1.3 Introduction

Note that if G is a d-regular graph then lifts of G are also d-regular. (In fact more generally, lifts preserve the degree distribution of the base graph.) Thus one might ask about the relationship between $\mathcal{L}_n(G)$ where G is a d-regular graph and $\mathcal{G}_{n,d}$, the graph model that selects an n-vertex d-regular graph uniformly at random. In general, one can show these are fundamentally different probability spaces; in particular, the probability that a random d-regular graph is a lift of a fixed d-regular G tends to zero whereas every $L_n \in \mathcal{L}_n(G)$ is a lift by construction. As a note to the reader, which we will not make precise, were one to allow loops (edges from one vertex to itself) and adjust the definition of a lift appropriately, one could show that $\mathcal{G}_{n,d}$ is essentially $\mathcal{L}_n(B_d)$ for even d, where B_d is the bouquet graph consisting of $d/2$ loops; as we will not consider graphs with loops, we do not pursue this idea any further. Nevertheless, there has been much work done to determine the chromatic number of a random d-regular graph from the uniform model $\mathcal{G}_{n,d}$ (see [2, 13, 21, 29, 41, 40]). For instance, Kemkes, Pérez-Giménez, and Wormald determined in [29] that the chromatic number of $G_{n,d}$ is asymptotically almost surely concentrated in a two-value window $\{k_d, k_d{+}1\}$, where k_d is the smallest integer k satisfying $d < 2k \log k$, and in fact for some values of d it is concentrated exactly at k_d. Coja-Oghlan, Efthymiou, and Hetterich extended this result in [13] by proving that the one-point concentration at k_d holds for all sufficiently large values of d.

In this thesis, based on work together with Xavier Pérez-Giménez, we will prove that for $L_n \in \mathcal{L}_n(K_{d+1})$, asymptotically almost surely $\chi(L)$ falls into the same two-value window $\{k_d, k_d + 1\}$. Furthermore, we will extend the upper bound to consider general d-regular graphs.

2.2 Main Results

In this section we give our main results of this chapter: we prove an asymptotic upper bound on the chromatic number of a random lift of G for any d-regular graph G as well as an asymptotic lower bound on the chromatic number of a random lift of the complete graph K_{d+1}. We prove the lower bound with a standard first moment argument. The upper bound requires a more delicate argument. We start by using a standard second moment argument culminating in an application of the Payley-Zygmund Inequality:

Theorem (Payley-Zygmund [38])**.** *If $Y \geq 0$ is a random variable then*

$$\Pr[Y > 0] \geq \frac{(\mathbf{E}Y)^2}{\mathbf{E}Y^2}.$$

Setting Y to be the number of k-colorings of our graph[1], we would want the ratio $(\mathbf{E}Y)^2/\mathbf{E}Y^2$ to tend to 1 as n tends to infinity so that asymptotically almost surely a coloring exists and our upper bound holds. Unfortunately, we find that it does not,

[1] We we actually define Y slightly more narrowly, but thinking of the number of k-colorings is sufficient for the moment.

tending instead towards a positive constant less than one. We thus appeal to the small subgraph conditioning method of Robinson and Wormald. A full exposition for the method can be found in [46], but at a high level the method explains that the second moment $\mathbf{E}Y^2$ is being artificially inflated by the presence of small cycles in our lift. By conditioning on the presence of these cycles, we prove that the probability that a coloring exists does indeed tend towards one as n approaches infinity.

Intuition might suggest fixing d and determining upper and lower bounds for the number of colors k as a function of d. Our arguments, however, work by fixing a number of colors k and determining for which d, as a function of k, lifts of d-regular graphs can be colored with k colors. After proving our bounds, we interpret their results.

We start with the lower bound of the chromatic number of a random lift of K_{d+1}. Define

$$u_k = \frac{2\log k}{\log k - \log(k-1)}. \tag{2.1}$$

Theorem 2.1. *If $k \geq 2$ and $d \geq u_k$, then asymptotically almost surely a random n-lift L of K_{d+1} is not k-colorable.*

Note that if asymptotically almost surely no coloring exists, then k colors are not sufficient and thus asymptotically almost surely the chromatic number is larger than k. Therefore, if the chromatic number of lifts of a clique is, in fact, k asymptotically almost surely, then $d < u_k$. In other words, u_k is the upper bound for the regularity d of a clique such that lifts of that clique have chromatic number k.

Let X be the number of k-colorings of a random n-lift of K_{d+1}. We prove Theorem 2.1 as a consequence of Theorem 2.2.

Theorem 2.2. *Suppose that* $\frac{d^2-1}{d\log d} < 2(k-1)$*. Then there is a constant* M *such that*

$$\mathbf{E}X = O\left(n^M\right)\left(\frac{(k-1)^d}{k^{d-2}}\right)^{(d+1)n/2}.$$

Proof of Theorem 2.1. It suffices to prove the statement for $d = \lceil u_k \rceil$. First note that if $k = 2$ then $u_k = 2$. A lemma of Linial [7] states that if $\chi(G) \geq 3$ then asymptotically almost surely a random lift of G has chromatic number at least three so the statement holds.

Now fix $k \geq 3$. Note that in this case $u_k \notin \mathbb{Z}$, so $u_k + 1 > d > u_k$. We have

$$\begin{aligned}
\frac{d^2-1}{d\log d} &< \frac{d}{\log d} \\
&< \frac{u_k+1}{\log u_k} \\
&< \frac{(2k-1)\log k + 1}{\log u_k} && \text{Lemma A.5} \\
&< \frac{2k\log k}{\log\log k^2 - \log\log\left(\frac{1}{1-1/k}\right)} && \log k \geq \log 3 > 1 \\
&< \frac{2k\log k}{\log\log k^2 + \log k - \frac{5}{8k}} && \text{Lemma A.6} \\
&< \frac{2k\log k}{\log k + \frac{\log k}{k-1}} && \text{Lemma A.7} \\
&= 2(k-1)
\end{aligned}$$

Then we can apply Theorem 2.2, and conclude that

$$\mathbf{E}X = O(n^M)\left(\frac{(k-1)^d}{k^{d-2}}\right)^{(d+1)n/2}.$$

Now we claim that $d > u_k$ implies $\frac{(k-1)^d}{k^{d-2}} < 1$. First note that for fixed $k > 0$,

$$\frac{(k-1)^x}{k^{x-2}} = k^2\left(1-\frac{1}{k}\right)^x$$

is a decreasing function of x. Thus as

$$k^2\left(\frac{k-1}{k}\right)^{u_k} = k^2\exp\bigl(u_k(\log(k-1)-\log k)\bigr) = k^2\exp(\log k^{-2}) = 1$$

and $d > u_k$ we conclude $\frac{(k-1)^d}{k^{d-2}} < 1$ and thus $\mathbf{E}X = o(1)$. As $\mathbf{E}X$ tends to zero, asymptotically almost surely there are zero k-colorings of L and so L is not k-colorable.

□

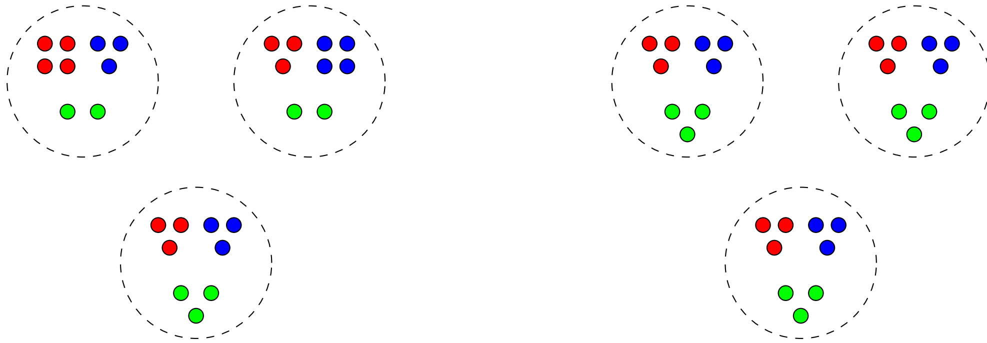
Now we turn to the upper bound for the chromatic number of a random lift which we prove for any d-regular graph G. Define

$$\ell_k = \frac{2(k-1)^3}{k(k-2)}\log(k-1). \tag{2.2}$$

Theorem 2.3. *Suppose $d < \ell_k$, and let G be any fixed d-regular graph. Then if k divides n, asymptotically almost surely a random n-lift of G is k-colorable.*

As before, if $d < \ell_k$ then, asymptotically almost surely, k colors suffice to color a random lift of G, so the chromatic number of such lifts is at most k. If the chromatic number is exactly k, then $k-1$ colors must not suffice, meaning $\ell_{k-1} \leq d$.

Let L be an n-lift of a fixed graph $G = G(V, E)$ with covering map Π. We call a proper k-coloring of L *strongly equitable* if for every $v \in V$, the set $\Pi^{-1}(v)$ is equitably colored: that is, each color is assigned to exactly n/k vertices in $\Pi^{-1}(v)$. Figure 2.2 contains an example of the difference between a typical coloring and a strongly equitable coloring. Strongly equitable colorings require n to be divisible by k, which we always assume when discussing them.



A typical 3-coloring A strongly equitable 3-coloring

Figure 2.2: Typical versus strongly equitable colorings

Let Y be the number of strongly equitable k-colorings of a random lift of G, where G is any fixed d-regular graph (not necessarily $G = K_{d+1}$). Let A be the adjacency matrix of G, and let $\alpha_1, \ldots, \alpha_{|V|}$ be the eigenvalues of A. (Note that several arguments will contain other matrices named A with various subscripts. When we refer to A alone, it will exclusively refer to the adjacency matrix of G.)

Before we prove Theorem 2.3, we require information regarding $\mathbf{E}Y$ and $\mathbf{E}Y^2$, as well as how the expectation is affected by small cycles, from the following theorems.

Theorem 2.4. *Let $k \geq 3$ and assume k divides n. Then*

$$\mathbf{E}Y \sim C_1 \left(2\pi n\right)^{-(k-1)|V|/2} \left(k^{|V|} \left(\frac{k-1}{k} \right)^{|E|} \right)^n,$$

where

$$C_1 = k^{k|V|/2} \left(\frac{(k-1)^2}{k(k-2)} \right)^{(k-1)|E|/2}.$$

For ease of notation, throughout the paper we use

$$\lambda = (k-1)^2 + 1 \quad \text{and} \quad \lambda' = (k-1)^2 - 1. \tag{2.3}$$

Theorem 2.5. *Suppose $d < \ell_k$ and k divides n. Then there is a constant $C_2 = C_2(d,k)$ such that*

$$\mathbf{E}Y^2 \sim C_2(2\pi n)^{-(k-1)|V|}\left(k^{|V|}\left(\frac{k-1}{k}\right)^{|E|}\right)^{2n}$$

where

$$C_2 = \frac{k^{(k^2-k+1)|V|}(k-1)^{(2k^2-2k)|E|}}{(\lambda)^{\frac{1}{2}(k-1)^2|E|}(\lambda')^{\frac{1}{2}(k^2-1)|E|}h(d,k)^{\frac{(k-1)^2}{2}}}$$

and

$$h(d,k) = \left(\frac{k^2}{\lambda\lambda'}\right)^{|V|}\prod_{i=1}^{|V|}(\lambda\lambda' + d - \alpha_i(k-1)^2) \tag{2.4}$$

Remark 2.6. *The restriction that n is divisible by k is an artifact of the proof which relies on the existence of strongly equitable colorings. Though not contained in this thesis, together with Xavier Pérez-Giménez I intend to use the techniques in [29], where the authors resolve a similar restriction, to extend the result to all n.*

For fixed $j \geq 3$, denote the number of j-cycles in a random lift by Z_j.

Theorem 2.7. *For $i = 1, \ldots, |V|$, let β_i^+ and β_i^- denote the roots of the quadratic $x^2 - \alpha_i x + d - 1 = 0$. That is,*

$$\beta_i^+ = \tfrac{1}{2}\alpha_i + \sqrt{\tfrac{1}{4}\alpha_i^2 - (d-1)} \quad \text{and} \quad \beta_i^- = \tfrac{1}{2}\alpha_i - \sqrt{\tfrac{1}{4}\alpha_i^2 - (d-1)}. \tag{2.5}$$

If k divides n, then for all $j \geq 3$ and $p_3, \ldots, p_j$, each a nonnegative integer,

$$\frac{\mathbf{E}(Y[Z_3]_{p_3}\cdots[Z_j]_{p_j})}{\mathbf{E}(Y)} \sim \prod_{\ell=3}^{j}(\lambda_\ell(1+\delta_\ell))^{p_\ell}$$

where

$$\lambda_j = \frac{(|E| - |V|)(1 + (-1)^j) + \sum_{i=1}^{|V|}((\beta_i^+)^j + (\beta_i^-)^j)}{2j}, \quad \delta_j = \frac{(-1)^j}{(k-1)^{j-1}}, \tag{2.6}$$

and $[A]_b$ *indicates the falling factorial moment* $A(A-1)\cdots(A-b+1)$.

We now prove Theorem 2.3 using the small subgraph conditioning method of Robinson and Wormald [46] which, for our purposes, amounts to verifying that

$$\frac{\mathbf{E}Y^2}{(\mathbf{E}Y)^2} \sim \exp\left(\sum_{j\geq 1} \lambda_j \delta_j^2\right).$$

Proof of Theorem 2.3. We begin with the right-hand side. We have

$$
\begin{aligned}
\sum_{j\geq 1}\lambda_j\delta_j^2 &= \sum_{j\geq 1}\lambda_j\cdot\frac{1}{(k-1)^{2j-2}}\\
&= \frac{(k-1)^2}{2}\Bigg((|E|-|V|)\sum_{j\geq 1}\frac{1}{j(k-1)^{2j}}+(|E|-|V|)\sum_{j\geq 1}\frac{(-1)^j}{j(k-1)^{2j}} \quad +\\
&\qquad + \quad \sum_{j\geq 1}\sum_{i=1}^{|V|}\frac{(\beta_i^+)^j+(\beta_i^-)^j}{j(k-1)^{2j}}\Bigg)\\
&= \frac{(k-1)^2}{2}\Bigg((|E|-|V|)\sum_{j\geq 1}\frac{1}{j(k-1)^{2j}}+(|E|-|V|)\sum_{j\geq 1}\frac{(-1)^j}{j(k-1)^{2j}} \quad +\\
&\qquad + \quad \sum_{i=1}^{|V|}\left(\sum_{j\geq 1}\frac{(\beta_i^+)^j}{j(k-1)^{2j}}+\sum_{j\geq 1}\frac{(\beta_i^-)^j}{j(k-1)^{2j}}\right)\Bigg)\\
&= \frac{(k-1)^2}{2}\log\Bigg(\left(\frac{(k-1)^2}{(k-1)^2-1}\right)^{|E|-|V|}\left(\frac{(k-1)^2}{(k-1)^2+1}\right)^{|E|-|V|} \quad\times\\
&\qquad \times \quad \prod_{i=1}^{|V|}\left(\frac{(k-1)^2}{(k-1)^2-\beta_i^+}\right)\left(\frac{(k-1)^2}{(k-1)^2-\beta_i^-}\right)\Bigg)\\
&= \frac{(k-1)^2}{2}\log\Bigg(\left(\frac{(k-1)^2}{\lambda'}\right)^{|E|-|V|}\left(\frac{(k-1)^2}{\lambda}\right)^{|E|-|V|} \quad\times\\
&\qquad \times \quad \prod_{i=1}^{|V|}\left(\frac{(k-1)^4}{(k-1)^4-(\beta_i^++\beta_i^-)(k-1)^2+\beta_i^+\beta_i^-}\right)\Bigg)\\
&= \frac{(k-1)^2}{2}\log\Bigg(\left(\frac{(k-1)^2}{\lambda'}\right)^{|E|-|V|}\left(\frac{(k-1)^2}{\lambda}\right)^{|E|-|V|} \quad\times\\
&\qquad \times \quad \prod_{i=1}^{|V|}\left(\frac{(k-1)^4}{\lambda\lambda'+d-\alpha_i(k-1)^2}\right)\Bigg)
\end{aligned}
$$

Exponentiating,

$$\exp\left(\sum_{j\geq 1}\lambda_j\delta_j^2\right) = \left(\frac{((k-1)^2)^{|E|-|V|+|E|-|V|+2|V|}}{(\lambda\lambda')^{|E|-|V|}\prod_{i=1}^{|V|}(\lambda\lambda'+d-\alpha_i(k-1)^2)}\right)^{\frac{1}{2}(k-1)^2}$$
$$= \left(\frac{((k-1)^2)^{2|E|}}{(\lambda\lambda')^{|E|-|V|}\prod_{i=1}^{|V|}(\lambda\lambda'+d-\alpha_i(k-1)^2)}\right)^{\frac{1}{2}(k-1)^2}$$

Now we turn to the left-hand side of the proposition. Using Theorems 2.4 and 2.5,

$$\frac{\mathbf{E}Y^2}{(\mathbf{E}Y)^2} \sim \frac{C_2(2\pi n)^{-(k-1)|V|}\left(k^{|V|}\left(\frac{k-1}{k}\right)^{|E|}\right)^{2n}}{\left(C_1(2\pi n)^{-(k-1)|V|/2}\left(k^{|V|}\left(\frac{k-1}{k}\right)^{|E|}\right)^{n}\right)^2} = \frac{C_2}{C_1^2}.$$

and

$$\frac{C_2}{C_1^2} = \frac{k^{(k^2-k+1)|V|-\frac{1}{2}(k^2-1)|E|}(k-1)^{(2k^2-2k)|E|}}{\lambda^{\frac{1}{2}(k-1)^2|E|}(k-2)^{\frac{1}{2}(k^2-1)|E|}h(d,k)^{\frac{(k-1)^2}{2}}\left(k^{k|V|/2}\left(\frac{(k-1)^2}{k(k-2)}\right)^{(k-1)|E|/2}\right)^2}$$
$$= \frac{k^{(k-1)^2|V|-\frac{1}{2}(k-1)^2|E|}(k-1)^{2(k-1)^2|E|}}{\lambda^{\frac{1}{2}(k-1)^2|E|}(k-2)^{\frac{1}{2}(k-1)^2|E|}h(d,k)^{\frac{(k-1)^2}{2}}}$$
$$= \frac{k^{(k-1)^2|V|}}{\left(\left(\frac{k^2}{\lambda\lambda'}\right)^{|V|}\prod_{i=1}^{|V|}(\lambda\lambda'+d-\alpha_i(k-1)^2)\right)^{\frac{(k-1)^2}{2}}}\cdot\left(\frac{(k-1)^{4|E|}}{(\lambda\lambda')^{|E|}}\right)^{(k-1)^2/2}$$
$$= \left(\frac{((k-1)^2)^{2|E|}}{(\lambda\lambda')^{|E|-|V|}\prod_{i=1}^{|V|}(\lambda\lambda'+d-\alpha_i(k-1)^2)}\right)^{\frac{1}{2}(k-1)^2}$$

from which we conclude

$$\frac{\mathbf{E}Y^2}{(\mathbf{E}Y)^2} \sim \exp\left(\sum_{j\geq 1}\lambda_j\delta_j^2\right)$$

as required. □

By applying our upper bound to the specific case $G = K_{d+1}$, we prove the following result which, as in the current best bounds for the chromatic number of $G \in \mathcal{G}_{n,d}$, gives two possible values for the chromatic number of random lifts of K_{d+1}:

Corollary 2.8. *Let u_k be as defined in* (2.1) *and ℓ_k as defined in* (2.2). *For each $d \geq 2$, asymptotically almost surely the chromatic number of a random lift of K_{d+1} is k_d or $k_d + 1$ where k_d is the smallest integer satisfying $d < u_{k_d}$. Furthermore, if $\lfloor u_{k_d} \rfloor < \lceil \ell_{k_d} \rceil$ then asymptotically almost surely the chromatic number of a random lift of K_{d+1} is exactly k_d.*

Proof. We are given d and asked for bounds on the chromatic number of a random lift of K_{d+1}. Theorems 2.1 and 2.3 imply that the chromatic number could be concentrated at k only if $d \in [\ell_{k-1}, u_k)$. How many such intervals could contain d? We claim at most two: if k satisfies $d \in [\ell_{k-1}, u_k)$ then $d \notin [\ell_{k+1}, u_{k+2})$ as

$$u_k < 2k \log k < \frac{2k^3}{k^2 - 1} \log k = \ell_{k+1}$$

where we've used Lemma A.5 to bound u_k.

Note that as d must be an integer, we get the tighter bounds $d \in [\lceil \ell_{k-1} \rceil, \lfloor u_k \rfloor)$. Thus if $\lfloor u_k \rfloor \leq \lceil \ell_k \rceil$, $d \notin [\lceil \ell_k \rceil, \lfloor u_{k+1} \rfloor)$ and k_d is the unique integer that satisfies both Theorem 2.1 and Theorem 2.3. □

Note that Corollary 2.8 does not resolve the question mentioned in Section 2.1.2 regarding the chromatic number of random lifts of K_5. If L is a random lift of K_5, our bounds give $3 \leq \chi(L) \leq 4$ asymptotically almost surely, matching the current best bounds [18]. The corollary does provide, however, a proof of the novel result that asymptotically almost surely the chromatic number of a random lift of K_7 is four, along with exactly determining the chromatic number for the lifts of many other values of complete graphs. Computer analysis suggests the unique concentration point condition is met by roughly half of the choices for d.

The remainder of this chapter contains the proofs of the theorems required for the arguments of our main results. In Section 2.3 we establish some results we will need for the other arguments. The goal of Section 2.4 is to prove Theorems 2.2 and 2.4. Then in Section 2.5 we prove Theorem 2.5. Finally, we prove Theorem 2.7 in Section 2.6.

2.3 Useful Tools

2.3.1 Optimization Over Stochastic Matrices

In this section we will introduce some inequalities concerning stochastic matrices. A matrix M is called *row-stochastic* if all of its entries are nonnegative and the sum of the entries in each row equals one, or equivalently, if each row of M defines a probability distribution. Similarly, M is *column-stochastic* if M^T is row-stochastic, and M is *doubly-stochastic* if it is both row- and column-stochastic.

Stochastic matrices occur naturally when estimating the second moment of the number of colorings of a random graph. For instance, suppose that C_1 and C_2 are two equitable k-vertex-colorings. Then we can describe how C_1 and C_2 correlate in terms of a doubly-stochastic matrix $M = (m_{i,j})_{i,j \in [k]}$, where each entry $m_{i,j}$ denotes the (appropriately rescaled) proportion of vertices that receive color i in C_1 and color j in C_2. In second moment calculations, one typically seeks a small set of pairs of colorings C_1, C_2 that contribute all but a negligible total of the moment, which can be formulated as an optimization problem over the set of doubly-stochastic $k \times k$ matrices.

The first inequality in this section, concerning square row-stochastic matrices, was proved by Achlioptas and Naor in [3]. They used this tool to obtain an accurate second moment estimate of the number of colorings of an Erdős-Rényi random graph of constant average degree. Their inequality has also been used in the context of uniform random regular graphs (see [2, 29]).

Before stating the result, we require some additional notation. Given a $q \times k$ matrix $M = (m_{i,j})_{i\in[q],j\in[k]}$, let

$$\rho(M) = \sum_{i=1}^{q}\sum_{j=1}^{k} m_{i,j}^2 \qquad \text{and} \qquad h(M) = -\sum_{i=1}^{q}\sum_{j=1}^{k} m_{i,j}\log m_{i,j}.$$

If, in addition, M is row-stochastic, then a standard convexity argument shows that

$$\frac{q}{k} \le \rho(M) \le q, \tag{2.7}$$

where the minimum $\rho(M) = q/k$ is uniquely attained when all $m_{i,j} = 1/k$, and the maximum $\rho(M) = q$ is achieved precisely by those row-stochastic matrices with all entries in $\{0, 1\}$.

For each integer $q \ge 3$, define

$$c_q = \frac{(q-1)^3}{q(q-2)}\log(q-1). \tag{2.8}$$

Proposition 2.9 (Achlioptas and Naor [3]). *Let $M = (m_{i,j})$ be a $q \times q$ row-stochastic matrix, where $q \ge 3$. Then, for any $c < c_q$,*

$$\frac{1}{q}h(M) + c\log\left(q^2 - 2q + \rho(M)\right) \le \log q + c\log\left((q-1)^2\right),$$

Furthermore, we have equality if and only if $m_{i,j} = \frac{1}{q}$ for all $i, j \in [q]$.

Remark 2.10. *The second claim in the proposition is not explicitly stated in [3]. However, it follows from their proof that equality holds if and only if $\rho(M) = 1$, which, in view of (2.7), corresponds to the case when all $m_{i,j} = \frac{1}{q}$.*

We will use Proposition 2.9 in our second moment calculations in Section 2.5 to compute Y, the number of strongly equitable k-colorings of a random lift of an arbitrary d-regular graph. Our estimate of $\mathbf{E}Y^2$ will involve solving an optimization problem which we will reduce to the inequality in Proposition 2.9. Furthermore, and somewhat surprisingly, while Achlioptas and Naor's inequality typically arises in second moment arguments, we will require a more general version of the inequality to handle a first moment calculation. Because we must concern ourselves with both a random lift and a random coloring of that lift, this first moment calculation takes on more of a second moment flavor. When we bound the expected number X of (not necessarily equitable) k-colorings of a random lift of K_{d+1} in Section 2.4, we will use a generalization of Proposition 2.9 that applies to rectangular $q \times k$ row-stochastic matrices.

Proposition 2.11. *Let $M = (m_{i,j})$ be a $q \times k$ row-stochastic matrix, where $q \geq k \geq 3$. Then, for any $c < \frac{k-1}{q-1} c_q$,*

$$\frac{1}{q} h(M) + c \log\left(kq - k - q + \frac{k}{q}\,\rho(M)\right) \leq \log k + c \log\left((q-1)(k-1)\right)$$

or

$$c \log\left(1 + \frac{\rho(M) \cdot \frac{k}{q} - 1}{(q-1)(k-1)}\right) \leq \log k - \frac{1}{q} h(M)$$

with equality if and only if $m_{i,j} = \frac{1}{k}$.

Proof. Our first step is to extend M to a $q \times q$ row-stochastic matrix $\hat{M}$ by adding $q - k$ extra columns and scaling each column appropriately. More precisely, let $\hat{M} = (\hat{m}_{i,j})_{i,j \in [q]}$ with

$$\hat{m}_{i,j} = \begin{cases} m_{i,j} \cdot \frac{k}{q} & 1 \leq j \leq k \\ \frac{1}{q} & k+1 \leq j \leq q. \end{cases}$$

As M was was row-stochastic, $\hat{M}$ is a $q \times q$ row-stochastic matrix. Hence, we can apply Proposition 2.9 to $\hat{M}$ with $\hat{c} := \frac{q-1}{k-1}c < c_q$, and obtain

$$\hat{c}\log\left(1+\frac{\rho(\hat{M})-1}{(q-1)^2}\right) \leq \log q - \frac{1}{q}h(\hat{M}), \tag{2.9}$$

which holds with equality if and only if $\hat{m}_{i,j} = \frac{1}{q}$ (which is in turn equivalent to $m_{i,j} = \frac{1}{k}$).

Our next task is to derive relations between $\rho(\hat{M})$ and $\rho(M)$ and also between $h(\hat{M})$ and $h(M)$, which will allow us to express the inequality in (2.9) in terms of $\rho(M)$ and $h(M)$. We have

$$\begin{aligned}
\rho(\hat{M}) &= \sum_{i,j=1}^{q} \hat{m}_{i,j}^2 \\
&= \frac{k^2}{q^2}\sum_{i,j=1}^{k} m_{i,j}^2 + q\,(q-k)\,\frac{1}{q^2} \\
&= \frac{k}{q}\left(\frac{\rho(M)k}{q}\right) + \frac{q-k}{q} \\
&= \frac{k}{q}\left(\frac{\rho(M)k}{q} - 1\right) + 1
\end{aligned} \tag{2.10}$$

Given a probability distribution $x = (x_1, \ldots, x_\ell)$, let $H(x) = -\sum_{j=1}^{k} x_j \log x_j$ denote the entropy of x. For each $i \in [q]$, let $m_i = (m_{i,1}, \ldots, m_{i,k})$ and $\hat{m}_i = (\hat{m}_{i,1}, \ldots, \hat{m}_{i,q})$. These are the i^{th} rows of matrices M and $\hat{M}$, which we regard as probability distributions. Also, let $u = (\frac{1}{q-k}, \ldots, \frac{1}{q-k})$ be the uniform distribution on a set of $q-k$ elements.

Note that $\hat{m}_i$ is a mixture of $m_i = (m_{i,1}, \ldots, m_{i,k})$ and $u = (\frac{1}{q-k}, \ldots, \frac{1}{q-k})$, in the sense that we can sample from $\hat{m}_i$ by sampling from m_i with probability $\frac{k}{q}$ and from

u with probability $\frac{q-k}{q}$. Hence, by the chain rule,

$$\begin{aligned}
H(\hat{m}_i) &= \frac{k}{q}H(m_i) + \frac{q-k}{q}H(u) + H(\tfrac{k}{q}, \tfrac{q-k}{q}) \\
&= \frac{k}{q}H(m_i) + \frac{q-k}{q}\log(q-k) + \frac{k}{q}(\log q - \log k) + \frac{q-k}{q}(\log q - \log(q-k)) \\
&= \log q - \frac{k}{q}\log k + \frac{k}{q}H(m_i).
\end{aligned}$$

Noting that $h(M) = \sum_{i=1}^{q} H(m_i)$ and $h(\hat{M}) = \sum_{i=1}^{q} H(\hat{m}_i)$, we conclude

$$\begin{aligned}
\log k - \frac{1}{q}h(M) &= \frac{1}{k}\sum_{i=1}^{q}\left(\frac{k}{q}\log k - \frac{k}{q}H(m_i)\right) \\
&= \frac{1}{k}\sum_{i=1}^{q}(\log q - H(\hat{m}_i)) \\
&= \frac{q}{k}\left(\log q - \frac{1}{q}h(\hat{M})\right) \qquad (2.11)
\end{aligned}$$

In view of (2.10) and (2.11), we can use the inequality in (2.9) to write

$$\begin{aligned}
\log k - \frac{1}{q}h(M) &= \frac{q}{k}\left(\log q - \frac{1}{q}h(\hat{M})\right) \\
&\geq \frac{q}{k}\hat{c}\log\left(1 + \frac{\rho(\hat{M}) - 1}{(q-1)^2}\right) \\
&= \frac{q}{k}\hat{c}\log\left(1 + \frac{\frac{k}{q}\left(\frac{\rho(M)k}{q} - 1\right)}{(q-1)^2}\right)
\end{aligned}$$

Note that $\frac{k}{q}\left(\frac{\rho(M)k}{q} - 1\right) > -\frac{k}{q} > -(q-1)^2$. Since $f(x) = \log\left(1 + \frac{x}{(q-1)^2}\right)$ is a concave function for $x > -(q-1)^2$ and $f(0) = 0$, Jensen's inequality gives

$$\log\left(1 + \frac{\frac{k}{q}\left(\frac{\rho(M)k}{q} - 1\right)}{(q-1)^2}\right) \geq \frac{k}{q}\log\left(1 + \frac{\left(\frac{\rho(M)k}{q} - 1\right)}{(q-1)^2}\right),$$

which combined with the inequality above yields

$$\log k - \frac{1}{q}h(M) \geq \hat{c}\log\left(1 + \frac{\left(\frac{\rho(M)k}{q} - 1\right)}{(q-1)^2}\right). \tag{2.12}$$

In order to further bound the right-hand side from below, we introduce a new function

$$g(y) = y\log\left(1 + \frac{a}{y}\right), \qquad \text{where} \quad a = \frac{1}{q-1}\left(\frac{\rho(M)k}{q} - 1\right).$$

From (2.7), we know $0 \leq a \leq 1$. We claim that $g(y)$ is nondecreasing for all $y > 0$. Indeed, its derivative satisfies

$$g'(y) = -\log\left(1 - \frac{a}{y+a}\right) - \frac{a}{y+a} \geq 0,$$

where we've used $\log(1+x) \leq x$ for $x > -1$. Hence, from (2.12),

$$\log k - \frac{1}{q}h(M) \geq \frac{\hat{c}}{q-1}\, g(q-1) \geq \frac{\hat{c}}{q-1}\, g(k-1) = c\log\left(1 + \frac{\frac{\rho(M)k}{q} - 1}{(q-1)(k-1)}\right),$$

which is equivalent to

$$\frac{1}{q}h(M) + c\log\left(kq - k - q + \frac{k}{q}\,\rho(M)\right) \leq \log k + c\log\left((q-1)(k-1)\right),$$

as desired. Finally, in view of our remark below (2.9), equality holds if and only if all $m_{i,j} = 1/k$. This completes the proof of the proposition. □

2.3.2 Laplace Summation Over Lattices

Here we will prove Theorem 2.13, which we will use in Sections 2.4.2 and 2.5.4 to precisely estimate expectations expressed as sums, as a consequence of a similar result of Greenhill, Janson and Ruciński [25].

We include here the definitions required to state their theorem, but refer the reader to [25] for additional background. A *lattice* Λ is an additive subgroup of $\mathbb{R}^N$ such that every bounded region in $\mathbb{R}^N$ contains a finite number of elements of Λ. It is well known that every lattice Λ is isomorphic to $\mathbb{Z}^r$, for some integer r with $0 \leq r \leq N$ which we call the *rank* of Λ. Moreover, every lattice Λ of rank $r \geq 1$ admits a *basis* $u_1, \ldots, u_r \in \Lambda$ such that every point in Λ can be uniquely represented as a linear combination $m_1 u_1 + \cdots + m_r u_r$ with integer coefficients m_i.

Given a lattice Λ with a basis $u_1, \ldots, u_r$, we define the *determinant* of the lattice as

$$\det(\Lambda) = \sqrt{\det U^T U},$$

where U is an $N \times r$ matrix with columns $u_1, \ldots, u_r$. This quantity does not depend on our choice of the basis. If $\mathbb{V} \subseteq \mathbb{R}^N$ is an r-dimensional vector space spanning Λ (and thus with basis $u_1, \ldots, u_r$), for any symmetric $N \times N$ matrix H we define

$$\det(H|_{\mathbb{V}}) = \frac{\det U^T H U}{\det U^T U}. \tag{2.13}$$

This value also does not depend on our choice of basis.

Theorem 2.12 (Greenhill, Janson and Ruciński [25])**.** *Suppose the following:*

(i) $\Lambda \subset \mathbb{R}^N$ *is a lattice with rank* $1 \leq r \leq N$*.*

(ii) $\mathbb{V} \subseteq \mathbb{R}^N$ *is the* r*-dimensional subspace spanned by* Λ*.*

(iii) $\mathbb{W} = \mathbb{V} + \boldsymbol{w}$ *is an affine subspace parallel to* $\mathbb{V}$*, for some* $\boldsymbol{w} \in \mathbb{R}^N$*.*

(iv) $K \subset \mathbb{R}^N$ *is a compact convex set with non-empty interior* K°*.*

(v) *$\phi : K \to \mathbb{R}$ is a continuous function and the restriction of ϕ to $K \cap \mathbb{W}$ has a unique maximum at some point $\boldsymbol{z}_0 \in K^\circ \cap \mathbb{W}$.*

(vi) *ϕ is twice continuously differentiable in a neighborhood of $\boldsymbol{z}_0$ and $H := D^2\phi(\boldsymbol{z}_0)$ is its Hessian at $\boldsymbol{z}_0$.*

(vii) *$\psi : K_1 \to \mathbb{R}$ is a continuous function on some neighborhood $K_1 \subseteq K$ of $\boldsymbol{z}_0$ with $\psi(\boldsymbol{z}_0) > 0$.*

(viii) *For each positive integer n there is a vector $\ell_n \in \mathbb{R}^N$ with $\ell_n/n \in \mathbb{W}$,*

(ix) *For each positive integer n there is a positive real number b_n and a function $a_n : (\Lambda + \ell_n) \cap nK \to \mathbb{R}$ such that, as $n \to \infty$,*

$$a_n(\ell) = O(b_n e^{n\phi(\ell/n)+o(n)}), \quad \ell \in (\Lambda + \ell_n) \cap nK$$

and

$$a_n(\ell) = b_n(\psi(\ell/n) + o(1))e^{n\phi(\ell/n)}, \quad \ell \in (\Lambda + \ell_n) \cap nK_1,$$

uniformly for ℓ in the indicated sets.

Then provided $\det(-H|_{\mathbb{V}}) \neq 0$, *as $n \to \infty$,*

$$\sum_{\ell \in (\Lambda+\ell_n)\cap nK} a_n(\ell) \sim \frac{(2\pi)^{r/2}\psi(\boldsymbol{z}_0)}{\det(\Lambda)\det(-H|_{\mathbb{V}})^{1/2}} b_n n^{r/2} e^{n\phi(\boldsymbol{z}_0)}.$$

In order to state our theorem, we need some additional definitions. Let $\Gamma = (V_\Gamma, E_\Gamma)$ be a non-empty multigraph (possibly with multiple edges, but no loops). Fix an arbitrary orientation of the edges in E_Γ. The *signed incidence matrix* of Γ (with respect to that orientation) is a $|V_\Gamma| \times |E_\Gamma|$ matrix $\tilde{D} = (\tilde{D}_{v,e})_{v\in V_\Gamma, e\in E_\Gamma}$, where $\tilde{D}_{v,e} = 1$ if v is the tail of e, $\tilde{D}_{v,e} = -1$ if v is the head of e and $\tilde{D}_{v,e} = 0$ otherwise.

Similarly, the *unsigned incidence matrix* of Γ is a matrix $D = (D_{v,e})_{v \in V_\Gamma, e \in E_\Gamma}$, where $D_{v,e} = |\tilde{D}_{v,e}|$, and does not depend on the orientation of the edges. Finally, recall the definition of $\det(H|_{\mathbb{V}})$ from (2.13).

Theorem 2.13. *Suppose the following:*

(i) $\Gamma = (V_\Gamma, E_\Gamma)$ *is a non-empty bipartite multigraph with at least one cycle.*

(ii) D *is the unsigned incidence matrix of* Γ *.*

(iii) $\tau(\Gamma)$ *is the number of maximal forests in* Γ*.*

(iv) $\mathbb{V} = \mathrm{Ker}(D) \subseteq \mathbb{R}^{|E_\Gamma|}$ *is a vector space of dimension* r*.*

(v) $\boldsymbol{y} \in \mathbb{R}^{|V_\Gamma|}$ *such that*

$$D\boldsymbol{x} = \boldsymbol{y} \tag{2.14}$$

is a consistent linear system.

(vi) $K \subset \mathbb{R}^{|E_\Gamma|}$ *is a compact convex set with non-empty interior* K°*.*

(vii) $\phi : K \to \mathbb{R}$ *is a continuous function and the maximum of* ϕ *in* K *subject to* (2.14) *is attained at a unique maximizer* $\hat{\boldsymbol{x}} \in K^\circ$*.*

(viii) ϕ *is twice continuously differentiable in a neighborhood of* $\hat{\boldsymbol{x}}$ *and* H *is its Hessian matrix at* $\hat{\boldsymbol{x}}$*.*

(ix) $\psi : K_1 \to \mathbb{R}$ *is a continuous function on some neighborhood* $K_1 \subseteq K$ *of* $\hat{\boldsymbol{x}}$ *with* $\psi(\hat{\boldsymbol{x}}) > 0$*.*

(x) For each positive integer n*,*

$$\mathbb{X}_n = \left\{ \boldsymbol{x} \in K \cap \frac{1}{n}\mathbb{Z}^{|E_\Gamma|} : D\boldsymbol{x} = \boldsymbol{y} \right\}$$

is non-empty, and there is a a positive real number c_n *and a function* $T_n : \mathbb{X}_n \to \mathbb{R}$ *such that, as* $n \to \infty$*,*

$$T_n(\boldsymbol{x}) = O(c_n e^{n\phi(\boldsymbol{x})+o(n)}), \quad \boldsymbol{x} \in \mathbb{X}_n$$

and

$$T_n(\boldsymbol{x}) = c_n(\psi(\boldsymbol{x}) + o(1))e^{n\phi(\boldsymbol{x})}, \quad \boldsymbol{x} \in \mathbb{X}_n \cap K_1,$$

uniformly for $\boldsymbol{x}$ *in the indicated sets.*

Then provided $\det(-H|_{\mathbb{V}}) \neq 0$*, as* $n \to \infty$*,*

$$\sum_{\boldsymbol{x} \in \mathbb{X}_n} T_n(\boldsymbol{x}) \sim \frac{\psi(\hat{\boldsymbol{x}})}{\tau(\Gamma)^{1/2} \det(-H|_{\mathbb{V}})^{1/2}} (2\pi n)^{r/2} c_n e^{n\phi(\hat{\boldsymbol{x}})}.$$

Furthermore, in the case that Γ *is not bipartite, the theorem remains valid if we replace* D *by the signed incidence matrix* $\tilde{D}$ *of* Γ *(with respect to a fixed orientation of* E_Γ*).*

Proof of Theorem 2.13. We start by proving the more general result where Γ is not bipartite in which we use the signed incidence matrix $\tilde{D}$.

Let $\Lambda = \mathbb{V} \cap \mathbb{Z}^{|E_\Gamma|}$ so that Λ is the set of all integer solutions of $\tilde{D}\boldsymbol{x} = \boldsymbol{0}$. Clearly, Λ is a lattice that spans $\mathbb{V}$. As it spans $\mathbb{V}$, we have $\dim \mathbb{V} = \operatorname{rank} \Lambda = r$. As Γ is nonempty, D is not the zero matrix and so $r \geq 1$.

Set $\mathbb{W} = \{\boldsymbol{x} : \tilde{D}\boldsymbol{x} = \boldsymbol{y}\}$. As we are guaranteed $\tilde{D}\boldsymbol{x} = \boldsymbol{y}$ is consistent, there is some $\boldsymbol{w} \in \mathbb{R}^N$ such that $\tilde{D}\boldsymbol{w} = \boldsymbol{y}$. For any $\boldsymbol{v} \in \mathbb{V}$ we have

$$\tilde{D}(\boldsymbol{v} + \boldsymbol{w}) = \tilde{D}\boldsymbol{v} + \tilde{D}\boldsymbol{w} = 0 + \boldsymbol{y} = \boldsymbol{y}$$

so $\mathbb{V} + \boldsymbol{w} \subseteq \mathbb{W}$. Furthermore, for any $\boldsymbol{x} \in \mathbb{W}$ we have

$$\tilde{D}(\boldsymbol{x} - \boldsymbol{w}) = \tilde{D}\boldsymbol{x} - \tilde{D}\boldsymbol{w} = \boldsymbol{y} - \boldsymbol{y} = 0$$

so $\boldsymbol{x} - \boldsymbol{w} \in \mathbb{V}$ and $\boldsymbol{x} = (\boldsymbol{x} - \boldsymbol{w}) + \boldsymbol{w} \in \mathbb{V} + \boldsymbol{w}$. We conclude $\mathbb{W} = \mathbb{V} + \boldsymbol{w}$ is an affine subspace parallel to $\mathbb{V}$.

Note that the conditions on K, ϕ and ψ in Theorem 2.13 exactly match those of Theorem 2.12 with $\hat{\boldsymbol{x}}$ replaced with $\boldsymbol{z}_0$.

For each positive integer n we have that $\mathbb{X}_n$ is nonempty, so choose $\boldsymbol{x}_n \in \mathbb{X}_n$ and set $\ell_n = n\boldsymbol{x}_n$. Then $\ell_n/n = \boldsymbol{x}_n$ satisfies $\tilde{D}\boldsymbol{x}_n = \boldsymbol{y}$ by the definition of $\mathbb{X}_n$ so $\ell_n/n \in \mathbb{W}$.

Let $b_n = c_n$ and define $a_n : (\Lambda + \ell_n) \cap nK \to \mathbb{R}$ by $a_n(\ell) = T(\ell/n)$. To see a_n is well defined, let $\ell \in (\Lambda + \ell_n) \cap nK$ and take $\boldsymbol{x} = \frac{1}{n}\ell$. Then certainly $\boldsymbol{x} \in K$. As $\ell_n = n\boldsymbol{x}_n$, we have $\boldsymbol{x} \in \frac{1}{n}\Lambda + \boldsymbol{x}_n$, and as $\Lambda \subseteq \mathbb{Z}^{|E_\Gamma|}$ and $\boldsymbol{x}_n \in \mathbb{X} \subseteq \frac{1}{n}\mathbb{Z}$ we get $\boldsymbol{x} \in \frac{1}{n}\mathbb{Z}$ as well. Finally, setting $\boldsymbol{x} = \frac{1}{n}\lambda + \boldsymbol{x}_n$ for some $\lambda \in \Lambda$, we have

$$\tilde{D}\boldsymbol{x} = \tilde{D}(\tfrac{1}{n}\lambda + \boldsymbol{x}_n) = \tfrac{1}{n}\tilde{D}\lambda + \tilde{D}\boldsymbol{x}_n = 0 + \boldsymbol{y} = \boldsymbol{y}$$

from which we conclude $\frac{1}{n}\ell = \boldsymbol{x} \in \mathbb{X}_n$ and a_n is well defined. Furthermore,

$$a_n(\ell) = T_n(\boldsymbol{x}) = O(c_n e^{n\phi(\boldsymbol{x})+o(n)}) = O(b_n e^{n\phi(\ell/n)+o(n)}), \quad \ell \in (\Lambda + \ell_n) \cap nK$$

and

$$a_n(\ell) = T_n(\boldsymbol{x}) = c_n(\psi(\boldsymbol{x})+o(1))e^{n\phi(\boldsymbol{x})} = b_n(\psi(\ell/n)+o(1))e^{n\phi(\ell/n)}, \quad \ell \in (\Lambda+\ell_n)\cap nK_1$$

uniformly for ℓ as required.

Thus as Theorem 2.13 requires $\det(-H|_{\mathbb{V}}) \neq 0$, we apply Theorem 2.12 to get

$$\begin{aligned}\sum_{x\in\mathbb{X}_n} T_n(\boldsymbol{x}) &= \sum_{\ell\in(\Lambda+\ell_n)\cap nK} a_n(\ell)\\ &\sim \frac{(2\pi)^{r/2}\psi(\boldsymbol{z}_0)}{\det(\Lambda)\det(-H|_{\mathbb{V}})^{1/2}} b_n n^{r/2} e^{n\phi(\boldsymbol{z}_0)}\\ &= \frac{\psi(\hat{\boldsymbol{x}})}{\det(\Lambda)\det(-H|_{\mathbb{V}})^{1/2}}(2\pi n)^{r/2} c_n e^{n\phi(\hat{\boldsymbol{x}})}\end{aligned}$$

All that remains to complete the proof is to show $\det(\Lambda) = \tau(\Gamma)^{1/2}$. Lemma 14.7.3 in [24] gives this result in the case where Γ is connected; to extend the result to disconnected Γ we apply the result to each component and take a product.

Lastly, we consider the case where Γ is bipartite. We claim we may use the unsigned incidence matrix D rather than $\tilde{D}$. Select one of the parts, say $L \subseteq V_\Gamma$, and orient E_Γ from L toward the other part $R = V_\Gamma \setminus L$. Then the coefficients in each of the first $|L|$ rows of $\tilde{D}$ are positive while the last $|R|$ rows contain all negative entries. Thus one can obtain D from $\tilde{D}$ by multiplying the last R rows by -1. These are elementary row operations and do not change the kernel, so we conclude $\operatorname{Ker} D = \operatorname{Ker} \tilde{D}$ and our definition of $\mathbb{V}$ is unchanged. Now suppose $\boldsymbol{y}$ is the vector we're given such that $D\boldsymbol{x} = \boldsymbol{y}$ is a consistent linear system. Define $\tilde{\boldsymbol{y}}$ by

$$\tilde{y}_i = \begin{cases} y_i & 1 \le i \le |L| \\ -y_i & |L|+1 \le i \le |L|+|R| \end{cases}$$

Then $\tilde{D}\boldsymbol{x} = \tilde{\boldsymbol{y}}$ if and only if $D\boldsymbol{x} = \boldsymbol{y}$ so we proceed with the rest of the proof above replacing $\boldsymbol{y}$ with $\hat{\boldsymbol{y}}$. □

2.4 First Moment Calculations

2.4.1 Coloring Optimization and Proof of Theorem 2.2

Let X be a random variable denoting the number of proper k-colorings of a random lift L of K_{d+1}. The main goal of this section is to prove Theorem 2.2 by providing an upper bound on $\mathbf{E}X$. Additionally, some of the ideas developed here will be utilized again in Section 2.4.2 when we study equitable k-colorings of a random lift of a general d-regular graph. Throughout this section, we use V and E to denote the vertex and edge sets of K_{d+1}, respectively, so in particular $|V| = d + 1$ and $|E| = \binom{d+1}{2}$. For convenience, we fix an arbitrary orientation of the edges in E, so that for each pair of different vertices $v, v' \in V$ exactly one of vv' and $v'v$ belongs to E.

To calculate $\mathbf{E}X$, we will count the number of pairs (L, C) such that L is a lift of K_{d+1} and C is a proper k-coloring of L, and then divide by the number of lifts of K_{d+1}. First observe that we can build any lift L of K_{d+1} by selecting a perfect matching between the fibers $\Pi^{-1}(v)$ and $\Pi^{-1}(v')$, for each edge $vv' \in E$. As a result, there are exactly $n!^{|E|}$ possible lifts.

Let us fix a pair (L, C), where L is a lift of K_{d+1} and C is a proper k-coloring of L. For each $v \in V$, we consider a (row) vector $a_v = (a_{v,i})_{i\in[k]}$, where $a_{v,i}$ denotes the proportion of vertices in fiber $\Pi^{-1}(v)$ that receive color i. By construction, the entries of each a_v are in $\frac{1}{n}\mathbb{Z}$ and define a probability distribution on the vertices in the fiber, which is to say

$$a_{v,i} \geq 0 \quad \forall v \in V, i \in [k] \qquad \text{and} \qquad \sum_{i\in[k]} a_{v,i} = 1 \quad \forall v \in V. \tag{2.15}$$

We write $\boldsymbol{a} = (a_v)_{v\in V} \in \frac{1}{n}\mathbb{Z}^{k|V|}$, which we also regard as a $|V|\times k$ row-stochastic matrix as each row $a_v \in \frac{1}{n}\mathbb{Z}^k$ is a probability distribution. Similarly, for each $e = vv' \in E$, we define $b_e = (b_{e,i,i'})_{i,i' \in [k], i \neq i'}$, where each $b_{e,i,i'}$ denotes the proportion of edges in $\Pi^{-1}(e)$ that connect a vertex of color i in $\Pi^{-1}(v)$ to a vertex of color i' in $\Pi^{-1}(v')$. We write $\boldsymbol{b} = (b_e)_{e \in E}$. The entries of each b_e must be in $\frac{1}{n}\mathbb{Z}$ and satisfy

$$\begin{aligned} b_{e,i,i'} &\geq 0 \quad \forall e \in E, i, i' \in [k], i \neq i' \\ \sum_{i' \neq i} b_{e,i,i'} &= a_{v,i} \quad \forall e = vv' \in E, i \in [k] \\ \sum_{i \neq i'} b_{e,i,i'} &= a_{v',i'} \quad \forall e = vv' \in E, i' \in [k]. \end{aligned} \tag{2.16}$$

For each choice of parameters $\boldsymbol{a} \in \frac{1}{n}\mathbb{Z}^{k|V|}$ and $\boldsymbol{b} \in \frac{1}{n}\mathbb{Z}^{k(k-1)|E|}$ satisfying (2.15) and (2.16), we will enumerate all pairs (L, C) which agree on those parameters and then sum over all possible choices of $\boldsymbol{a}$ and $\boldsymbol{b}$.

Given $\boldsymbol{a}$ and $\boldsymbol{b}$, we generate a pair (L, C) in three steps. First, we assign colors to the vertices of L so that, for each $v \in V$ and $i \in [k]$, exactly $a_{v,i}n$ vertices in $\Pi^{-1}(v)$ receive color i. There are

$$\prod_{v \in V} \frac{n!}{\prod_{i \in [k]} (a_{v,i}n)!} = \prod_{v \in V} \binom{n}{a_v n}$$

ways to do this. Next, for every $e = vv' \in E$ and distinct colors $i, i' \in [k]$, we need to decide which sets of $b_{e,i,i'}n$ vertices in $\Pi^{-1}(v)$ and $\Pi^{-1}(v')$ will be matched. We can do that in

$$\prod_{\substack{e \in E \\ e = vv'}} \left(\prod_{i \in [k]} \frac{(a_{v,i}n)!}{\prod_{i' \neq i} (b_{e,i,i'}n)!} \right) \left(\prod_{i' \in [k]} \frac{(a_{v',i'}n)!}{\prod_{i \neq i'} (b_{e,i,i'}n)!} \right) = \prod_{\substack{e \in E \\ e = vv'}} \frac{(a_v n)!}{(b_e n)!} \frac{(a_{v'} n)!}{(b_e n)!}$$

many ways. Finally, we need to choose a perfect matching between these sets, which can be done in

$$\prod_{e\in E}\prod_{\substack{i,i'\in[k]\\ i\neq i'}}(b_{e,i,i'}n)! = \prod_{e\in E}(b_e n)!$$

different ways. Putting everything together, we get

$$\mathbf{E}X = \frac{1}{n!^{|E|}}\sum_{\boldsymbol{a},\boldsymbol{b}}\prod_{v\in V}\binom{n}{a_v n}\prod_{\substack{e\in E\\ e=vv'}}\frac{(a_v n)!(a_{v'}n)!}{(b_e n)!}, \tag{2.17}$$

where the sum is over all $\boldsymbol{a} \in \frac{1}{n}\mathbb{Z}^{k|V|}$ and $\boldsymbol{b} \in \frac{1}{n}\mathbb{Z}^{k(k-1)|E|}$ satisfying (2.15) and (2.16).

Recall that one can write Stirling's formula as $x! = \xi(x)(x/e)^x$, where ξ is a function satisfying $\xi(x) \sim \sqrt{2\pi x}$ as $x \to \infty$ and $\xi(x) \geq 1$ for all $x \geq 0$. Then, using (2.15) and (2.16), and after some tedious but simple calculations,

$$\begin{aligned}
\mathbf{E}X &= \big(\xi(n)(n/e)^n\big)^{-|E|}\sum_{\boldsymbol{a},\boldsymbol{b}}\prod_{v\in V}\frac{\xi(n)(n/e)^n}{\prod_i \xi(a_{v,i}n)(a_{v,i}n/e)^{a_{v,i}n}} \quad \times \\
&\qquad \times \quad \prod_{\substack{e\in E\\ e=vv'}}\frac{\big(\prod_i \xi(a_{v,i}n)(a_{v,i}n/e)^{a_{v,i}n}\big)\big(\prod_{i'} \xi(a_{v',i'}n)(a_{v',i'}n/e)^{a_{v',i'}n}\big)}{\prod_{\substack{i,i'\\ i\neq i'}} \xi(b_{e,i,i'}n)(b_{e,i,i'}n/e)^{b_{e,i,i'}n}} \\
&= \sum_{\boldsymbol{a},\boldsymbol{b}}\prod_{v\in V}\frac{\xi(n)}{\prod_i \xi(a_{v,i}n){a_{v,i}}^{a_{v,i}n}} \quad \times \\
&\qquad \times \quad \prod_{\substack{e\in E\\ e=vv'}}\frac{\big(\prod_i \xi(a_{v,i}n)\big)\big(\prod_{i'} \xi(a_{v',i'}n)\big)\left(\prod_{\substack{i,i'\\ i\neq i'}}(a_{v,i}a_{v',i'})^{b_{e,i,i'}n}\right)}{\xi(n)\prod_{\substack{i,i'\\ i\neq i'}}\xi(b_{e,i,i'}n){b_{e,i,i'}}^{b_{e,i,i'}n}} \\
&= \sum_{\boldsymbol{a},\boldsymbol{b}} p(\boldsymbol{a},\boldsymbol{b},n)e^{nf(\boldsymbol{a},\boldsymbol{b})},
\end{aligned} \tag{2.18}$$

where

$$p(\boldsymbol{a},\boldsymbol{b},n) = \prod_{v\in V}\frac{\xi(n)}{\prod_i \xi(a_{v,i}n)}\prod_{\substack{e\in E\\ e=vv'}}\frac{\big(\prod_i \xi(a_{v,i}n)\big)\big(\prod_{i'} \xi(a_{v',i'}n)\big)}{\xi(n)\prod_{\substack{i,i'\\ i\neq i'}}\xi(b_{e,i,i'}n)} \tag{2.19}$$

and

$$f(\boldsymbol{a},\boldsymbol{b}) = -\sum_{v\in V}\sum_{i} a_{v,i}\log a_{v,i} + \sum_{\substack{e\in E\\ e=vv'}}\sum_{\substack{i,i'\\ i\neq i'}} b_{e,i,i'}\log\left(\frac{a_{v,i}a_{v',i'}}{b_{e,i,i'}}\right). \tag{2.20}$$

In order to bound the exponential behaviour of $\mathbf{E}X$, we will maximize $f(\boldsymbol{a},\boldsymbol{b})$ and show that the main contribution to the sum in (2.18) comes from the term where all the parameters are equal. Let $\hat{a} = (\hat{a}_i)_{i\in[k]}$ with all $\hat{a}_i = 1/k$, and $\hat{b} = (\hat{b}_{i,i'})_{i,i'\in[k],i\neq i'}$ with all $\hat{b}_{i,i'} = 1/k(k-1)$. Also, define $\hat{\boldsymbol{a}} = (\hat{a})_{v\in V}$ and $\hat{\boldsymbol{b}} = (\hat{b})_{e\in E}$. Note that $\hat{\boldsymbol{a}}, \hat{\boldsymbol{b}}$ are only a valid set of parameters if $k(k-1) \mid n$, since otherwise their entries are not in $\frac{1}{n}\mathbb{Z}$. However, since we are only interested in an upper bound on $\mathbf{E}X$ we may proceed with these values by bounding $\mathbf{E}X$ in a larger space that contains $\frac{1}{n}\mathbb{Z}$.

Proposition 2.14. *Suppose that $\frac{d^2-1}{d\log d} < 2(k-1)$. Let $f(\boldsymbol{a},\boldsymbol{b})$ be defined as in (2.20). Then the maximum of $f(\boldsymbol{a},\boldsymbol{b})$ for $\boldsymbol{a} \in \mathbb{R}^{k|V|}$ and $\boldsymbol{b} \in \mathbb{R}^{k(k-1)|E|}$ subject to (2.15) and (2.16) is uniquely attained at the point in which $\boldsymbol{a} = \hat{\boldsymbol{a}}$ and $\boldsymbol{b} = \hat{\boldsymbol{b}}$, and equals*

$$f(\hat{\boldsymbol{a}},\hat{\boldsymbol{b}}) = \log\left(\frac{(k-1)^d}{k^{d-2}}\right)^{(d+1)/2}.$$

Before proving Proposition 2.14, we need the following technical result.

Lemma 2.15. *For any $\boldsymbol{a} = (a_v)_{v\in V}$ with $a_v = (a_{v,i})_{i\in[k]} \in \mathbb{R}^k$ satisfying (2.15),*

$$\sum_{\substack{e\in E\\ e=vv'}} \log\big(1-\langle a_v, a_{v'}\rangle\big) \le \binom{d+1}{2}\log\left(1-\frac{d+1}{dk}+\frac{1}{d(d+1)}\sum_{v\in V}\langle a_v,a_v\rangle\right).$$

Proof. Let $c = \sum_{v\in V} a_v$ and $j = (1,\ldots,1) \in \mathbb{R}^k$. From (2.15), we have that $\langle j, a_v\rangle = 1$ and thus $\langle j, c\rangle = d+1$. In particular,

$$1-\langle a_v, a_{v'}\rangle \ge 1-\langle a_v, j\rangle = 0, \tag{2.21}$$

so the left-hand side of the inequality in the lemma is defined, if possibly $-\infty$. Moreover, by the Cauchy-Schwarz inequality,

$$\langle c, c\rangle = \frac{1}{k}\langle j, j\rangle\langle c, c\rangle \geq \frac{1}{k}\langle j, c\rangle^2 = \frac{(d+1)^2}{k}.$$

Hence, by writing

$$\rho = \sum_{v\in V}\langle a_v, a_v\rangle\,,$$

we obtain

$$\sum_{\substack{v,v'\in V\\ v\neq v'}}\langle a_v, a_{v'}\rangle = \sum_{v,v'\in V}\langle a_v, a_{v'}\rangle - \sum_{v\in V}\langle a_v, a_v\rangle = \langle c, c\rangle - \rho \geq \frac{(d+1)^2}{k} - \rho. \tag{2.22}$$

Before we proceed to prove our main inequality, recall that E is the edge set of K_{d+1} (with a fixed orientation). Thus we can write

$$\begin{aligned}\sum_{\substack{e\in E\\ e=vv'}}\log\big(1-\langle a_v, a_{v'}\rangle\big) &= \frac{1}{2}\sum_{\substack{v,v'\in V\\ v\neq v'}}\log\big(1-\langle a_v, a_{v'}\rangle\big)\\ &= \frac{1}{2}\log\prod_{\substack{v,v'\in V\\ v\neq v'}}\big(1-\langle a_v, a_{v'}\rangle\big)\\ &= \binom{d+1}{2}\log\left(\prod_{\substack{v,v'\in V\\ v\neq v'}}\big(1-\langle a_v, a_{v'}\rangle\big)\right)^{1/d(d+1)}.\end{aligned}$$

We apply the inequality of arithmetic and geometric means to all $1-\langle a_v, a_{v'}\rangle$

above (which are nonnegative, as observed in (2.21)), and conclude that

$$\sum_{\substack{e\in E\\ e=vv'}} \log\left(1-\langle a_v, a_{v'}\rangle\right) \le \binom{d+1}{2} \log\left(\frac{1}{d(d+1)} \sum_{\substack{v,v'\in V\\ v\neq v'}} \left(1-\langle a_v, a_{v'}\rangle\right)\right).$$

Combining this inequality with (2.22) yields

$$\sum_{\substack{e\in E\\ e=vv'}} \log\left(1-\langle a_v, a_{v'}\rangle\right) \le \binom{d+1}{2} \log\left(\frac{1}{d(d+1)}\left(d(d+1) - \frac{(d+1)^2}{k} + \rho\right)\right),$$

which completes the proof of the lemma. □

Proof of Proposition 2.14. Assume throughout the proof that $\frac{d^2-1}{d\log d} < 2(k-1)$. Fix any $\boldsymbol{a} \in \mathbb{R}^{k|V|}$ satisfying (2.15), and let

$$g(\boldsymbol{a}) = h(\boldsymbol{a}) + \binom{d+1}{2} \log\left(\frac{d(d+1) - (d+1)^2/k + \rho(\boldsymbol{a})}{d(d+1)}\right),$$

where

$$h(\boldsymbol{a}) = -\sum_{v\in V}\sum_{i\in[k]} a_{v,i}\log a_{v,i} \qquad \text{and} \qquad \rho(\boldsymbol{a}) = \sum_{v\in V} \langle a_v, a_v\rangle .$$

We will maximize $f(\boldsymbol{a}, \boldsymbol{b})$ for such fixed $\boldsymbol{a}$ and with $\boldsymbol{b} \in \mathbb{R}^{k(k-1)|E|}$ subject only to

$$b_{e,i,i'} \ge 0 \qquad \text{and} \qquad \sum_{\substack{i,i'\in[k]\\ i\neq i'}} b_{e,i,i'} = 1. \tag{2.23}$$

In view of this relaxation, we can regard each $b_e = (b_{e,i,i'})_{i,i'\in[k],i\neq i'}$ as an arbitrary probability distribution, and maximize each term $\sum_{\substack{i,i'\\ i\neq i'}} b_{e,i,i'} \log\left(\frac{a_{v,i}a_{v',i'}}{b_{e,i,i'}}\right)$ in (2.20) separately. For each $e = vv' \in E$, we define another probability distribution given by

$$b^*_{e,i,i'} = \frac{a_{v,i}a_{v',i'}}{z_e}, \qquad \text{for} \quad i, i' \in [k], i \neq i',$$

where

$$z_e = \sum_{\substack{i,i' \\ i \neq i'}} a_{v,i}a_{v',i'}$$

is the normalizing factor, and we write $b^*_e = (b^*_{e,i,i'})_{i,i' \in [k], i \neq i'}$. Then

$$\sum_{\substack{i,i' \\ i \neq i'}} b_{e,i,i'} \log \left(\frac{a_{v,i}a_{v',i'}}{b_{e,i,i'}} \right) = \log z_e - D_{KL}(b_e \| b^*_e),$$

where $D_{KL}(b_e \| b^*_e) = \sum_{\substack{i,i' \\ i \neq i'}} b_{e,i,i'} \log \left(\frac{b_{e,i,i'}}{b^*_{e,i,i'}} \right)$ is the Kullback-Leibler divergence from b_e to b^*_e. By Gibb's inequality, $D_{KL}(b_e \| b^*_e) \geq 0$ with equality if and only if $b_e = b^*_e$. As a result,

$$\max_{\boldsymbol{b} \text{ s.t. } (2.23)} f(\boldsymbol{a}, \boldsymbol{b}) = h(\boldsymbol{a}) + \sum_{e \in E} \log z_e, \tag{2.24}$$

with one unique maximizer at $\boldsymbol{b} = \boldsymbol{b}^* := (b^*_e)_{e \in E}$. Note that if $\boldsymbol{a} = \hat{\boldsymbol{a}}$ then $\boldsymbol{b}^* = \hat{\boldsymbol{b}}$. We proceed to bound the right-hand side of (2.24). In view of (2.15), we can write

$$\log z_e = \log \sum_{i \in [k]} a_{v,i} \left(1 - a_{v',i}\right) = \log \left(1 - \sum_{i \in [k]} a_{v,i}a_{v',i} \right) = \log(1 - \langle a_v, a_{v'} \rangle),$$

and then, by Lemma 2.15,

$$\sum_{e \in E} \log z_e = \sum_{\substack{e \in E \\ e = vv'}} \log(1 - \langle a_v, a_{v'} \rangle) \leq \binom{d+1}{2} \log \left(1 - \frac{d+1}{dk} + \frac{1}{d(d+1)} \rho(\boldsymbol{a}) \right),$$

so

$$h(\boldsymbol{a}) + \sum_{e \in E} \log z_e \leq g(\boldsymbol{a}).$$

Combining this with (2.24) yields

$$\max_{\boldsymbol{b} \text{ s.t. (2.23)}} f(\boldsymbol{a}, \boldsymbol{b}) \leq g(\boldsymbol{a}). \tag{2.25}$$

We will bound $g(\boldsymbol{a})$ by applying Proposition 2.11 to $\boldsymbol{a} = (a_{v,i})_{v \in V, i \in [k]}$, which is a row-stochastic $(d+1) \times k$ matrix. Note that our assumption $\frac{d^2-1}{d \log d} < 2(k-1)$ implies, with some algebraic manipulation,

$$\frac{d}{2} < \frac{k-1}{d} \frac{d^3 \log d}{d^2 - 1} = \frac{k-1}{d} c_{d+1},$$

where c_{d+1} is defined as in (2.8). Therefore, Proposition 2.11 (with $q = d+1$, $c = d/2$ and $M = \boldsymbol{a}$) yields

$$\frac{d}{2} \log \left(\frac{d(k-1) + \frac{k}{d+1} \rho(\boldsymbol{a}) - 1}{d(k-1)} \right) \leq \log k - \frac{1}{d+1} h(\boldsymbol{a}),$$

with equality if and only if $\boldsymbol{a} = \hat{\boldsymbol{a}}$. After some manipulation, we can rewrite this as

$$h(\boldsymbol{a}) + \binom{d+1}{2} \log \left(\frac{d(k-1) + \frac{k}{d+1} \rho(\boldsymbol{a}) - 1}{dk} \right) \leq (d+1) \log k + \binom{d+1}{2} \log \left(\tfrac{d(k-1)}{dk} \right),$$

which immediately implies

$$h(\boldsymbol{a}) + \binom{d+1}{2} \log \left(1 - \frac{d+1}{dk} + \frac{\rho(\boldsymbol{a})}{d(d+1)} \right) \leq (d+1) \log k + \binom{d+1}{2} \log \left(\frac{k-1}{k} \right).$$

Therefore, noting that $\rho(\hat{\boldsymbol{a}}) = (d+1)/k$,

$$g(\boldsymbol{a}) \leq g(\hat{\boldsymbol{a}}) = \log \left(\frac{(k-1)^d}{k^{d-2}} \right)^{(d+1)/2} = f(\hat{\boldsymbol{a}}, \hat{\boldsymbol{b}}),$$

with equality if and only if $\boldsymbol{a} = \hat{\boldsymbol{a}}$. Combining this with (2.25), we obtain the desired bound

$$\max_{\boldsymbol{a},\boldsymbol{b} \text{ s.t. (2.15) and (2.16)}} f(\boldsymbol{a}, \boldsymbol{b}) \leq \max_{\boldsymbol{a} \text{ s.t. (2.15)}} \max_{\boldsymbol{b} \text{ s.t. (2.23)}} f(\boldsymbol{a}, \boldsymbol{b}) \leq \max_{\boldsymbol{a} \text{ s.t. (2.15)}} g(\boldsymbol{a}) = f(\hat{\boldsymbol{a}}, \hat{\boldsymbol{b}}).$$

Note that for $\boldsymbol{a} \neq \hat{\boldsymbol{a}}$, we have $g(\boldsymbol{a}) < g(\hat{\boldsymbol{a}})$ and thus $f(\boldsymbol{a}, \boldsymbol{b}) < f(\hat{\boldsymbol{a}}, \hat{\boldsymbol{b}})$. Furthermore, if $\boldsymbol{a} = \hat{\boldsymbol{a}}$ but $\boldsymbol{b} \neq \hat{\boldsymbol{b}}$ then, recalling that the maximum in (2.24) is uniquely attained at $\boldsymbol{b}^* = \hat{\boldsymbol{b}}$, we conclude that $f(\hat{\boldsymbol{a}}, \boldsymbol{b}) < g(\hat{\boldsymbol{a}}) = f(\hat{\boldsymbol{a}}, \hat{\boldsymbol{b}})$. This finishes the proof. □

We now proceed to prove the main theorem in this section.

Proof of Theorem 2.2. Recall from (2.18) that

$$\mathbf{E}X = \sum_{\boldsymbol{a},\boldsymbol{b}} p(\boldsymbol{a}, \boldsymbol{b}, n) e^{n f(\boldsymbol{a}, \boldsymbol{b})},$$

where we sum over all $\boldsymbol{a} \in \frac{1}{n}\mathbb{Z}^{k|V|}$ and $\boldsymbol{b} \in \frac{1}{n}\mathbb{Z}^{k(k-1)|E|}$ satisfying (2.15) and (2.16). In particular, a crude upper bound on the number of terms is given by

$$O\left(n^{k|V|+k(k-1)|E|}\right),$$

since each entry in $\boldsymbol{a}$ or $\boldsymbol{b}$ can take at most $n+1$ values. Moreover, from (2.19) and since $1 \leq \xi(x) = O(\sqrt{x})$ as $x \to \infty$, we can bound the polynomial factor $p(\boldsymbol{a}, \boldsymbol{b}, n)$ in each term by

$$p(\boldsymbol{a}, \boldsymbol{b}, n) = O\left(n^{|V|+2k|E|}\right), \tag{2.26}$$

where the hidden constant in the big O notation does not depend on $\boldsymbol{a}$ or $\boldsymbol{b}$. Hence, by Proposition 2.14,

$$\mathbf{E}X \le e^{nf(\hat{\boldsymbol{a}},\hat{\boldsymbol{b}})} \sum_{\boldsymbol{a},\boldsymbol{b}} p(\boldsymbol{a},\boldsymbol{b},n) = O\left(n^{(d+1)(k+1)+\binom{d+1}{2}k(k+1)}\right)\left(\frac{(k-1)^d}{k^{d-2}}\right)^{(d+1)n/2},$$

which completes the proof of the theorem. □

2.4.2 Strongly Equitable Colorings and Proof of Theorem 2.4

In this section we prove Theorem 2.4 regarding the expected number of strongly equitable colorings of a random lift. Here we allow G to be any fixed d-regular graph, not necessarily $G = K_{d+1}$, and always assume that n is divisible by k. Let Y be the number of strongly equitable k-colorings of a random n-lift of G.

Note that the only place in Section 2.4.1 where we used the fact that $G = K_{d+1}$ was in the proof of Lemma 2.15. Therefore, equation (2.18) is still valid for a general d-regular graph G. In particular, restricting the sum to the terms in which $\boldsymbol{a} = \hat{\boldsymbol{a}}$ gives the expected number of strongly equitable colorings, that is

$$\mathbf{E}Y = \sum_{\boldsymbol{b}} p(\hat{\boldsymbol{a}},\boldsymbol{b},n)e^{nf(\hat{\boldsymbol{a}},\boldsymbol{b})}, \tag{2.27}$$

with

$$p(\hat{\boldsymbol{a}},\boldsymbol{b},n) = \left(\frac{\xi(n)}{\xi(n/k)^k}\right)^{|V|}\left(\frac{\xi(n/k)^{2k}}{\xi(n)}\right)^{|E|}\prod_{e\in E}\prod_{\substack{i,i'\\ i\neq i'}}\frac{1}{\xi(b_{e,i,i'}n)} \tag{2.28}$$

$$f(\boldsymbol{a},\boldsymbol{b}) = |V|\log k - |E|\log k^2 - \sum_{e\in E}\sum_{\substack{i,i'\\ i\neq i'}} b_{e,i,i'}\log b_{e,i,i'}, \tag{2.29}$$

and where the sum is over all $\boldsymbol{b} \in \frac{1}{n}\mathbb{Z}^{k(k-1)|E|}$ satisfying

$$\begin{aligned} b_{e,i,i'} &\geq 0 \quad \forall e \in E, i, i' \in [k], i \neq i' \\ \sum_{i' \neq i} b_{e,i,i'} &= 1/k \quad \forall e \in E, i \in [k] \\ \sum_{i \neq i'} b_{e,i,i'} &= 1/k \quad \forall e \in E, i' \in [k]. \end{aligned} \tag{2.30}$$

Moreover, the discussion in the proof of Proposition 2.14 leading to (2.24) still holds for general G, and gives for $\boldsymbol{a} = \hat{\boldsymbol{a}}$

$$\max_{\boldsymbol{b} \text{ s.t. } (2.23)} f(\hat{\boldsymbol{a}}, \boldsymbol{b}) = h(\hat{\boldsymbol{a}}) + \sum_{e \in E} \log z_e = |V| \log k + |E| \log \left(\frac{k-1}{k} \right) = f(\hat{\boldsymbol{a}}, \hat{\boldsymbol{b}}),$$

with one unique maximizer at $\boldsymbol{b} = \hat{\boldsymbol{b}}$. (Note that we did not use Lemma 2.15 to bound $\sum_{e \in E} \log z_e$, since we are only concerned about the case $\boldsymbol{a} = \hat{\boldsymbol{a}}$.) Since $\hat{\boldsymbol{b}}$ trivially satisfies (2.30), we obtain the following analogue of Proposition 2.14 for general d-regular G but restricted to strongly equitable colorings.

Theorem 2.16. *Let $f(\hat{\boldsymbol{a}}, \boldsymbol{b})$ be defined as in* (2.20)*. Then the maximum of $f(\hat{\boldsymbol{a}}, \boldsymbol{b})$ subject to* (2.30) *is uniquely attained at the point in which $\boldsymbol{b} = \hat{\boldsymbol{b}}$, and equals*

$$f(\hat{\boldsymbol{a}}, \hat{\boldsymbol{b}}) = \log \left(k^{|V|} \left(\frac{k-1}{k} \right)^{|E|} \right).$$

We will estimate the sum in (2.27) by using our version of Laplace's method, Theorem 2.13.

Proof of Theorem 2.4. We begin by defining a bipartite graph $\Gamma = \Gamma(V_\Gamma, E_\Gamma)$ so that we may express the equality constraints in (2.30) in terms of its unsigned incidence matrix D and use Theorem 2.13. The idea is to associate each equation in (2.30) to a vertex of Γ and every variable to an edge in a way that preserves the incidence relations. To do this, we assign label $w_{e,1,i}$ to equation $\sum_{i' \neq i} b_{e,i,i'} = 1/k$ and label $w_{e,2,i'}$ to equation $\sum_{i \neq i'} b_{e,i,i'} = 1/k$. The vertex set of Γ is $V_\Gamma = V_{\Gamma,1} \cup V_{\Gamma,2}$, where

$$V_{\Gamma,1} = \{w_{e,1,i} : e \in E, i \in [k]\} \qquad \text{and} \qquad V_{\Gamma,2} = \{w_{e,2,i'} : e \in E, i' \in [k]\}$$

are the two sides of a bipartition. The edge set is

$$E_\Gamma = \{b_{e,i,i'} : e \in E, i, i' \in [k], i \neq i'\},$$

where each edge $b_{e,i,i'}$ has endpoints $w_{e,1,i}$ and $w_{e,2,i'}$ (i.e. the labels of the two equations in which variable $b_{e,i,i'}$ appears). Then the equality constraints in (2.30) are equivalent to

$$D\boldsymbol{b} = \boldsymbol{y}, \tag{2.31}$$

where D is the unsigned incidence matrix of Γ and $\boldsymbol{y}$ is the vector in $\mathbb{R}^{|V_\Gamma|}$ whose entries are all $1/k$. The equations in (2.31) are consistent, since they admit the solution

$$\begin{aligned} b_{e,i,i'} &= 0 \quad \forall e \in E, i, i' \in [k], i' \notin \{i, i+1\} \\ b_{e,i,i+1} &= \frac{1}{k} \quad \forall e \in E, i \in [k], \end{aligned} \tag{2.32}$$

where we use arithmetic modulo k for indices i, i'. We observe a few easy facts about Γ. First,

$$|V_\Gamma| = 2k|E| \qquad \text{and} \qquad |E_\Gamma| = k(k-1)|E|$$

Also, Γ has exactly $|E|$ connected components. More precisely, for each $e \in E$, the set of all vertices of the form $w_{e,1,i}$ or $w_{e,2,i'}$ induces a connected component of Γ.

Each of these components is isomorphic to $F := K_{k,k} - M$, the complete bipartite graph $K_{k,k}$ minus one perfect matching M. In particular, Γ has at least one cycle (since $k \geq 3$). Since Γ is bipartite, it is well known (see e.g. Theorem 8.2.1 in [24]) that D has rank $|V_\Gamma| - |E|$, and therefore $\mathbb{V} = \mathrm{Ker}(D)$ has dimension

$$r = |E_\Gamma| - |V_\Gamma| + |E| = (k^2 - 3k + 1)|E|.$$

Now we calculate $\tau(\Gamma)$. Since each maximal forest in Γ is bijectively determined by selecting a spanning tree in each component, we conclude that the number of maximal forests in Γ is

$$\tau(\Gamma) = \tau(F)^{|E|}.$$

Naturally, next we count the number of spanning trees of F.

Let I_k and J_k denote the $k \times k$ identity matrix and the $k \times k$ matrix whose entries are all 1s, respectively. With an appropriate ordering of the vertices, the adjacency matrix of F is

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes (J_k - I_k),$$

and has eigenvalues

$$\{k - 1, (1)_{k-1}, (-1)_{k-1}, 1 - k\},$$

where the subindices indicate multiplicities. Therefore, the Laplacian matrix

$$Q = (k - 1)I_{2k} - A$$

of F has eigenvalues

$$\{0, (k - 2)_{k-1}, (k)_{k-1}, 2k - 2\}.$$

By Kirchhoff's Matrix Tree Theorem [12], the number of spanning trees of F is

$$\tau(F) = \frac{1}{2k}(k-2)^{k-1}k^{k-1}(2k-2) = (k-2)^{k-1}k^{k-2}(k-1)$$

and thus

$$\tau(\Gamma) = \left((k-1)k^{k-2}(k-2)^{k-1}\right)^{|E|}. \tag{2.33}$$

Let

$$K = \{\boldsymbol{b} \in \mathbb{R}^{|E_\Gamma|} : 0 \le b_{e,i,i'} \le 1/k\}$$

and

$$K_1 = \left\{\boldsymbol{b} \in \mathbb{R}^{|E_\Gamma|} : \tfrac{0.9}{k(k-1)} \le b_{e,i,i'} \le \tfrac{1.1}{k(k-1)}\right\}.$$

Clearly, K is a compact convex set with non-empty interior K°, and any choice of $\boldsymbol{b}$ that satisfies (2.30) lies inside K. Let

$$\phi(\boldsymbol{b}) = f(\hat{\boldsymbol{a}}, \boldsymbol{b}) = |V| \log k - |E| \log k^2 - \sum_{e \in E} \sum_{\substack{i,i' \\ i \neq i'}} b_{e,i,i'} \log b_{e,i,i'},$$

which is continuous on K, and

$$\psi(\boldsymbol{b}) = \prod_{e \in E} \prod_{\substack{i,i' \\ i \neq i'}} \frac{1}{\sqrt{b_{e,i,i'}}},$$

which is continuous and positive on K_1. By Theorem 2.16, the maximum of $\phi(\boldsymbol{b})$ in K subject to (2.31) is uniquely attained at $\hat{\boldsymbol{b}} \in K_1 \subset K^\circ$. Moreover, $\phi(\boldsymbol{b})$ is twice continuously differentiable in the interior K°, and its partial derivatives are

$$\frac{\partial \phi}{\partial b_{e,i,i'}} = -(\log b_{e,i,i'} + 1)$$

and

$$\frac{\partial^2 \phi}{\partial b_{e,i,i'} \partial b_{e',j,j'}} = \begin{cases} -1/b_{e,i,i'} & \text{if } (e,i,i') = (e',j,j') \\ 0 & \text{otherwise.} \end{cases}$$

Hence, the Hessian matrix of ϕ at $\boldsymbol{b} = \hat{\boldsymbol{b}}$ is

$$H = -k(k-1)I_{|E_\Gamma|}.$$

Then, for any $|E_\Gamma| \times r$ matrix U whose columns are a basis of $\mathbb{V}$,

$$\det(-H|_{\mathbb{V}}) = \frac{\det(-U^T HU)}{\det U^T U} = \frac{\det(k(k-1)U^T U)}{\det U^T U} = \big(k(k-1)\big)^r \neq 0.$$

Let

$$\mathbb{X}_n = \left\{ \boldsymbol{b} \in K \cap \frac{1}{n}\mathbb{Z}^{|E_\Gamma|} : D\boldsymbol{b} = \boldsymbol{y} \right\}.$$

The solution described in (2.32) belongs to K and, since $k \mid n$, also to $\frac{1}{n}\mathbb{Z}^{|E_\Gamma|}$, so $\mathbb{X}_n$ is not empty. For each $\boldsymbol{b} \in \mathbb{X}_n$, let

$$T_n(\boldsymbol{v}) = p(\hat{\boldsymbol{a}}, \boldsymbol{b}, n)e^{nf(\hat{\boldsymbol{a}}, \boldsymbol{b})}$$

and

$$c_n = k^{k|V|/2 - k|E|}(2\pi n)^{-(k-1)|V|/2 - r/2}.$$

First, in view of (2.26), for $\boldsymbol{b} \in \mathbb{X}_n$,

$$p(\hat{\boldsymbol{a}}, \boldsymbol{b}, n)/c_n = O(n^{|V|+2k|E|}/c_n) = e^{o(n)}$$

and combined with Theorem 2.16 this gives

$$T_n(\boldsymbol{x}) = c_n(p(\hat{\boldsymbol{a}}, \boldsymbol{b}, n)/c_n)e^{nf(\hat{\boldsymbol{a}},\boldsymbol{b})} = O(c_n e^{n\phi(\hat{\boldsymbol{b}})+o(n)}).$$

Now note that we have chosen c_n such that due to (2.28) and the fact that $\xi(x) \sim \sqrt{2\pi x}$ as $x \to \infty$, and after a few computations, we have that for $\boldsymbol{b} \in \mathbb{X}_n \cap K_1$,

$$p(\hat{\boldsymbol{a}}, \boldsymbol{b}, n) = c_n(\psi(\boldsymbol{b}) + o(1)).$$

Finally, as we've met all of the conditions of Theorem 2.13,

$$\begin{aligned}
\mathbf{E}Y &= \sum_{\boldsymbol{b}\in\mathbb{X}_n} T_n(\boldsymbol{b}) \\
&\sim \frac{\psi(\hat{\boldsymbol{b}})}{\tau(\Gamma)^{1/2}\det(-H|_{\mathbb{V}})^{1/2}}(2\pi n)^{r/2}c_n e^{n\phi(\hat{\boldsymbol{b}})} \\
&= \frac{(k(k-1))^{k(k-1)|E|/2}\,k^{k|V|/2-k|E|}}{((k-1)k^{k-2}(k-2)^{k-1})^{|E|/2}\left(k(k-1)\right)^{r/2}}(2\pi n)^{-(k-1)|V|/2}\left(k^{|V|}\left(\frac{k-1}{k}\right)^{|E|}\right)^n \\
&= k^{k|V|/2}\left(\frac{(k-1)^2}{k(k-2)}\right)^{(k-1)|E|/2}(2\pi n)^{-(k-1)|V|/2}\left(k^{|V|}\left(\frac{k-1}{k}\right)^{|E|}\right)^n.
\end{aligned}$$

This completes the proof of Theorem 2.4. □

2.5 Second Moment Calculations

As in the previous section, let Y denote the number of strongly equitable k-colorings of a random n-lift of G. In this section we continue to assume that G is some fixed d-regular graph, not necessarily K_{d+1}, and that k divides n. Our goal in this section is to prove Theorem 2.5.

Our proof is similar in nature to the proof of Theorem 2.4. However, in that proof we fixed $\boldsymbol{a} = \hat{\boldsymbol{a}}$ and approximated the single summation over the $\boldsymbol{b}$s. In this proof, we will not be able to avoid a double summation. We therefore require a more intricate argument, arranged as follows: In Section 2.5.1, we give a counting argument for $\mathbf{E}Y^2$ similar to the argument in Section 2.4.1. We next optimize the exponential contribution in Section 2.5.2. Then we approximate the inner sum in Section 2.5.3 before completing the proof by approximating the outer sum in Section 2.5.4.

2.5.1 Counting Argument

In order to calculate $\mathbf{E}Y^2$, we count pairs of balanced colorings. To each $v \in V$ we assign a $k \times k$ matrix $A_v = (a_{v,i,j})_{i=1}^{k}{}_{j=1}^{k}$ where $a_{v,i,j}$ is the proportion of the vertices in $\Pi^{-1}(v)$ that receive color (i, j) (that is, color i in the first coloring and color j in the second coloring). Each matrix A_v must satisfy

$$\begin{aligned} a_{v,i,j} &\geq 0, \quad \forall v \in V, i, j \in [k] \\ \sum_j a_{v,i,j} &= 1/k, \quad \forall v \in V, i \in [k] \\ \sum_i a_{v,i,j} &= 1/k, \quad \forall v \in V, j \in [k]. \end{aligned} \tag{2.34}$$

In particular, kA_v is a doubly-stochastic matrix.

For each $e = (v, v') \in E$, let $b_{e,i,j,i',j'}$ denote the proportion of edges in $\Pi^{-1}(e)$ which join a vertex of $\Pi^{-1}(v)$ with color (i, j) to a vertex of $\Pi^{-1}(v')$ with color (i', j'). Here $i, j, i', j' \in [k]$, but we require i to be distinct from i' and j to be distinct from j' to assure the colorings are proper. Hence, to each $e = (v, v') \in E$, we assign a four-dimensional array $B_e = (b_{e,i,j,i',j'})_K$, where

$$K = \{(i, j, i', j') \in [k]^4 : i \neq i', j \neq j'\}.$$

Furthermore, each B_e must satisfy:

$$\begin{aligned} b_{e,i,j,i',j'} &\geq 0, \quad \forall e = (v,v') \in E, (i,j,i',j') \in K \\ \sum_{i' \neq i, j' \neq j} b_{e,i,j,i',j'} &= a_{v,i,j}, \quad \forall e = (v,v') \in E, i,j \in [k] \\ \sum_{i \neq i', j \neq j'} b_{e,i,j,i',j'} &= a_{v',i',j'}, \quad \forall e = (v,v') \in E, i',j' \in [k]. \end{aligned} \tag{2.35}$$

We will write $\boldsymbol{A} = (A_v)_{v \in V}$ and $\boldsymbol{B} = (B_e)_{e \in E}$. Note that in addition to (2.34) and (2.35), each entry of A_v and B_e must be in $\frac{1}{n}\mathbb{Z}$.

In the following calculation, we obtain $\mathbf{E}Y^2$ by summing, for every valid choice of $\boldsymbol{A}$ and $\boldsymbol{B}$, the number of triples (lift, coloring 1, coloring 2) compatible with such $\boldsymbol{A}$ and $\boldsymbol{B}$ divided by the total number of lifts.

$$\begin{aligned} \mathbf{E}Y^2 &= \frac{1}{n!^{|E|}} \sum_{\boldsymbol{A},\boldsymbol{B}} \prod_{v \in V} \binom{n}{nA_v} \prod_{e \in E} \frac{(nA_v)!(nA_{v'})!}{(nB_e)!} \\ &= \frac{1}{n!^{|E|}} \sum_{\boldsymbol{A}} \prod_{v \in V} \binom{n}{nA_v} \prod_{e \in E} (nA_v)!(nA_{v'})! \sum_{\boldsymbol{B}} \prod_{e \in E} \frac{1}{(nB_e)!} \\ &= \sum_{\boldsymbol{A},\boldsymbol{B}} \operatorname{poly}(n) \prod_{v \in V} \frac{1}{\prod_{i,j} {a_{v,i,j}}^{na_{v,i,j}}} \prod_{e \in E} \prod_{(i,j,i',j') \in K} \left(\frac{a_{v,i,j} a_{v',i',j'}}{b_{e,i,j,i',j'}} \right)^{nb_{e,i,j,i',j'}} \\ &= \sum_{\boldsymbol{A},\boldsymbol{B}} \operatorname{poly}(n) \exp\big(nf(\boldsymbol{A},\boldsymbol{B})\big) \end{aligned} \tag{2.36}$$

where poly(n) is some function that is polynomial in n and

$$f(\boldsymbol{A},\boldsymbol{B}) = -\sum_{v \in V} \sum_{i,j} a_{v,i,j} \log a_{v,i,j} + \sum_{e \in E} \sum_{i,j,i',j'} b_{e,i,j,i',j'} \log \left(\frac{a_{v,i,j} a_{v',i',j'}}{b_{e,i,j,i',j'}} \right) \tag{2.37}$$

is the function we optimize in the next section.

2.5.2 Optimization

We will show that the exponential part in $\mathbf{E}Y^2$ is maximized by the term in which each $a_{v,i,j} = 1/k^2$ and each $b_{e,i,j,i',j'} = 1/k^2(k-1)^2$. We introduce some notation. Let $\hat{A} = (\hat{a}_{i,j})_{i,j\in[k]}$ with all $\hat{a}_{i,j} = 1/k^2$, and $\hat{B} = (\hat{b}_{i,j,i',j'})_{(i,j,i',j')\in K}$ with all $\hat{b}_{i,j,i',j'} = 1/k^2(k-1)^2$. We also write $\hat{\boldsymbol{A}} = (\hat{A},\dots,\hat{A})$ and $\hat{\boldsymbol{B}} = (\hat{B},\dots,\hat{B})$. As in Section 2.4.1, we note $\hat{\boldsymbol{A}}$ and $\hat{\boldsymbol{B}}$ are only valid assignments to $\boldsymbol{A}$ and $\boldsymbol{B}$, respectively, if $k^2(k-1)^2$ divides n, but as we merely seek an upper bound for f we may do so on a larger space that does include $\hat{\boldsymbol{A}}$ and $\hat{\boldsymbol{B}}$.

Theorem 2.17. *Suppose $d < \ell_k$. Let $f(\boldsymbol{A},\boldsymbol{B})$ be defined as in* (2.37). *Then the maximum of $f(\boldsymbol{A},\boldsymbol{B})$ subject to* (2.34) *and* (2.35) *is uniquely attained at the point in which $\boldsymbol{A} = \hat{\boldsymbol{A}}$ and $\boldsymbol{B} = \hat{\boldsymbol{B}}$, and equals*

$$f(\hat{\boldsymbol{A}},\hat{\boldsymbol{B}}) = \log\left(k^{|V|}\left(\frac{k-1}{k}\right)^{|E|}\right)^{2n}.$$

Proof. This is easier than for $\mathbf{E}X$.

Fix any $\boldsymbol{A}$ satisfying (2.34), and let

$$g(\boldsymbol{A}) = \sum_{v\in V}\left(h(A_v) + \frac{d}{2}\log(1 - 2/k + \rho(A_v))\right),$$

where

$$h(A_v) = -\sum_{i,j\in[k]} a_{v,i,j}\log a_{v,i,j} \qquad \text{and} \qquad \rho(A_v) = \sum_{i,j\in[k]} {a_{v,i,j}}^2.$$

We will maximize $f(\boldsymbol{A},\boldsymbol{B})$ for such fixed $\boldsymbol{A}$ and with $\boldsymbol{B}$ subject only to

$$b_{e,i,j,i',j'} \ge 0, \qquad \sum_{i,j,i',j'} b_{e,i,j,i',j'} = 1. \tag{2.38}$$

In view of this relaxation, we can regard each $B_e = (b_{e,i,j,i',j'})_{(i,j,i',j')\in K}$ as an arbitrary probability distribution, and maximize each term $\sum_{i,j,i',j'} b_{e,i,j,i',j'} \log\left(\frac{a_{v,i,j}a_{v',i',j'}}{b_{e,i,j,i',j'}}\right)$ in (2.37) separately. For each $e \in E$, we define another probability distribution given by

$$b^*_{e,i,j,i',j'} = \frac{a_{v,i,j}a_{v',i',j'}}{z_e}, \qquad \text{for} \quad (i,j,i',j') \in K,$$

where

$$z_e = \sum_{(i,j,i',j')\in K} a_{v,i,j}a_{v',i',j'}$$

is the normalizing factor, and we write $B^*_e = (b^*_{e,i,j,i',j'})_{(i,j,i',j')\in K}$. Then

$$\sum_{i,j,i',j'} b_{e,i,j,i',j'} \log\left(\frac{a_{v,i,j}a_{v',i',j'}}{b_{e,i,j,i',j'}}\right) = \log z_e - D_{KL}(B_e \| B^*_e),$$

where $D_{KL}(B_e\|B^*_e) = \sum_{i,j,i',j'} b_{e,i,j,i',j'} \log\left(\frac{b_{e,i,j,i',j'}}{b^*_{e,i,j,i',j'}}\right)$ is the Kullback-Leibler divergence from B_e to B^*_e. By Gibb's inequality, $D_{KL}(B_e\|B^*_e) \geq 0$ with equality if and only if $B_e = B^*_e$. As a result,

$$\max_{\boldsymbol{B} \text{ s.t. (2.38)}} f(\boldsymbol{A}, \boldsymbol{B}) = \sum_{v\in V} h(A_v) + \sum_{e\in E} \log z_e, \tag{2.39}$$

with one unique maximizer at $\boldsymbol{B} = \boldsymbol{B}^* := (B^*_e)_{e\in E}$. Note that if $\boldsymbol{A} = \hat{\boldsymbol{A}}$ then $\boldsymbol{B}^* = \hat{\boldsymbol{B}}$. We proceed to bound $\log z_e$. Using inclusion-exclusion, the fact that $\boldsymbol{A}$

satisfies (2.34) and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
\log z_e &= \log \sum_{i,j\in[k]} a_{v,i,j} \left(\sum_{i',j'\in[k]} a_{v',i',j'} - \sum_{j'\in[k]} a_{v',i,j'} - \sum_{i'\in[k]} a_{v',i',j} + a_{v',i,j} \right) \\
&= \log \sum_{i,j\in[k]} a_{v,i,j} \left(1 - 2/k + a_{v',i,j}\right) \\
&= \log \left(1 - 2/k + \sum_{i,j\in[k]} a_{v,i,j} a_{v',i,j} \right) \\
&\leq \log \left(\sqrt{1 - 2/k + \sum_{i,j\in[k]} a_{v,i,j}^2} \sqrt{1 - 2/k + \sum_{i,j\in[k]} a_{v',i,j}^2} \right) \\
&= \frac{1}{2} \log(1 - 2/k + \rho(A_v)) + \frac{1}{2} \log(1 - 2/k + \rho(A_{v'})).
\end{aligned}$$

Since each $v \in V$ has degree d,

$$\sum_{e\in E} \log z_e \leq \frac{d}{2} \log(1 - 2/k + \rho(A_v)),$$

and combining this with (2.39) yields

$$\max_{\boldsymbol{B} \text{ s.t. } (2.38)} f(\boldsymbol{A}, \boldsymbol{B}) \leq g(\boldsymbol{A}). \tag{2.40}$$

We will bound $g(\boldsymbol{A})$ by applying Proposition 2.9 to matrix kA_v for each $v \in V$, which is a doubly-stochastic matrix. Note that $d < \ell_k$ implies $d/2 < c_k$, where c_k is defined in (2.8). Therefore, Proposition 2.9 (with $q = k$, $c = d/2$ and $A = kA_v$) yields

$$\frac{d}{2} \log \left(\frac{(k-1)^2 + \rho(kA_v) - 1}{(k-1)^2} \right) \leq \log k - \frac{1}{k} h(kA_v),$$

with equality if and only if $A_v = \hat{A}$. Noting that $\rho(kA_v) = k^2 \rho(A_v)$ and $\frac{1}{k} h(kA_v) =$

$h(A_v) - \log k$, we obtain

$$h(A_v) + \frac{d}{2}\log\left(\frac{(k-1)^2 + k^2\rho(A_v) - 1}{k^2}\right) \leq 2\log k + \frac{d}{2}\log\left(\frac{(k-1)^2}{k^2}\right).$$

Thus, after summing over $v \in V$ and simplifying, we get

$$g(\boldsymbol{A}) \leq g(\hat{\boldsymbol{A}}) = \log\left(k^{|V|}\left(\frac{k-1}{k}\right)^{|E|}\right)^{2n} = f(\hat{\boldsymbol{A}}, \hat{\boldsymbol{B}}),$$

with equality if and only if $\boldsymbol{A} = \hat{\boldsymbol{A}}$. Combining this with (2.40), we obtain the desired bound

$$\begin{aligned} \max_{\boldsymbol{A},\boldsymbol{B} \text{ s.t. } (2.34) \text{ and } (2.35)} f(\boldsymbol{A}, \boldsymbol{B}) &\leq \max_{\boldsymbol{A} \text{ s.t. } (2.34)} \max_{\boldsymbol{B} \text{ s.t. } (2.38)} f(\boldsymbol{A}, \boldsymbol{B}) \\ &\leq \max_{\boldsymbol{A} \text{ s.t. } (2.34)} g(\boldsymbol{A}) = f(\hat{\boldsymbol{A}}, \hat{\boldsymbol{B}}). \end{aligned}$$

Note that for $\boldsymbol{A} \neq \hat{\boldsymbol{A}}$, we have $g(\boldsymbol{A}) < g(\hat{\boldsymbol{A}})$ and thus $f(\boldsymbol{A}, \boldsymbol{B}) < f(\hat{\boldsymbol{A}}, \hat{\boldsymbol{B}})$. Furthermore, if $\boldsymbol{A} = \hat{\boldsymbol{A}}$ but $\boldsymbol{B} \neq \hat{\boldsymbol{B}}$ then, recalling that the maximum in (2.39) is uniquely attained at $\boldsymbol{B}^* = \hat{\boldsymbol{B}}$, we conclude that $f(\hat{\boldsymbol{A}}, \boldsymbol{B}) < g(\hat{\boldsymbol{A}}) = f(\hat{\boldsymbol{A}}, \hat{\boldsymbol{B}})$. This finishes the proof. □

2.5.3 Inner Sum

In order to accurately approximate $\mathbf{E}Y^2$, we now return to (2.36) and find asymptotics for the inner sum over the $\boldsymbol{B}_e$. In particular, we prove the following proposition:

Proposition 2.18. *Fix a choice of* $\boldsymbol{A}$ *such that each* $a_{v,i,j}$ *is close to* $\frac{1}{k^2}$*. Then*

$$\begin{aligned}\mathcal{I}(\boldsymbol{A}) &:= \sum_{\boldsymbol{B}} \prod_{\substack{e\in E\\(i,j,i',j')\in K}} \frac{1}{(b_{e,i,j,i',j'}n)!}\\ &\sim (n/e)^{-n|E|}\gamma(n,k)^{\frac{1}{2}|E|} \exp\left(\frac{-nk^2(k-1)^2}{2} \quad\times\right.\\ &\qquad\times \quad \left(\sum_{vv'\in E}\left(\frac{1}{2\lambda}\sum_{i,j}\left(a_{v,i,j}+a_{v',i,j}-\frac{2}{k^2}\right)^2 \quad +\right.\right.\\ &\qquad+ \quad \left.\left.\left.\frac{1}{2\lambda'}\sum_{i,j}(a_{v,i,j}-a_{v',i,j})^2+\frac{2}{k^2(k-1)^2}\log\frac{1}{k^2(k-1)^2}\right)\right)\right)\end{aligned}$$

where

$$\gamma(n,k) := \frac{k^{(3k^2+1)}(k-1)^{4k(k-1)}}{(2\pi n)^{(2k^2-1)}((k-1)^2+1)^{(2k^2-1)}(k-2)^{(k^2-1)}} \tag{2.41}$$

and λ *and* λ' *are as defined in* (2.3)*.*

Proof. We prove the proposition using the saddle point method. We start by expressing the sum as the coefficient of a generating function.

Consider the generating function on variables $\{x_{e,i,j}\} \cup \{x'_{e,i',j'}\}$ for each $e \in E$ and $(i,j,i',j') \in K$:

$$\prod_{e\in E}\prod_{(i,j,i',j')\in K}\sum_{t\in\mathbb{N}_0}\frac{(x_{e,i,j}x'_{e,i',j'})^t}{t!}$$

We want to extract the coefficient when $t = b_{e,i,j,i',j'}n$. Recalling (2.35), we have

$$\sum_{\boldsymbol{B}} \prod_{\substack{e\in E \\ (i,j,i',j')\in K}} \frac{1}{(b_{e,i,j,i',j'}n)!} = \left[\prod_{e=(v,v')} \left(\prod_{i,j\in[k]} {x_{e,i,j}}^{a_{v,i,j}n} \prod_{i',j'\in[k]} {x'_{e,i',j'}}^{a_{v',i',j'}n}\right)\right] \quad \times$$

$$\times \quad \prod_{e\in E} \prod_{(i,j,i',j')\in K} \sum_{t\in\mathbb{N}_0} \frac{(x_{e,i,j}x'_{e,i',j'})^t}{t!}$$

$$= \left[\prod_{e=(v,v')} \left(\prod_{i,j\in[k]} {x_{e,i,j}}^{a_{v,i,j}n} \prod_{i',j'\in[k]} {x'_{e,i',j'}}^{a_{v',i',j'}n}\right)\right] \quad \times$$

$$\times \quad \exp\left(\sum_{e\in E} \sum_{(i,j,i',j')\in K} x_{e,i,j}x'_{e,i',j'}\right)$$

We extract this coefficient using the residue theorem:

$$\frac{1}{(2\pi i)^{|E|2k^2}} \int \frac{\exp(\sum_{(i,j,i',j')\in K} z_{e,i,j}z'_{e,i',j'})}{\prod_e \left(\prod_{i,j} {z_{e,i,j}}^{a_{v,i,j}n+1} \prod_{i',j'} {z'_{e,i',j'}}^{a_{v',i',j'}n+1}\right)} d\boldsymbol{z}$$

where we've set $x_{e,i,j} = z_{e,i,j}$ and $x'_{e,i',j'} = z'_{e,i',j'}$ to emphasize that they are complex variables.

In applying the saddle point method, we will use a circular path of radius ρ. We use the same radius in every dimension. The method permits us to make this choice, and we do so because it works.

We can now set $z_{e,i,j} = \rho e^{i\theta_{e,i,j}}$ and $z'_{e,i',j'} = \rho e^{i\theta'_{e,i',j'}}$. The change of variables gives $dz_{e,i,j} = i\rho e^{i\theta_{e,i,j}} d\theta_{e,i,j}$ (and similarly for the primed variables) so we cancel all $|E|2k^2$ copies of i in the denominator as well as one copy of each $z_{e,i,j}$ and $z'_{e,i',j'}$ from the denominator and get

$$\frac{1}{(2\pi)^{|E|2k^2}}\int\frac{\exp(\rho^2\sum_e\sum_{(i,j,i',j')\in K}e^{i(\theta_{e,i,j}+\theta'_{e,i',j'})})}{\rho^{|E|2n}\exp\left(in\sum_e(\sum_{i,j}\theta_{e,i,j}a_{v,i,j}+\sum_{i',j'}\theta'_{e,i',j'}a_{v',i',j'})\right)}d\boldsymbol{\theta}$$

where $\boldsymbol{\theta}$ is a vector of all of the θs and θ's.

Now we set all of the θs to zero to find the value on the real line and choose ρ to optimize (the log of) this value:

$$\left(\rho^2|E|k^2(k-1)^2-|E|2n\log\rho\right)'=0$$

which is accomplished, after some elementary calculus, by

$$\rho=\sqrt{\frac{n}{k^2(k-1)^2}}.$$

Plugging in ρ, we can rewrite the equation above as:

$$\frac{1}{(2\pi)^{|E|2k^2}}\left(\frac{k^2(k-1)^2}{n}\right)^{|E|n}\int e^{h(\boldsymbol{\theta})}\ d\boldsymbol{\theta}$$

so that we may analyze $h(\boldsymbol{\theta})$:

$$\begin{aligned}h(\boldsymbol{\theta})=\frac{n}{k^2(k-1)^2}\sum_e\sum_{(i,j,i',j')\in K}e^{i(\theta_{e,i,j}+\theta'_{e,i',j'})}\quad-\\ -\quad in\sum_e\left(\sum_{i,j}\theta_{e,i,j}a_{v,i,j}+\sum_{i',j'}\theta'_{e,i',j'}a_{v',i',j'}\right)\end{aligned}$$

Consider

$$\begin{aligned}
|e^{h(\boldsymbol{\theta})}| &= \exp\left(\Re\left(\frac{n}{k^2(k-1)^2}\sum_e \sum_{(i,j,i',j')\in K} e^{i(\theta_{e,i,j}+\theta'_{e,i',j'})} \right.\right. \quad - \\
&\qquad \left.\left. - \quad in\sum_e\left(\sum_{i,j}\theta_{e,i,j}a_{v,i,j}+\sum_{i',j'}\theta'_{e,i',j'}a_{v',i',j'}\right)\right)\right) \\
&= \exp\left(\frac{n}{k^2(k-1)^2}\sum_e \sum_{(i,j,i',j')\in K}\Re(e^{i(\theta_{e,i,j}+\theta'_{e,i',j'})})\right) \\
&= \exp\left(\frac{n}{k^2(k-1)^2}\sum_e \sum_{(i,j,i',j')\in K}\cos(\theta_{e,i,j}+\theta'_{e,i',j'})\right)
\end{aligned}$$

where we use $\Re(z)$ to denote the real part of z.

Thus in order to maximize $|e^{h(\boldsymbol{\theta})}|$, we should make $\theta_{e,i,j}+\theta'_{e,i',j'}$ close to zero for every pair $\theta_{e,i,j},\theta'_{e,i',j'}$. If some pair has sum far from zero, we should be able to show $|e^{h(\boldsymbol{\theta})}|$ is negligible. We now formalize this notion.

For each edge e, define

$$\theta_e = \frac{1}{2k^2}\left(\sum_{i,j}\theta_{e,i,j} - \sum_{i',j'}\theta'_{e,i',j'}\right)$$

to be a weighted average of the θs. Then define $\delta_{e,i,j}$ and $\delta'_{e,i',j'}$ by

$$\theta_{e,i,j} = \theta_e + \delta_{e,i,j} \qquad \text{and} \qquad \theta'_{e,i',j'} = -\theta_e - \delta'_{e,i',j'}.$$

Clearly, any choice of the $\theta_{e,i,j}$ and $\theta'_{e,i',j'}$ determines θ_e and each $\delta_{e,i,j}$ and $\delta'_{e,i',j'}$.

Furthermore, as

$$\begin{aligned}\theta_e &= \frac{1}{2k^2}\left(\sum_{i,j}\theta_{e,i,j} - \sum_{i',j'}\theta'_{e,i',j'}\right)\\ &= \frac{1}{2k^2}\left(\sum_{i,j}(\theta + \delta_{e,i,j}) - \sum_{i',j'}(-\theta - \delta'_{e,i',j'})\right)\\ &= \theta_e + \frac{1}{2k^2}\left(\sum_{i,j}\delta_{e,i,j} + \sum_{i',j'}\delta'_{e,i',j'}\right)\end{aligned}$$

we see that every choice of θ_e and the $\delta_{e,i,j}$ and $\delta'_{e,i',j'}$ satisfying the linear constraints

$$\left(\sum_{i,j}\delta_{e,i,j} + \sum_{i',j'}\delta'_{e,i',j'}\right) = 0$$

determines a valid choice for the $\theta_{e,i,j}$ and $\theta'_{e,i',j'}$.

Let $R \subseteq \mathbb{R}^{|E|(2k^2+1)}$ be the subspace of choices of θ_e and the $\delta_{e,i,j}$ and $\delta'_{e,i',j'}$ that correspond to valid choices of the $\theta_{e,i,j}$ and $\theta'_{e,i',j'}$. Partition R into R_1 and R_2, where each point in R_1 satisfies

$$|\delta_{e,i,j}|, |\delta'_{e,i',j'}| \leq \frac{\log n}{\sqrt{n}} =: \varepsilon$$

for all e, i, j, i' and j' (and θ_e takes any value in $[0, 2\pi]$) and R_2 is the complementary region.

We claim that

$$\int_{R_2} e^{h(\boldsymbol{\theta})}\ d\theta d\boldsymbol{\delta} = o\left(\left(\frac{e^n}{\sqrt{n}}\right)^{|E|}\right).$$

For each point in R_2, there is some pair $(\delta_{e,i,j}, \delta'_{e,i',j'})$ such that

$$|\delta_{e,i,j} - \delta'_{e,i',j'}| > 2\varepsilon.$$

Then

$$
\begin{aligned}
|e^{h(\boldsymbol{\theta})}| &= \exp\left(\frac{n}{k^2(k-1)^2}\sum_e \sum_{(i,j,i',j')\in K} \cos(\theta_{e,i,j}+\theta'_{e,i',j'})\right)\\
&= \exp\left(\frac{n}{k^2(k-1)^2}\sum_e \sum_{(i,j,i',j')\in K} \cos(\theta+\delta_{e,i,j}-\theta-\delta'_{e,i',j'})\right)\\
&< \exp\left(\frac{n}{k^2(k-1)^2}\sum_e \sum_{(i,j,i',j')\in K} \cos(2\varepsilon)\right)\\
&= \exp\left(\frac{n}{k^2(k-1)^2}\sum_e \sum_{(i,j,i',j')\in K} 1-\Omega(\varepsilon^2)\right)\\
&= \exp\left(n|E|\left(1-\Omega\left(\frac{\log^2 n}{n}\right)\right)\right)\\
&= \exp\left(|E|\left(n-\Omega(\log^2 n)\right)\right)
\end{aligned}
$$

and so

$$
\left|\int_{R_2} e^{h(\boldsymbol{\theta})}\, d\boldsymbol{\theta}\right| \le (2\pi\rho)^{|E|2k^2} e^{|E|\left(n-\Omega(\log^2 n)\right)} = o\left(\left(\frac{e^n}{\sqrt{n}}\right)^{|E|}\right)
$$

as each $\theta_{e,i,j}, \theta'_{e,i',j'} \in [0, 2\pi\rho]$, $\rho = O(\sqrt{n})$ and $n^{-\Omega(\log^2 n)} = o(\frac{1}{n})$.

Next, we show that this value is negligible compared to the integral of the function over R_1.

For points in R_1, we use a Taylor expansion for the exponential term:

$$
\begin{aligned}
h(\boldsymbol{\theta}) = \frac{n}{k^2(k-1)^2}\sum_e \sum_{(i,j,i',j')\in K} \left(1+i(\theta_{e,i,j}-\theta'_{e,i',j'})-\tfrac{1}{2}(\theta_{e,i,j}+\theta'_{e,i',j'})^2+O(\varepsilon^3)\right) \quad - \\
- \quad in\sum_e\left(\sum_{i,j}\theta_{e,i,j}a_{v,i,j}+\sum_{i',j'}\theta'_{e,i',j'}a_{v',i',j'}\right)
\end{aligned}
$$

as

$$
|\theta_{e,i,j}+\theta'_{e,i',j'}| = |\theta+\delta_{e,i,j}-\theta-\delta'_{e,i',j'}| \le |\delta_{e,i,j}|+|\delta'_{e,i',j'}| \le 2\varepsilon.
$$

Now we sum the relevant terms to get

$$h(\boldsymbol{\theta}) = \frac{n}{k^2(k-1)^2} \cdot |E| k^2 (k-1)^2 + \frac{n}{k^2(k-1)^2} \cdot (k-1)^2 \left(\sum_{i,j} \theta_{e,i,j} + \sum_{i',j'} \theta'_{e,i',j'} \right) \quad -$$

$$- \quad \frac{n}{2k^2(k-1)^2} \sum_e \sum_{(i,j,i',j') \in K} (\theta_{e,i,j} + \theta'_{e,i',j'})^2 + O(n\varepsilon^3) \quad -$$

$$- \quad in \sum_e \left(\sum_{i,j} \theta_{e,i,j} a_{v,i,j} + \sum_{i',j'} \theta'_{e,i',j'} a_{v',i',j'} \right)$$

or, cleaning up and combining terms,

$$h(\boldsymbol{\theta}) = \sum_e \left(n + in \left(\sum_{i,j} \theta_{e,i,j} (\tfrac{1}{k^2} - a_{v,i,j}) + \sum_{i',j'} \theta'_{e,i',j'} (\tfrac{1}{k^2} - a_{v',i',j'}) \right) \quad - \right.$$

$$\left. - \quad \frac{n}{2k^2(k-1)^2} \sum_{(i,j,i',j') \in K} (\theta_{e,i,j} + \theta'_{e,i',j'})^2 \right) + O(n\varepsilon^3)$$

Note that

$$n\varepsilon^3 = n \frac{\log^3 n}{n^{3/2}} = \frac{\log^3 n}{\sqrt{n}} = o(1).$$

Recall that $h(\boldsymbol{\theta})$ is the exponent of the integral in which we're interested. Thus we define h_e to be the term corresponding to edge e and note that

$$e^{h(\boldsymbol{\theta})} = (1 + O(1)) \prod_e e^{h_e}.$$

For convenience we now drop the e from the subscripts of $\theta_{e,i,j}$, $\theta'_{e,i',j'}$, $a_{e,i,j}$ and $a'_{e,i',j'}$; each variable will be associated with the edge indicated by h_e. Furthermore, let $\alpha_{i,j} = a_{v,i,j} - \frac{1}{k^2}$ and $\alpha'_{i',j'} = \alpha_{v',i',j'} - \frac{1}{k^2}$. Then

$$h_e = n - in\left(\sum_{i,j}\theta_{i,j}\alpha_{i,j} + \sum_{i',j'}\theta'_{i',j'}\alpha'_{i',j'}\right) - \frac{n}{2k^2(k-1)^2}\sum_{(i,j,i',j')\in K}(\theta_{i,j}+\theta'_{i',j'})^2$$

Let $\boldsymbol{\alpha}_{\text{all}}$ be a vector of each α and α' so that we may write

$$\begin{aligned} h_e = n - in\langle\boldsymbol{\theta},\boldsymbol{\alpha}_{\text{all}}\rangle - \frac{n}{2k^2(k-1)^2} \quad \times \\ \times \quad \left((k-1)^2\sum_{i,j}\theta_{i,j}^2 + (k-1)^2\sum_{i',j'}\theta'^2_{i',j'} + 2\sum_{(i,j,i',j')\in K}\theta_{i,j}\theta'_{i',j'}\right) \end{aligned}$$

The last term of h_e is a quadratic form. With I_m and J_m denoting the $m\times m$ identity matrix and the $m\times m$ matrix whose entries are all 1s, respectively, set

$$B = (k-1)^2 I_{2k^2} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes (J_k - I_k)^{\otimes 2}$$

so that

$$h_e = n - in\langle\boldsymbol{\theta},\boldsymbol{\alpha}_{\text{all}}\rangle - \frac{n}{2k^2(k-1)^2}\boldsymbol{\theta}^T B\boldsymbol{\theta}$$

We now analyze the spectrum of B. First

$$\operatorname{spec}\{J_k - I_k\} = \{k-1, (-1)_{k-1}\}$$

as J_k has eigenvalues k with multiplicity 1 and 0 with multiplicity $k-1$ while I_k has eigenvalue 1 with multiplicity k. Then

$$\operatorname{spec}\{(J_k - I_k)^{\otimes 2}\} = \{(k-1)^2, (1-k)_{2k-2}, 1_{(k-1)^2}\}$$

as the Kronecker product of two matrices has an eigenvalue for each pair of eigenvalues of base matrices. Similarly,

$$\begin{aligned}\operatorname{spec}\{\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]\otimes(J_k-I_k)^{\otimes 2}\} = \{&(k-1)^2, -(k-1)^2, (1-k)_{2k-2},\\ &(k-1)_{2k-2}, 1_{(k-1)^2}, (-1)_{(k-1)^2}\}\end{aligned}$$

as $\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]$ has eigenvalues -1 and 1.

Finally, as $(k-1)^2 I_{2k}$ has eigenvalue $(k-1)^2$ with multiplicity $2k$,

$$\begin{aligned}\operatorname{spec}\{B\} = \{&2(k-1)^2, 0, ((k-1)^2+(1-k))_{2k-2}, ((k-1)^2+(k-1))_{2k-2},\\ &((k-1)^2+1)_{(k-1)^2}, ((k-1)^2-1)_{(k-1)^2}\}\end{aligned}$$

Let $\boldsymbol{f}_{i,j}$ be vectors in $\mathbb{R}^{k^2}$ as defined in Lemma 6 from [29]. They are an orthonormal basis of eigenvectors of $(J_k-I_k)^{\otimes 2}$. Let $\boldsymbol{w}_{i,j} = \begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix} \otimes \boldsymbol{f}_{i,j} = \left[\begin{smallmatrix}\boldsymbol{f}_{i,j}/\sqrt{2}\\ \boldsymbol{f}_{i,j}/\sqrt{2}\end{smallmatrix}\right]$ and $\boldsymbol{w'}_{i',j'} = \begin{bmatrix}1/\sqrt{2}\\-1/\sqrt{2}\end{bmatrix} \otimes \boldsymbol{f}_{i',j'} = \left[\begin{smallmatrix}\boldsymbol{f}_{i',j'}/\sqrt{2}\\ -\boldsymbol{f}_{i',j'}/\sqrt{2}\end{smallmatrix}\right]$. The $\boldsymbol{w}_{i,j}$ and the $\boldsymbol{w'}_{i',j'}$ are an orthonormal basis of $\mathbb{R}^{2k^2}$ and moreover are eigenvectors of B. The corresponding eigenvalues are

$$\lambda_{i,j} = \begin{cases}(k-1)^2+1 & i,j\neq k\\ (k-1)^2-(k-1) = (k-1)(k-2) & (i=k) \text{ xor } (j=k)\\ (k-1)^2+(k-1)^2 = 2(k-1)^2 & i=j=k\end{cases}$$

and

$$\lambda'_{i',j'} = \begin{cases} (k-1)^2 - 1 = k(k-2) & i', j' \neq k \\ (k-1)^2 + (k-1) = k(k-1) & (i' = k) \text{ xor } (j' = k) \\ (k-1)^2 - (k-1)^2 = 0 & i' = j' = k \end{cases}$$

Express $\boldsymbol{\theta}$ in terms of this new basis:

$$\boldsymbol{\theta} = \sum_{i,j} \tau_{i,j} \boldsymbol{w}_{i,j} + \sum_{i',j'} \tau'_{i',j'} \boldsymbol{w}'_{i,j}$$

so that we may write

$$h_e = n - in \left(\sum_{i,j} \tau_{i,j} \langle \hat{\boldsymbol{w}}_{i,j}, \boldsymbol{\alpha}_{\text{all}} \rangle + \sum_{i',j'} \tau'_{i',j'} \langle \boldsymbol{w}'_{i',j'}, \boldsymbol{\alpha}_{\text{all}} \rangle \right) \quad - \\ - \quad \frac{n}{2k^2(k-1)^2} \left(\sum_{i,j} \lambda_{i,j} \tau_{i,j}^2 + \sum_{i',j'} \lambda'_{i',j'} {\tau'}^2_{i',j'} \right).$$

Recall that our definition of

$$\theta = \theta_e = \frac{1}{2k^2} \left(\sum_{i,j} \theta_{i,j} - \sum_{i',j'} \theta'_{i',j'} \right)$$

forced

$$\sum_{i,j} \delta_{i,j} - \sum_{i',j'} \delta'_{i',j'} = 0.$$

Noting that $\boldsymbol{w}'_{k,k} = \frac{1}{\sqrt{2k^2}} \begin{bmatrix} \mathbf{1}_{k^2} \\ -\mathbf{1}_{k^2} \end{bmatrix}$, this is equivalent to

$$\langle \boldsymbol{\delta}, \boldsymbol{w}'_{k,k} \rangle = 0.$$

Thus

$$\boldsymbol{\theta} = \theta\sqrt{2k^2}\boldsymbol{w}'_{k,k} + \boldsymbol{\delta}.$$

We conclude $\tau'_{k,k} = \theta\sqrt{2k^2}$ and that we can express

$$\boldsymbol{\delta} = \sum_{i,j} \tau_{i,j}\boldsymbol{w}_{i,j} + \sum_{\substack{i',j' \\ (i',j')\neq(k,k)}} \tau'_{i',j'}\boldsymbol{w}'_{i,j}$$

In order to integrate over R_1, we must express R_1 in terms of the τ. As θ can take any value in $[0, 2\pi]$, we have $\tau'_{k,k} \in [0, 2\pi\sqrt{2k^2}]$. For each other τ and τ', we make the following argument:

Because we are in R_1, each $\delta_{i,j}$ and $\delta'_{i',j'}$ satisfies $|\delta_{i,j}|, |\delta'_{i',j'}| \leq \varepsilon$. Therefore, $||\boldsymbol{\delta}||_\infty \leq \varepsilon$. Note that

$$||\boldsymbol{\delta}||_2^2 = \sum_{i,j} \delta_{i,j}^2 + \sum_{i',j'} {\delta'_{i',j'}}^2 \leq 2k^2||\boldsymbol{\delta}||_\infty^2 \leq 2k^2\varepsilon^2$$

and thus $||\boldsymbol{\delta}||_2 = O(\varepsilon)$. As the $\boldsymbol{w}$ are orthonormal, we see $||\boldsymbol{\delta}||_2 = ||\boldsymbol{\tau}^-||_2$, where $\boldsymbol{\tau}^-$ is a vector of the $\tau_{i,j}$ and $\tau'_{i',j'}$ without $\tau'_{k,k}$. Finally, as

$$||\boldsymbol{\tau}^-||_2^2 = \sum_{i,j} \tau_{i,j}^2 + \sum_{\substack{i',j' \\ (i',j')\neq(k,k)}} {\tau'_{i',j'}}^2 \geq \max\left(\max_{i,j} \tau_{i,j}^2, \max_{\substack{i',j' \\ (i',j')\neq(k,k)}} {\tau'_{i',j'}}^2 \right) = ||\boldsymbol{\tau}^-||_\infty^2$$

we conclude

$$||\boldsymbol{\tau}^-||_\infty \leq ||\boldsymbol{\tau}^-||_2 = O(\varepsilon)$$

so that each τ and τ' (except $\tau'_{k,k}$) is $O(\varepsilon)$.

Note further that as the $\boldsymbol{w}$ are orthogonal, we have $d\boldsymbol{\theta} = d\boldsymbol{\tau}$, so by setting

$$I_{i,j} = \int_{-O(\varepsilon)}^{O(\varepsilon)} e^{-in\tau_{i,j}\langle \boldsymbol{w}_{i,j}, \boldsymbol{\alpha}\rangle - \frac{n}{2k^2(k-1)^2}\lambda_{i,j}\tau_{i,j}^2} \, d\tau_{i,j},$$

$$I'_{i',j'} = \int_{-O(\varepsilon)}^{O(\varepsilon)} e^{-in\tau'_{i',j'}\langle \boldsymbol{w}'_{i',j'},\boldsymbol{\alpha}\rangle - \frac{n}{2k^2(k-1)^2}\lambda'_{i',j'}\tau'_{i',j'}{}^2} \; d\tau'_{i',j'}$$

for $(i',j') \neq (k,k)$ and

$$I'_{k,k} = \int_{0}^{2\pi\sqrt{2k^2}} e^{-in\tau'_{k,k}\langle \boldsymbol{w}'_{k,k},\boldsymbol{\alpha}\rangle - \frac{n}{2k^2(k-1)^2}\lambda'_{k,k}\tau'_{k,k}{}^2} \; d\tau'_{k,k}$$

We have

$$\mathcal{I}(\boldsymbol{A}) \sim \frac{1}{(2\pi)^{|E|2k^2}} \left(\frac{k^2(k-1)^2}{n}\right)^{|E|n} \prod_{e} \left(e^n \prod_{i,j} I_{i,j} \prod_{i',j'} I'_{i',j'} \right).$$

We wish to approximate each integral. First, we claim that for $i = k$ or $j = k$, we have $\langle \boldsymbol{w}_{i,j}, \boldsymbol{\alpha}_{\text{all}}\rangle = 0$ and similarly for $i' = k$ or $j' = k$, we have $\langle \boldsymbol{w}'_{i',j'}, \boldsymbol{\alpha}_{\text{all}}\rangle = 0$. This can be seen in the proof of Lemma 7 of [29]. Furthermore, note that $\lambda'_{k,k} = 0$. Thus

$$I'_{k,k} = \int_{0}^{2\pi\sqrt{2k^2}} e^0 \; d\tau'_{k,k} = 2\pi\sqrt{2k^2}.$$

For the other integrals, we make another substitution $x_{i,j} = \sqrt{n}\tau_{i,j}$ and $x'_{i',j'} = \sqrt{n}\tau'_{i',j'}$, so noting $\sqrt{n}\varepsilon = O(\log n)$, the limits of the new integral go from $-O(\log n)$ to $O(\log n)$, which we approximate by $-\infty$ and ∞, to get

$$I_{i,j} \sim \frac{1}{\sqrt{n}} \int_{-\infty}^{\infty} e^{-i\sqrt{n}\langle \boldsymbol{w}_{i,j},\boldsymbol{\alpha}\rangle x_{i,j} - \frac{\lambda_{i,j}}{2k^2(k-1)^2}x_{i,j}^2} \; dx_{i,j}$$

and

$$I'_{i',j'} \sim \frac{1}{\sqrt{n}} \int_{-\infty}^{\infty} e^{-i\sqrt{n}\langle \boldsymbol{w}'_{i',j'},\boldsymbol{\alpha}\rangle x'_{i',j'} - \frac{\lambda'_{i',j'}}{2k^2(k-1)^2}x'_{i',j'}{}^2} \; dx'_{i',j'}$$

for $(i',j') \neq (k,k)$.

Using the equation

$$\int_{-\infty}^{\infty} e^{ax-bx^2} = \sqrt{\frac{\pi}{b}} \exp\left(\frac{a^2}{4b}\right)$$

we get

$$I_{i,j} = \sqrt{\frac{2\pi k^2(k-1)^2}{n\lambda_{i,j}}} \exp\left(-\frac{nk^2(k-1)^2\langle \boldsymbol{w}_{i,j}, \boldsymbol{\alpha}\rangle^2}{2\lambda_{i,j}}\right)$$

and

$$I'_{i',j'} = \sqrt{\frac{2\pi k^2(k-1)^2}{n\lambda'_{i',j'}}} \exp\left(-\frac{nk^2(k-1)^2\langle \boldsymbol{w}'_{i',j'}, \boldsymbol{\alpha}\rangle^2}{2\lambda'_{i',j'}}\right)$$

for $(i',j') \neq (k,k)$.

Thus

$$\begin{aligned}
\prod_{i,j} I_{i,j} \prod_{i',j'} I'_{i',j'} \sim 2\pi\sqrt{2k^2} \prod_{i,j} \sqrt{\frac{2\pi k^2(k-1)^2}{n\lambda_{i,j}}} \exp\left(-\frac{nk^2(k-1)^2\langle \boldsymbol{w}_{i,j}, \boldsymbol{\alpha}\rangle^2}{2\lambda_{i,j}}\right) \quad \times \\
\times \prod_{\substack{i',j' \\ (i',j')\neq(k,k)}} \sqrt{\frac{2\pi k^2(k-1)^2}{n\lambda'_{i',j'}}} \exp\left(-\frac{nk^2(k-1)^2\langle \hat{\boldsymbol{w}}'_{i',j'}, \boldsymbol{\alpha}\rangle^2}{2\lambda'_{i',j'}}\right) \\
= \frac{2\pi}{\det \mathcal{L}} \left(\sqrt{\frac{2\pi k^2(k-1)^2}{n}}\right)^{2k^2-1} \prod_{i,j} \exp\left(-\frac{nk^2(k-1)^2\langle \boldsymbol{w}_{i,j}, \boldsymbol{\alpha}\rangle^2}{2\lambda_{i,j}}\right) \quad \times \\
\times \prod_{\substack{i',j' \\ (i',j')\neq(k,k)}} \exp\left(-\frac{n(k-1)^2\langle \boldsymbol{w}'_{i',j'}, \boldsymbol{\alpha}\rangle^2}{2\lambda'_{i',j'}}\right)
\end{aligned}$$

where

$$\det \mathcal{L} = \sqrt{\frac{1}{2k^2} \prod_{i,j} \lambda_{i,j} \prod_{\substack{i',j' \\ (i',j')\neq(k,k)}} \lambda'_{i',j'}}$$

Recall that for $i, i' = k$ or $j, j' = k$, we have

$$\langle \boldsymbol{w}_{i,j}, \boldsymbol{\alpha}_{\text{all}}\rangle = 0 = \langle \hat{\boldsymbol{w}}'_{i',j'}, \boldsymbol{\alpha}_{\text{all}}\rangle.$$

Thus we need only consider the eigenvalues we underlined earlier; the numerator in the other terms is zero. This gives us common denominators, allowing us to write

$$\prod_{i,j} I_{i,j} \prod_{i',j'} I_{i',j'} \sim \frac{2\pi}{\det \mathcal{L}} \left(\sqrt{\frac{2\pi k^2(k-1)^2}{n}} \right)^{2k^2-1} \times$$
$$\times \quad \exp\left(-nk^2(k-1)^2 \left(\frac{\sum_{i,j} \langle \boldsymbol{w}_{i,j}, \boldsymbol{\alpha} \rangle^2}{2\lambda} + \frac{\sum_{i',j'} \langle \boldsymbol{w}'_{i',j'}, \boldsymbol{\alpha} \rangle^2}{2\lambda'} \right) \right)$$

with λ and λ' as defined in (2.3).

We may now return to our evaluation of $\mathcal{I}(\boldsymbol{A})$:

$$\mathcal{I}(\boldsymbol{A}) \sim \frac{1}{(2\pi)^{|E|2k^2}} \left(\frac{k^2(k-1)^2}{n} \right)^{|E|n} \prod_e \left(e^n \frac{2\pi}{\det \mathcal{L}} \left(\sqrt{\frac{2\pi k^2(k-1)^2}{n}} \right)^{2k^2-1} \times \right.$$
$$\left. \times \quad \exp\left(-nk^2(k-1)^2 \left(\frac{\sum_{i,j} \langle \boldsymbol{w}_{i,j}, \boldsymbol{\alpha} \rangle^2}{2\lambda} + \frac{\sum_{i',j'} \langle \boldsymbol{w}'_{i',j'}, \boldsymbol{\alpha} \rangle^2}{2\lambda'} \right) \right) \right)$$
$$= \left(\frac{(k^2(k-1)^2)^{n+k^2-\frac{1}{2}}}{(^n/_e)^n (\sqrt{2\pi n})^{2k^2-1} \det \mathcal{L}} \right)^{|E|} \times$$
$$\times \quad \prod_e \left(\exp\left(\frac{-nk^2(k-1)^2}{2} \times \right.\right.$$
$$\left.\left. \times \quad \left(\frac{\sum_{i,j} \langle \hat{\boldsymbol{w}}_{i,j}, \boldsymbol{\alpha}_{\text{all}} \rangle^2}{\lambda} + \frac{\sum_{i',j'} \langle \boldsymbol{w}'_{i',j'}, \boldsymbol{\alpha}_{\text{all}} \rangle^2}{\lambda'} \right) \right) \right)$$

Recall that $\boldsymbol{w}_{i,j} = \left[\begin{smallmatrix} \boldsymbol{f}_{i,j}/\sqrt{2} \\ \boldsymbol{f}_{i,j}/\sqrt{2} \end{smallmatrix}\right]$, $\boldsymbol{w'}_{i',j'} = \left[\begin{smallmatrix} \boldsymbol{f}_{i',j'}/\sqrt{2} \\ -\boldsymbol{f}_{i',j'}/\sqrt{2} \end{smallmatrix}\right]$ and $\boldsymbol{\alpha}_{\text{all}}$ is a column vector of the αs atop the α's.We now break $\boldsymbol{\alpha}_{\text{all}}$ into two column vectors, $\boldsymbol{\alpha}$ containing the $\alpha_{i,j}$ and $\boldsymbol{\alpha}'$ containing the $\alpha'_{i',j'}$ so that we may write:

$$\langle \boldsymbol{w}_{i,j}, \boldsymbol{\alpha}_{\text{all}} \rangle = \frac{1}{\sqrt{2}} \left(\langle \boldsymbol{f}_{i,j}, \boldsymbol{\alpha} \rangle + \langle \boldsymbol{f}_{i,j}, \boldsymbol{\alpha}' \rangle \right) = \frac{1}{\sqrt{2}} \langle \boldsymbol{f}_{i,j}, \boldsymbol{\alpha} + \boldsymbol{\alpha}' \rangle$$

and

$$\langle \boldsymbol{w}'_{i,j}, \boldsymbol{\alpha}_{\text{all}} \rangle = \frac{1}{\sqrt{2}} \left(\langle \boldsymbol{f}_{i,j}, \boldsymbol{\alpha} \rangle - \langle \boldsymbol{f}_{i,j}, \boldsymbol{\alpha}' \rangle \right) = \frac{1}{\sqrt{2}} \langle \boldsymbol{f}_{i,j}, \boldsymbol{\alpha} - \boldsymbol{\alpha}' \rangle$$

Furthermore, as the $\boldsymbol{f}_{i,j}$ form a basis,

$$\sum_{i,j} \langle \boldsymbol{f}_{i,j}, \boldsymbol{\alpha} + \boldsymbol{\alpha}' \rangle = ||\boldsymbol{\alpha} + \boldsymbol{\alpha}'||^2 \quad \text{and} \quad \sum_{i,j} \langle \boldsymbol{f}_{i,j}, \boldsymbol{\alpha} - \boldsymbol{\alpha}' \rangle = ||\boldsymbol{\alpha} - \boldsymbol{\alpha}'||^2$$

This simplification gives

$$\begin{aligned}
\mathcal{I}(\boldsymbol{A}) \sim & \left(\frac{(k^2(k-1)^2)^{n+k^2-\frac{1}{2}}}{(n/e)^n(\sqrt{2\pi n})^{2k^2-1} \det \mathcal{L}} \right)^{|E|} \quad \times \\
& \times \quad \prod_e \left(\exp\left(\frac{-nk^2(k-1)^2}{2} \left(\frac{||\boldsymbol{\alpha} + \boldsymbol{\alpha}'||^2}{2\lambda} + \frac{||\boldsymbol{\alpha} - \boldsymbol{\alpha}'||^2}{2\lambda'} \right) \right) \right) \\
= & \left(\frac{(k^2(k-1)^2)^{n+k^2-\frac{1}{2}}}{(n/e)^n(\sqrt{2\pi n})^{2k^2-1} \det \mathcal{L}} \right)^{|E|} \quad \times \\
& \times \quad \exp\left(\frac{-nk^2(k-1)^2}{2} \sum_e \left(\frac{||\boldsymbol{\alpha} + \boldsymbol{\alpha}'||^2}{2\lambda} + \frac{||\boldsymbol{\alpha} - \boldsymbol{\alpha}'||^2}{2\lambda'} \right) \right) \\
= & (n/e)^{-n|E|} \left(\frac{(k^2(k-1)^2)^{k^2-\frac{1}{2}}}{(2\pi n)^{\frac{1}{2}(2k^2-1)} \det \mathcal{L}} \right)^{|E|} (k^2(k-1)^2)^{n|E|} \quad \times \\
& \times \quad \exp\left(\frac{-nk^2(k-1)^2}{2} \sum_e \left(\frac{||\boldsymbol{\alpha} + \boldsymbol{\alpha}'||^2}{2\lambda} + \frac{||\boldsymbol{\alpha} - \boldsymbol{\alpha}'||^2}{2\lambda'} \right) \right) \\
= & (n/e)^{-n|E|} \left(\frac{(k^2(k-1)^2)^{k^2-\frac{1}{2}}}{(2\pi n)^{\frac{1}{2}(2k^2-1)} \det \mathcal{L}} \right)^{|E|} \exp\big(n|E| \log(k^2(k-1)^2) \quad - \\
& - \quad \frac{nk^2(k-1)^2}{2} \sum_e \left(\frac{||\boldsymbol{\alpha} + \boldsymbol{\alpha}'||^2}{2\lambda} + \frac{||\boldsymbol{\alpha} - \boldsymbol{\alpha}'||^2}{2\lambda'} \right) \bigg) \\
= & (n/e)^{-n|E|} \left(\frac{(k^2(k-1)^2)^{k^2-\frac{1}{2}}}{(2\pi n)^{\frac{1}{2}(2k^2-1)} \det \mathcal{L}} \right)^{|E|} \exp\bigg(-\frac{nk^2(k-1)^2}{2} \quad \times \\
& \times \quad \sum_e \left(\frac{||\boldsymbol{\alpha} + \boldsymbol{\alpha}'||^2}{2\lambda} + \frac{||\boldsymbol{\alpha} - \boldsymbol{\alpha}'||^2}{2\lambda'} + \frac{2}{k^2(k-1)^2} \log \frac{1}{k^2(k-1)^2} \right) \bigg)
\end{aligned} \tag{2.42}$$

Now recall

$$\begin{aligned}
\det \mathcal{L} &= \sqrt{\frac{1}{2k^2} \prod_{i,j} \lambda_{i,j} \prod_{\substack{i',j' \\ (i',j') \neq (k,k)}} \lambda'_{i',j'}} \\
&= \left(\frac{1}{2k^2} (\lambda)^{(k-1)^2} ((k-1)(k-2))^{2(k-1)} (2(k-1)^2)(\lambda')^{(k-1)^2} (k(k-1))^{2(k-1)} \right)^{1/2} \\
&= (\lambda)^{\frac{1}{2}(k-1)^2} (k-2)^{(k-1)+\frac{1}{2}(k-1)^2} (k-1)^{2(k-1)-1} k^{\frac{1}{2}(k-1)^2+(k-1)-1}
\end{aligned}$$

so

$$\begin{aligned}
\left(\frac{(k^2(k-1)^2)^{k^2-\frac{1}{2}}}{(2\pi)^{\frac{1}{2}(2k^2-1)} \det \mathcal{L}} \right)^{|E|} &= \left(\frac{k^{\frac{1}{2}(3k^2+1)} (k-1)^{2k^2-2k}}{(2\pi n)^{\frac{1}{2}(2k^2-1)} (\lambda)^{\frac{1}{2}(k-1)^2} (k-2)^{\frac{1}{2}(k^2-1)}} \right)^{|E|} \\
&= \left(\frac{k^{(3k^2+1)} (k-1)^{4k(k-1)}}{(2\pi n)^{(2k^2-1)} (\lambda)^{(k-1)^2} (k-2)^{(k^2-1)}} \right)^{\frac{1}{2}|E|} \\
&= \gamma(n,k)^{\frac{1}{2}|E|}
\end{aligned}$$

In addition, we have $\alpha_{i,j} = a_{v,i,j} - \frac{1}{k^2}$ and $\alpha'_{i',j'} = a_{v',i',j'} - \frac{1}{k^2}$. Making these substitutions into (2.42) completes the proof. □

2.5.4 Outer Sum and Proof of Theorem 2.5

Having estimated the inner sum over the $\boldsymbol{B}$s, we now return to (2.36). Using Proposition 2.18, the definitions of λ and λ' from (2.3), and applying Stirling's approximation gives

$$\begin{aligned}
\mathbf{E}Y^2 &= \frac{1}{n!^{|E|}} \sum_{\boldsymbol{A}} \prod_{v\in V} \binom{n}{nA_v} \prod_{e\in E} (nA_v)!(nA_{v'})! \sum_{\boldsymbol{B}} \prod_{e\in E} \frac{1}{(nB_e)!} \\
&\sim \frac{1}{n!^{|E|}} \sum_{\boldsymbol{A}} \prod_{v\in V} \binom{n}{nA_v} \prod_{e\in E} (nA_v)!(nA_{v'})!({}^{n}\!/_{e})^{-n|E|} \gamma(n,k)^{\frac{1}{2}|E|} \quad \times \\
&\qquad \times \quad \exp\left(\frac{-nk^2(k-1)^2}{2} \left(\sum_{vv'\in E} \left(\frac{1}{2\lambda} \sum_{i,j} \left(a_{v,i,j} + a_{v',i,j} - \frac{2}{k^2} \right)^2 \quad + \right.\right.\right. \\
&\qquad + \quad \left.\left.\left. \frac{1}{2\lambda'} \sum_{i,j} (a_{v,i,j} - a_{v',i,j})^2 + \frac{2}{k^2(k-1)^2} \log \frac{1}{k^2(k-1)^2} \right)\right)\right) \\
&\sim (\xi(n))^{|E|} ({}^{n}\!/_{e})^{-2n|E|} \gamma(n,k)^{\frac{1}{2}|E|} \sum_{\boldsymbol{A}} \prod_{v\in V} \frac{\xi(n)(n/e)^n}{\prod_{i,j} \xi(a_{v,i,j}n)(a_{v,i,j}n/e)^{a_{v,i,j}n}} \quad \times \\
&\qquad \times \quad \prod_{\substack{e\in E \\ e=vv'}} \left(\prod_{i,j} \xi(a_{v,i,j}n) \left(\tfrac{a_{v,i,j}n}{e} \right)^{a_{v,i,j}n} \right) \quad \times \\
&\qquad \times \quad \left(\prod_{i',j'} \xi(a_{v',i',j'}n) \left(\tfrac{a_{v',i',j'}n}{e} \right)^{a_{v',i',j'}n} \right) \quad \times \\
&\qquad \times \quad \exp\left(\frac{-nk^2(k-1)^2}{2} \left(\sum_{vv'\in E} \left(\frac{1}{2\lambda} \sum_{i,j} \left(a_{v,i,j} + a_{v',i,j} - \frac{2}{k^2} \right)^2 + \right.\right.\right. \\
&\qquad + \quad \left.\left.\left. \frac{1}{2\lambda'} \sum_{i,j} (a_{v,i,j} - a_{v',i,j})^2 + \frac{2}{k^2(k-1)^2} \log \frac{1}{k^2(k-1)^2} \right)\right)\right) \\
&\sim \sum_{\boldsymbol{A}} \prod_{v\in V} \prod_{i,j} a_{v,i,j}^{(d-1)a_{v,i,j}n} \prod_{v\in V} \frac{\xi(n)}{\prod_{i,j} \xi(a_{v,i,j}n)} \quad \times \\
&\qquad \times \quad \prod_{\substack{e\in E \\ e=vv'}} \frac{\left(\prod_{i,j} \xi(a_{v,i,j}n) \right) \left(\prod_{i',j'} \xi(a_{v',i',j'}n) \right)}{\xi(n)} \gamma(n,k)^{\frac{1}{2}|E|} \quad \times \\
&\qquad \times \quad \exp\left(\frac{-nk^2(k-1)^2}{2} \left(\sum_{vv'\in E} \left(\frac{1}{2\lambda} \sum_{i,j} \left(a_{v,i,j} + a_{v',i,j} - \frac{2}{k^2} \right)^2 + \right.\right.\right. \\
&\qquad + \quad \left.\left.\left. \frac{1}{2\lambda'} \sum_{i,j} (a_{v,i,j} - a_{v',i,j})^2 + \frac{2}{k^2(k-1)^2} \log \frac{1}{k^2(k-1)^2} \right)\right)\right) \\
&= \sum_{\boldsymbol{A}} P(\boldsymbol{A}, n) e^{nF(\boldsymbol{A})}
\end{aligned}$$

where

$$P(\boldsymbol{A}, n) = \gamma(n,k)^{\frac{1}{2}|E|} \prod_{v\in V} \frac{\xi(n)}{\prod_{i,j} \xi(a_{v,i,j} n)} \prod_{vv'\in E} \frac{\left(\prod_{i,j} \xi(a_{v,i,j} n)\right)\left(\prod_{i',j'} \xi(a_{v',i',j'} n)\right)}{\xi(n)} \tag{2.43}$$

$$\begin{aligned} F(\boldsymbol{A}) = (d-1)\sum_{v\in V}\sum_{i,j} a_{v,i,j}\log a_{v,i,j} \quad & - \\ - \quad & \frac{k^2(k-1)^2}{2}\left(\sum_{vv'\in E}\left(\frac{1}{2\lambda}\sum_{i,j}\left(a_{v,i,j}+a_{v',i,j}-\frac{2}{k^2}\right)^2 + \right.\right. \\ + \quad & \left.\left. \frac{1}{2\lambda'}\sum_{i,j}(a_{v,i,j}-a_{v',i,j})^2 + \frac{2}{k^2(k-1)^2}\log\frac{1}{k^2(k-1)^2}\right)\right) \end{aligned} \tag{2.44}$$

First we bound $F(\boldsymbol{A})$. Note that in our proof of Theorem 2.17 we relaxed the constraints on $\boldsymbol{B}$ in order to bound $f(\boldsymbol{A}, \boldsymbol{B})$. However, in Section 2.5.3 we have shown that

$$F(\boldsymbol{A}) = \max_{\boldsymbol{B} \text{ s.t. } (2.35)} f(\boldsymbol{A}, \boldsymbol{B}) \le \max_{\boldsymbol{B} \text{ s.t. } (2.38)} f(\boldsymbol{A}, \boldsymbol{B})$$

as the constraints of (2.35) are more strict than those of (2.38). Thus combining Theorem 2.17 and Proposition 2.18 gives the following corollary:

Corollary 2.19. *Suppose $d < \ell_k$. The maximum of $F(\boldsymbol{A})$ subject to* (2.16) *is uniquely attained at the point $\boldsymbol{A} = \hat{\boldsymbol{A}}$ and equals*

$$F(\hat{\boldsymbol{A}}) = f(\hat{\boldsymbol{A}}, \hat{\boldsymbol{B}}) = \log\left(k^{|V|}\left(\frac{k-1}{k}\right)^{|E|}\right)^{2n}.$$

Now we will once again use Theorem 2.13 to bound the outer sum. As in the proof of Theorem 2.4, we define a bipartite graph $\Gamma = \Gamma(V_\Gamma, E_\Gamma)$ that allows us to express the equality constraints in (2.34) in terms of its unsigned incidence matrix D. The idea is to associate each equation in (2.34) to a vertex of Γ and every variable to an edge in a way that preserves the incidence relations. To do this, we assign label $w_{v,1,i}$ to equation $\sum_j a_{v,i,j} = 1/k$ and label $w_{v,2,j}$ to equation $\sum_i a_{v,i,j} = 1/k$. The vertex set of Γ is $V_\Gamma = V_{\Gamma,1} \cup V_{\Gamma,2}$, where

$$V_{\Gamma,1} = \{w_{v,1,i} : v \in V, i \in [k]\} \qquad \text{and} \qquad V_{\Gamma,2} = \{w_{v,2,j} : v \in V, j \in [k]\}$$

are the two sides of a bipartition. The edge set is

$$E_\Gamma = \{a_{v,i,j} : v \in V, i, j \in [k]\},$$

where each edge $a_{v,i,j}$ has endpoints $w_{v,1,i}$ and $w_{v,2,j}$ (i.e. the labels of the two equations in which variable $a_{v,i,j}$ appears). Note that unlike in the proof of Theorem 2.4, here we do not require $i \neq j$. Then the equality constraints in (2.34) are equivalent to

$$D\boldsymbol{b} = \boldsymbol{y}, \tag{2.45}$$

where D is the unsigned incidence matrix of Γ and $\boldsymbol{y}$ is the vector in $\mathbb{R}^{|V_\Gamma|}$ whose entries are all $1/k$. The equations in (2.45) are consistent, since they admit the solution

$$\begin{aligned} a_{v,i,j} &= 0 \quad \forall v \in V, i, j \in [k], j \neq i \\ a_{v,i,i} &= \frac{1}{k} \quad \forall v \in V, i \in [k], \end{aligned} \tag{2.46}$$

We observe a few easy facts about Γ. First,

$$|V_\Gamma| = 2k|V| \qquad \text{and} \qquad |E_\Gamma| = k^2|V|$$

Also, Γ has exactly $|V|$ connected components. More precisely, for each $v \in V$, the set of all vertices of the form $w_{v,1,i}$ or $w_{v,2,j}$ induces a connected component of Γ. Each of these components is isomorphic to the complete bipartite graph $K_{k,k}$. In particular, Γ has at least one cycle (since $k \geq 3$). Since Γ is bipartite, it is well known (see e.g. Theorem 8.2.1 in [24]) that D has rank $|V_\Gamma| - |V|$, and therefore $\mathbb{V} = \mathrm{Ker}(D)$ has dimension

$$r = |E_\Gamma| - |V_\Gamma| + |V| = (k^2 - 2k + 1)|V| = (k-1)^2|V|.$$

Now we calculate $\tau(\Gamma)$. Since each maximal forest in Γ is bijectively determined by selecting a spanning tree in each component, we conclude that the number of maximal forests in Γ is

$$\tau(\Gamma) = \tau(K_{k,k})^{|V|} = \left(k^{2k-2}\right)^{|V|}. \tag{2.47}$$

Let

$$K = \{\boldsymbol{A} \in \mathbb{R}^{|E_\Gamma|} : 0 \leq a_{v,i,j} \leq 1/k\} \qquad \text{and} \qquad K_1 = \left\{\boldsymbol{A} \in \mathbb{R}^{|E_\Gamma|} : \tfrac{0.9}{k^2} \leq a_{v,i,j} \leq \tfrac{1.1}{k^2}\right\}.$$

Clearly, K is a compact convex set with non-empty interior K°, and any choice of $\boldsymbol{A}$ that satisfies (2.34) lies inside K. Let $\phi(\boldsymbol{A}) = F(\boldsymbol{A})$, which is continuous on K, and

$$\psi(\boldsymbol{A}) = \prod_{v \in V} \left(\prod_{i,j} \sqrt{a_{v,i,j}} \right)^{d-1},$$

which is continuous and positive on K_1. By Corollary 2.19, the maximum of $\phi(\boldsymbol{A})$ in K subject to (2.45) is uniquely attained at $\hat{\boldsymbol{A}} \in K_1 \subset K^\circ$. Moreover, $\phi(\boldsymbol{A})$ is twice continuously differentiable in the interior K°. Set $\phi = (d-1)G_1 - \frac{k^2(k-1)^2}{2}G_2$. Then

$$\frac{\partial^2 G_1}{\partial a_{\nu,\ell,m}\partial a_{\nu',\ell',m'}} = \begin{cases} \frac{1}{a_{\nu,\ell,m}} & \nu=\nu', \ell=\ell', m=m' \\ 0 & \text{otherwise} \end{cases}$$

The second derivatives of G_2 are more difficult. First,

$$\frac{\partial G_2}{\partial a_{\nu,\ell,m}} = \sum_{v' \text{ s.t. } \nu v' \in E} \left(\frac{1}{\lambda}(\alpha_{\nu,\ell,m} + \alpha_{v',\ell,m}) + \frac{1}{\lambda'}(\alpha_{\nu,\ell,m} - \alpha_{v',\ell,m}) \right)$$

Then

$$\frac{\partial^2 G_2}{\partial a_{\nu,\ell,m}\partial a_{\nu',\ell',m'}} = \begin{cases} d(\frac{1}{\lambda} + \frac{1}{\lambda'}) & \nu=\nu', \ell=\ell', m=m' \\ \frac{1}{\lambda} - \frac{1}{\lambda'} & \nu\nu' \in E, \ell=\ell', m=m' \\ 0 & \ell \neq \ell' \text{ or } m \neq m' \end{cases}$$

Noting that

$$\frac{1}{\lambda} + \frac{1}{\lambda'} = \frac{1}{(k-1)^2+1} + \frac{1}{(k-1)^2-1} = \frac{2(k-1)^2}{((k-1)^2+1)((k-1)^2-1)} = \frac{2(k-1)^2}{\lambda\lambda'}$$

and

$$\frac{1}{\lambda} - \frac{1}{\lambda'} = \frac{1}{(k-1)^2+1} - \frac{1}{(k-1)^2-1} = \frac{-2}{((k-1)^2+1)((k-1)^2-1)} = \frac{-2}{\lambda\lambda'}$$

we get

$$\frac{\partial^2 \phi}{\partial a_{\nu,\ell,m}\partial a_{\nu',\ell',m'}} = \begin{cases} \frac{d-1}{a_{\nu,\ell,m}} - d\frac{k^2(k-1)^4}{\lambda\lambda'} & \nu=\nu', \ell=\ell', m=m' \\ \frac{k^2(k-1)^2}{\lambda\lambda'} & \nu\nu' \in E, \ell=\ell', m=m' \\ 0 & \ell \neq \ell' \text{ or } m \neq m' \end{cases}$$

Hence, recalling that A is the adjacency matrix of G, the Hessian matrix of ϕ at

$\boldsymbol{A} = \hat{\boldsymbol{A}}$ is

$$H = \left((d-1)k^2 - d\frac{k^2(k-1)^4}{\lambda\lambda'}\right) I_{|V|} \otimes I_{k^2} + \left(\frac{k^2(k-1)^2}{\lambda\lambda'}\right) A \otimes I_{k^2}.$$

Consider the structure of D, the unsigned incidence matrix of Γ. As each component of Γ is isomorphic to $K_{k,k}$, we may write $D = I_{|V|} \otimes \hat{D}$, where $\hat{D}$ is a $2k \times k^2$ incidence matrix for $K_{k,k}$. Note that as $\hat{D}$ is bipartite with one connected component, $\text{rank}(\hat{D}) = 2k-1$ and so $\dim(\ker(D)) = k^2 - (2k-1) = (k-1)^2$. Let U be any $k^2 \times (k-1)^2$ matrix whose columns form a basis of $\ker(\hat{D})$. Then we claim the columns of $I_{|V|} \otimes U$ form a basis of $\mathbb{V}$. Every column of $I_{|V|} \otimes U$ is in the kernel of D as the corresponding column of U is in the kernel of $\hat{D}$. The columns of U are linearly independent, as they form a basis, so the columns of $I_{|V|} \otimes U$ are as well. Finally, there are $|V|r = \dim(\mathbb{V})$ columns in $I_{|V|} \otimes U$. Then, making use of several facts about the Kronecker product (see, for example, [47] Theorem 4.5 and Problem 4.3.3a) we have

$$
\begin{aligned}
\det(-H|_{\mathbb{V}}) &= \frac{\det((I_{|V|}\otimes U)^T(-H)(I_{|V|}\otimes U))}{\det((I_{|V|}\otimes U)^T(I_{|V|}\otimes U))} \\
&= \frac{\det\left(\left(-(d-1)k^2+d\frac{k^2(k-1)^4}{\lambda\lambda'}\right)I_{|V|}\otimes U^TU-\left(\frac{k^2(k-1)^2}{\lambda\lambda'}\right)A\otimes U^TU\right)}{\det(I_{|V|}\otimes U^TU))} \\
&= \det\left(\left(-(d-1)k^2+d\frac{k^2(k-1)^4}{\lambda\lambda'}\right)I_{|V|}-\frac{k^2(k-1)^2}{\lambda\lambda'}A\right)^{(k-1)^2} \times \\
&\qquad \times \quad \frac{\det(U^TU)^{|V|}}{1^{(k-1)^2}\det(U^TU)^{|V|}} \\
&= \left(\prod_{i=1}^{|V|}\left(-(d-1)k^2+d\frac{k^2(k-1)^4}{\lambda\lambda'}\right)-\frac{k^2(k-1)^2}{\lambda\lambda'}\alpha_i\right)^{(k-1)^2} \\
&= \left(\left(\frac{k^2}{\lambda\lambda'}\right)^{|V|}\prod_{i=1}^{|V|}(-\lambda\lambda'(d-1)+d(k-1)^4-\alpha_i(k-1)^2\right)^{(k-1)^2} \\
&= \left(\left(\frac{k^2}{\lambda\lambda'}\right)^{|V|}\prod_{i=1}^{|V|}(\lambda\lambda'+d-\alpha_i(k-1)^2\right)^{(k-1)^2} \\
&= (h(d,k))^{(k-1)^2}
\end{aligned}
$$

where $h(d,k)$ is as defined in (2.4).

Let

$$\mathbb{X}_n = \left\{\boldsymbol{A}\in K\cap\frac{1}{n}\mathbb{Z}^{|E_\Gamma|} : D\boldsymbol{A}=\boldsymbol{y}\right\}.$$

The solution described in (2.46) belongs to K and, since $k \mid n$, also to $\frac{1}{n}\mathbb{Z}^{|E_\Gamma|}$, so $\mathbb{X}_n$ is not empty. For each $\boldsymbol{A}\in\mathbb{X}_n$, let

$$T_n(\boldsymbol{A}) = P(\boldsymbol{A},n)e^{nF(\boldsymbol{A})}$$

and

$$c_n = \gamma(n,k)^{\frac{1}{2}|E|}(2\pi n)^{-\frac{1}{2}(k^2-1)|V|+\frac{1}{2}(2k^2-1)|E|}.$$

First, from (2.43) and since $1 \leq \xi(x) = O(\sqrt{x})$ as $x \to \infty$, we can bound the polynomial factor $P(\boldsymbol{A}, n)$ in each term by

$$P(\boldsymbol{A}, n) = O\left(n^{|V|+2k^2|E|}\right), \tag{2.48}$$

where the hidden constant in the big O notation does not depend on $\boldsymbol{A}$.

In view of (2.48), for $\boldsymbol{A} \in \mathbb{X}_n$,

$$P(\boldsymbol{A}, n)/c_n = O(n^{|V|+2k^2|E|}/c_n) = e^{o(n)}$$

and combined with Corollary 2.19 this gives

$$T_n(\boldsymbol{A}) = c_n(P(\boldsymbol{A}, n)/c_n)e^{nF(\boldsymbol{A})} = O(c_n e^{n\phi(\hat{\boldsymbol{A}})+o(n)}).$$

Now note that we have chosen c_n such that because $\xi(x) \sim \sqrt{2\pi x}$ as $x \to \infty$, and after a few computations, we have that for $\boldsymbol{A} \in \mathbb{X}_n \cap K_1$,

$$P(\boldsymbol{A}, n) = c_n(\psi(\boldsymbol{A}) + o(1)).$$

Finally, as we've met all of the conditions of Theorem 2.13,

$$
\begin{aligned}
\mathbf{E}Y^2 &= \sum_{\boldsymbol{A}\in\mathbb{X}_n} T_n(\boldsymbol{A}) \\
&\sim \frac{\psi(\hat{\boldsymbol{A}})}{\tau(\Gamma)^{1/2}\det(-H|_{\mathbb{V}})^{1/2}}(2\pi n)^{r/2} c_n e^{n\phi(\hat{\boldsymbol{A}})} \\
&= \frac{k^{-k^2(d-1)|V|}}{(k^{2k-2})^{\frac{1}{2}|V|} h(d,k)^{\frac{(k-1)^2}{2}}}(2\pi n)^{\frac{1}{2}(k-1)^2|V|}\gamma(n,k)^{\frac{1}{2}|E|} \quad \times \\
&\qquad \times \quad (2\pi n)^{-\frac{1}{2}(k^2-1)|V|+\frac{1}{2}(2k^2-1)|E|}\left(k^{|V|}\left(\frac{k-1}{k}\right)^{|E|}\right)^{2n} \\
&= \frac{\gamma(n,k)^{\frac{1}{2}|E|}k^{-2k^2|E|+(k^2-k+1)|V|}}{h(d,k)^{\frac{(k-1)^2}{2}}}(2\pi n)^{-(k-1)|V|+\frac{1}{2}(2k^2-1)|E|}\left(k^{|V|}\left(\frac{k-1}{k}\right)^{|E|}\right)^{2n} \\
&= \frac{k^{(k^2-k+1)|V|-\frac{1}{2}(k^2-1)|E|}(k-1)^{(2k^2-2k)|E|}}{\lambda^{\frac{1}{2}(k-1)^2|E|}(k-2)^{\frac{1}{2}(k^2-1)|E|}h(d,k)^{\frac{(k-1)^2}{2}}}(2\pi n)^{-(k-1)|V|}\left(k^{|V|}\left(\frac{k-1}{k}\right)^{|E|}\right)^{2n}
\end{aligned}
$$

This completes the proof of Theorem 2.5.

2.6 Joint Moment Calculations and Proof of Theorem 2.7

Recall that Y counts the number of strongly equitable k-colorings of a random n-lift of G. In this section we allow G to be any d-regular graph. Recall that for fixed $j \geq 3$, we denote the number of j-cycles in a random lift by Z_j. In this section, we estimate the expected value of the joint moment YZ_j by proving the following proposition:

Proposition 2.20. *If Y is the number of strongly equitable k-colorings of a random n-lift L of G and, for $j \geq 3$, Z_j counts the number of j-cycles in L, then*

$$\frac{\mathbf{E}(YZ_j)}{\mathbf{E}Y} \sim \lambda_j(1+\delta_j)$$

where λ_j and δ_j are as defined in (2.6).

In order to prove Proposition 2.20 we require two lemmas. The first is a lemma of Friedman, originally proved in [20], as presented in [25]:

Lemma 2.21. *Suppose that G is d-regular with $d \geq 3$ and let $\alpha_1, \ldots, \alpha_{|V|}$ be the eigenvalues of the adjacency matrix of G. For $i = 1, \ldots, |V|$, let ${\beta_i}^+$ and ${\beta_i}^-$ denote the roots of the quadratic $x^2 - \alpha_i x + d - 1 = 0$. That is,*

$$ {\beta_i}^+ = \tfrac{1}{2}\alpha_i + \sqrt{\tfrac{1}{4}\alpha_i^2 - (d-1)}, \quad \beta_i^- = \tfrac{1}{2}\alpha_i - \sqrt{\tfrac{1}{4}\alpha_i^2 - (d-1)}. $$

Then the number of non-backtracking closed j-walks in G is given by

$$ \begin{aligned} c_j &:= \frac{1}{2}|V|(d-2)(1+(-1)^j) + \sum_{i=1}^{|V|}((\beta_i^+)^j + (\beta_i^-)^j) \\ &= (|E| - |V|)(1+(-1)^j) + \sum_{i=1}^{|V|}((\beta_i^+)^j + (\beta_i^-)^j) \end{aligned} $$

where we note that in a d-regular graph we have

$$ 2|E| = \sum_{v \in V} d(v) = d|V| $$

and thus

$$ \frac{1}{2}|V|(d-2) = \frac{1}{2}(d|V| - 2|V|) = \frac{1}{2}(2|E| - 2|V|) = |E| - |V|. $$

The second lemma considers properly coloring cycles.

Lemma 2.22. *The number of ways to properly k-color a rooted, directed cycle of length j is*

$$ (k-1)^j + (k-1)(-1)^j. $$

Proof. Assign a unique color to each vertex in K_k. Each proper k-coloring of the j-cycle corresponds to a closed walk of length j in K_k: starting with the color of the root, travel in the direction of the cycle to the vertex corresponding to the next color. The total number of closed j-walks in K_k is $\text{tr}(A^j)$, where $A = J_k - I_k$ is the adjacency matrix of K_k. As A has eigenvalues $(k-1)$ with multiplicity one and -1 with multiplicity $k-1$, the number of directed, rooted j-walks is $(k-1)^j + (k-1)(-1)^j$. □

We now prove Proposition 2.20.

Proof of Propositon 2.20. We estimate $\mathbf{E}(YZ_j)$ by counting ordered triples of the form (lift, j-cycle, coloring), where the lift contains the j-cycle and is properly colored by the coloring, and then dividing by the total number of lifts.

First, it will be convenient to consider rooted, oriented cycles. Each cycle of length j contains $2j$ rooted, oriented cycles, so we simply correct our answer by a factor of $2j$.

For any rooted, oriented cycle $C = (c_1, \ldots, c_j)$, we define the *cycle type* of C to be the sequence $(v_1, \ldots, v_j)$ such that $c_i \in \Pi^{-1}(v_i)$; that is, the cycle type is the sequence of fibers containing the vertices of the cycle. Note that every cycle type corresponds to a closed, non-backtracking walk in G. Thus our first step in counting ordered triples is to choose a closed, non-backtracking walk w to determine the cycle type of our cycle.

Next we fix a coloring, Q, of the cycle, after which we choose the vertices in the lift to realize the cycle of the appropriate cycle type and color them according to Q. If the cycle type revisits a fiber, there may be fewer than n options for each vertex. However, as the cycle has fixed length j and n goes to infinity, there are $(1 - o(1))n$ choices for each vertex, for a total contribution of $(1 - o(1))n^j$.

Having specified w, Q and the vertices of the cycle, we now follow a process very similar to that of Section 2.4.1 resulting in (2.17). For each $e = vv' \in E$ we define $b^*_e = (b^*_{e,i,i'})_{i,i' \in [k], i \neq i'}$ where $b^*_{e,i,i'}$ denotes the proportion of edges in $\Pi^{-1}(e)$ that connect a vertex of color i in $\Pi^{-1}(v)$ to a vertex of color i' in $\Pi^{-1}(v')$, excluding edges already prescribed by w and Q. We set $\boldsymbol{b^*} = (b^*_e)_{e \in E}$. The entries of each b^*_e must be in $\frac{1}{n}\mathbb{Z}$ and satisfy

$$\begin{aligned} b^*_{e,i,i'} &\geq 0 \quad \forall e \in E, i, i' \in [k], i \neq i' \\ \sum_{i' \neq i} b^*_{e,i,i'} &= a(v, i, w, Q) \quad \forall e = vv' \in E, i \in [k] \\ \sum_{i \neq i'} b^*_{e,i,i'} &= a(v', i, w, Q) \quad \forall e = vv' \in E, i' \in [k]. \end{aligned} \tag{2.49}$$

where $a(v, i, w, Q)$ is the proportion of vertices in $\Pi^{-1}(v)$ that still need to receive color i, after accounting for the vertices already colored by Q according to w, so that each color is assigned to $\frac{1}{k}$ of the vertices in $\Pi^{-1}(v)$ (as Y is a strongly equitable coloring). Specifically, if ℓ_i vertices in $\Pi^{-1}(v)$ have already been assigned color i by w and Q, then

$$a(v, i, w, Q) = \frac{1}{k} - \frac{\ell_i}{n}.$$

For each vertex v in G, we must color the uncolored vertices in the fiber $\Pi^{-1}(v)$. Define $\epsilon(v, w)$ to be the number number of times v is encountered in w, and let

$$a^*_v = (a^*_{v,i})_{i \in [k]} = (a(v, i, w, Q))_{i \in [k]}.$$

Then the number of ways to color the vertices is

$$\prod_{v \in V} \binom{n - \epsilon(v, w)}{a^*_v n}.$$

We then decide, for every $e = vv' \in E$ and distinct colors $i, i' \in [k]$, which sets of $b_{e,i,i'}n$ in $\Pi^{-1}(v)$ and $\Pi^{-1}(v')$ will be matched. For each $e = vv'$, define $a^{**}_{e,v} = (a^{**}_{e,v,i})_{i\in[k]}$ to be the sequence of vertices in $\Pi^{-1}(v)$ that have been assigned color i and have not already been matched to a vertex in $\Pi^{-1}(v')$ by w. Then the number of ways to choose sets of vertices to be matched is

$$\prod_{\substack{e\in E\\ e=vv'}} \left(\prod_{i\in[k]} \frac{(a^{**}_{e,v,i}n)!}{\prod_{i'\neq i}(b^{*}_{e,i,i'}n)!} \right) \left(\prod_{i'\in[k]} \frac{(a^{**}_{e,v',i'}n)!}{\prod_{i\neq i'}(b^{*}_{e,i,i'}n)!} \right) = \prod_{\substack{e\in E\\ e=vv'}} \frac{(a^{**}_{e,v}n)!}{(b^{*}_{e}n)!} \frac{(a^{**}_{e,v'}n)!}{(b^{*}_{e}n)!}$$

Finally, we need to choose a perfect matching between these sets, which can be done in

$$\prod_{e\in E} \prod_{\substack{i,i'\in[k]\\ i\neq i'}} (b^{*}_{e,i,i'}n)! = \prod_{e\in E}(b^{*}_{e}n)!$$

different ways. Putting everything together, we get

$$\mathbf{E}(YZ_j) = \frac{1}{2j} \sum_{w} \sum_{Q} (1+o(1)) n^j \frac{1}{n!^{|E|}} \prod_{v\in V} \binom{n-\epsilon(v,w)}{a^{*}_{v}n} \times \\ \times \sum_{\boldsymbol{b^*} \text{ s.t. } (2.49)} \prod_{\substack{e\in E\\ e=vv'}} \frac{(a^{**}_{e,v}n)!(a^{**}_{e,v'}n)!}{(b^{*}_{e}n)!} \tag{2.50}$$

We now seek estimates for the equation above. First, note that for any walk w,

$$\sum_{v\in V} \epsilon(v,w) = j$$

as the walk is of length j. Similarly, across all $v \in V$, there are exactly j vertices already colored with some color. As Y is a strongly equitable coloring, we have

$$\prod_{v\in V} \binom{n-\epsilon(v,w)}{a^{*}_{v}n} \sim \frac{\left(\frac{n}{k}\right)^j}{n^j} \cdot \frac{(n!)^{|V|}}{\left(\left(\frac{n}{k}\right)!\right)^{|V|}}. \tag{2.51}$$

Furthermore, for any choice of $\boldsymbol{b}^*$, the product over all edges of the $a^{**}_{e,v}$ encounters all j edges of w. For $t = 1, \ldots, j$, set $b_t = b_{e,i,i'}$ where e is the edge that connects vertices t and $t+1$ of w and Q colors those vertices with colors i and i', respectively, where the $b_{e,i,i'}$ are as defined in (2.30) (as opposed to the $b^*_{e,i,i'}$ defined in (2.49)). Then

$$\prod_{\substack{e\in E\\ e=vv'}} \frac{(a^{**}_{e,v}n)!(a^{**}_{e,v'}n)!}{(b^*_e n)!} \sim \prod_{t=1}^{j} \frac{b_t n}{(\frac{n}{k})^2} \prod_{\substack{e\in E\\ e=vv'}} \frac{((\frac{n}{k})!)^2}{(b_e n)!}. \tag{2.52}$$

Putting these facts together allows us to rewrite (2.50) in terms of $p(\hat{\boldsymbol{a}}, \boldsymbol{b}, n)$ and $f(\boldsymbol{a}, \boldsymbol{b})$ as defined in (2.28) and (2.29), respectively:

$$\mathbf{E}(YZ_j) \sim \frac{1}{2j} \sum_w \sum_Q k^j \sum_{\boldsymbol{b} \text{ s.t. } (2.30)} \left(\prod_{t=1}^{j} b_t \right) p(\hat{\boldsymbol{a}}, \boldsymbol{b}, n) e^{nf(\boldsymbol{a},\boldsymbol{b})}. \tag{2.53}$$

Note that we may bound

$$\prod_{t=1}^{j} b_t n \leq 1$$

and that, furthermore, when $\boldsymbol{b} = \hat{\boldsymbol{b}}$, none of the b_t vanish. Therefore, Theorem 2.16 holds. Though we omit the details, we may thus use Theorem 2.13 almost identically as in Section 2.4.2 to get

$$\mathbf{E}(YZ_j) \sim \frac{1}{2j} \sum_w \sum_Q \left(\frac{n}{k}\right)^{-j} \left(\frac{n}{k(k-1)}\right)^j \mathbf{E}(Y).$$

Finally, as the terms no longer depend on w or Q, we apply Lemmas 2.21 and 2.22

to get

$$\begin{aligned}\mathbf{E}(YZ_j) &= \frac{1}{2j} \cdot c_j \cdot ((k-1)^j + (k-1)(-1)^j) \cdot \frac{1}{(k-1)^j}\mathbf{E}(Y) \\ &= \frac{c_j}{2j}\left(1 + \frac{(-1)^j}{(k-1)^{j-1}}\right)\mathbf{E}(Y) \\ &= \lambda_j(1+\delta_j)\mathbf{E}(Y)\end{aligned}$$

Dividing by $\mathbf{E}(Y)$ completes the proof. □

In order to apply the small subgraph conditioning method, we need estimations for higher moments. The argument for Theorem 2.7 is just an extension of the proof of Proposition 2.20. We count ordered tuples (lift, cycle, ..., cycle, coloring) so that the lift is properly colored by the coloring and contains cycles of the specified length. We claim the contribution from cases where the cycles intersect turn out to be negligible, adapting an argument of [29]. Suppose that the cycles form a subgraph H with ν vertices and μ edges. If the cycles are disjoint, then $\nu = \mu$. If they overlap, then as the minimum degree in H is at least two and some vertex has degree at least three, we have $\nu < \mu$. We then follow the same argument as in the proof above with the following changes. When choosing vertices for the cycle, we have $(1+o(1))n^{\nu}$ choices. In (2.51), the coefficient is $(\frac{n}{k})^{\nu} n^{-\nu}$. Then in (2.52), the product ranges from 1 to μ. Thus unlike in the proposition, where by (2.53) all of the ns have cancelled, we get a $\Theta(n^{\nu-\mu})$ term. In the disjoint case the ns do cancel, but, as there are finitely many isomorphism types of H, the contribution from terms with overlapping cycles are on the order $\frac{1}{n}$ times the rest. Finally, the disjoint terms decompose into a product of factors corresponding to the individual cycles, giving the desired result.

Appendix A

Technical Lemmas

This section contains the proofs of several technical lemmas used throughout the thesis. The proofs for these lemmas are often rote calculation or uninspiring inequalities but are included for the sake of completeness.

We frequently use the following inequality of Bollobás in this section:

Lemma A.1 (Bollobás, [8])**.** *For* $t > -1$*,*

$$\log(1+t) \leq t - \frac{1}{2}t^2 + \frac{1}{3}t^3$$

where $\log x$ *represents the natural logarithm.*

The proof relies of the Taylor expansion of $\log(1+x)$.

Lemma A.2. *If* $t \geq 3r-1$ *then*

$$\frac{2}{t+1} \leq \frac{2}{3} \cdot \frac{1}{r}$$

and

$$-\left(1 - \frac{2}{t+1}\right)^{t-1} \leq -\left(1 - \frac{2}{3r}\right)^{t-1} = -\left(\frac{r-1}{r} + \frac{1}{3r}\right)^{t-1}.$$

Proof. We have

$$t \geq 3r - 1 \implies \frac{t+1}{2} \geq \frac{3}{2}r \implies \frac{2}{t+1} \leq \frac{2}{3} \cdot \frac{1}{r}$$

and

$$\begin{aligned} \frac{2}{t+1} \leq \frac{2}{3r} &\implies \left(1 - \frac{2}{t+1}\right)^{t-1} \geq \left(1 - \frac{2}{3r}\right)^{t-1} \\ &\implies -\left(1 - \frac{2}{t+1}\right)^{t-1} \leq -\left(1 - \frac{2}{3r}\right)^{t-1} = -\left(\frac{r-1}{r} + \frac{1}{3r}\right)^{t-1}. \end{aligned}$$

□

Lemma A.3. *For* $t \geq 5$*,*

$$p(t) = t \cdot \left(\frac{2}{3}\right)^t - \frac{2}{3} < 0.$$

Proof. Note that

$$p(5) = 5 \cdot \left(\frac{2}{3}\right)^5 - \frac{2}{3} = -\frac{2}{243} < 0$$

and that

$$\frac{dp}{dt} = \left(\frac{2}{3}\right)^t [1 + t(\log 2 - \log 3)]$$

which is less than zero for $t > -\frac{1}{\log 2 - \log 3} \approx 2.466 < 5$. Thus p is negative at $t = 5$ and decreasing on $[5, \infty)$. □

Lemma A.4. *For* $r \geq 26$ *and any* $c \in [r-1, r]$*,*

$$-x + c \ln\left(1 + \frac{x}{c}\right) + 1 < 0$$

for $x = 1 + \sqrt{2r}$*.*

Proof. Using Lemma A.1,

$$-x + c\ln\left(1+\frac{x}{c}\right) + 1 \leq -x + c\left(\frac{x}{c} - \frac{x^2}{2c^2} + \frac{x^3}{3c^3}\right) + 1 = 1 - \frac{x^2}{2c} + \frac{x^3}{3c^2}.$$

With $x = 1 + \sqrt{2r}$ and $c \in [r-1, r]$ we have

$$-\frac{x^2}{2c} = -\frac{1+2\sqrt{2r}+2r}{2c} \leq -\frac{1+2\sqrt{2r}+2r}{2r} = -1 - \frac{1+2\sqrt{2}\sqrt{r}}{2r}$$

and

$$\frac{x^3}{3c^2} = \frac{1+3\sqrt{2r}+6r+2^{3/2}r^{3/2}}{3c^2} \leq \frac{1+3\sqrt{2r}+6r+2^{3/2}r^{3/2}}{3(r-1)^2}.$$

Thus

$$\begin{aligned} -x + c\ln\left(1+\frac{x}{c}\right) + 1 &\leq 1 - 1 - \frac{1+2\sqrt{2}\sqrt{r}}{2r} + \frac{1+3\sqrt{2r}+6r+2^{3/2}r^{3/2}}{3(r-1)^2} \\ &= \frac{-2\sqrt{2}r^{5/2} + 9r^2 + 18\sqrt{2}r^{3/2} + 8r - 2\sqrt{2}\sqrt{r} - 1}{6r(r-1)^2} \\ &\leq \frac{-2\sqrt{2}r^{3/2} + 9r + 18\sqrt{2}\sqrt{r} + 8}{6(r-1)^2}. \end{aligned}$$

Note that the denominator is positive, so in order for this quantity to be negative we need

$$-2\sqrt{2}r^{3/2} + 9r + 18\sqrt{2}\sqrt{r} + 8 < 0.$$

At $r = 26$, we have

$$-2\sqrt{2}(26)^{3/2} + 9\cdot 26 + 18\sqrt{2}\sqrt{26} + 8 = -104\sqrt{13} + 234 + 36\sqrt{13} + 8 < -3.177 < 0.$$

Furthermore, for $r \geq 26$ we have

$$\begin{aligned}\frac{d}{dr}[-2\sqrt{2}r^{3/2} + 9r + 18\sqrt{2}\sqrt{r} + 8] &= -3\sqrt{2}\sqrt{r} + 9 + \frac{9\sqrt{2}}{\sqrt{r}} \\ &\leq -6\sqrt{13} + 9 + \frac{9\sqrt{13}}{13} \\ &< -10.137 < 0\end{aligned}$$

so this quantity is decreasing. □

Lemma A.5. *For all $k \geq 3$,*

$$u_k = \frac{2\log k}{\log k - \log(k-1)} < (2k-1)\log k.$$

Proof. First note, via L'Hôpital's rule, $\lim_{k\to\infty}(2k-1)(\log k - \log(k-1)) = 2$. Then from Lemma A.1 we have

$$\begin{aligned}\frac{d}{dk}[(2k-1)(\log k - \log(k-1))] &= 2\log\left(1 + \frac{1}{k-1}\right) - \frac{2k-1}{k(k-1)} \\ &< 2\left(\frac{1}{k-1} - \frac{1}{2(k-1)^2} + \frac{1}{3(k-1)^3}\right) - \frac{2k-1}{k(k-1)} \\ &= \frac{-k+3}{3k(k-1)^3}\end{aligned}$$

which is nonpositive for $k \geq 3$. We conclude $(2k-1)(\log k - \log(k-1)) > 2$ for $k \geq 3$.

Thus as $\log k > 0$ for $k \geq 3$ we have

$$\begin{aligned}2 < (2k-1)(\log k - \log(k-1)) &\implies 2\log k < (2k-1)\log k(\log k - \log(k-1)) \\ &\implies u_k < (2k-1)\log k.\end{aligned}$$

□

Lemma A.6. *For $k \geq 3$ we have*

$$-\log\log\left(\frac{1}{1-\frac{1}{k}}\right) > \log k - \frac{5}{8k} > 0.$$

Proof. First, for $k \geq 3$ we have

$$8k\log k \geq 24\log 3 > 26.3 > 5$$

and thus $\log k - \frac{5}{8k} > 0$.

We verify the other inequality for $k = 3$ and $k = 4$ manually:

$$-\log\log\left(\frac{1}{1-\frac{1}{3}}\right) > 0.9 > 0.891 > \log 3 - \frac{5}{8(3)}$$

and

$$-\log\log\left(\frac{1}{1-\frac{1}{4}}\right) > 1.24 > 1.231 > \log 4 - \frac{3}{8(4)}.$$

Now take $k \geq 5$. The polynomial $p(k) = 3k^3 - 17k^2 + 21k - 15$ has one real zero at approximately $4.32 < 5$. Thus, as its leading term is $3 > 0$, $p(k)$ is positive for $k \geq 5$. Then

$$\frac{5}{8k} - \frac{1}{k-1} + \frac{k}{2(k-1)^2} - \frac{k}{3(k-1)^3} = \frac{3k^3 - 17k^2 + 21k - 15}{24(k-1)^3 k} > 0,$$

or

$$\frac{5}{8k} > \frac{1}{k-1} - \frac{k}{2(k-1)^2} + \frac{k}{3(k-1)^3}$$

on $k \geq 5$ as well. Now, using the Taylor expansion for e^x,

$$\begin{aligned} e^{\frac{5}{8k}} &> 1 + \frac{5}{8k} \\ &> 1 + \frac{1}{k-1} - \frac{k}{2(k-1)^2} + \frac{k}{3(k-1)^3} \\ &= k\left(\frac{1}{k-1} - \frac{1}{2(k-1)^2} + \frac{1}{3(k-1)^3}\right) \\ &\geq k \log\left(1 + \frac{1}{k-1}\right) && \text{Lemma A.1} \\ &= k \log\left(\frac{1}{1-\frac{1}{k}}\right) \end{aligned}$$

Taking the logarithm of both sides and negating the inequality gives

$$-\frac{5}{8k} < -\log k - \log\log\left(\frac{1}{1-\frac{1}{k}}\right) \implies -\log\log\left(\frac{1}{1-\frac{1}{k}}\right) > \log k - \frac{5}{8k}.$$

□

Lemma A.7. *For $k \geq 3$ we have*

$$\frac{\log k}{k-1} < \left(\log\log k^2 - \frac{5}{8k}\right).$$

Proof. Let

$$f(k) = 2\log k - k^{1/(k-1)} \exp\left(\frac{5}{8k}\right).$$

First we demonstrate that $f(k)$ is an increasing function on $k \geq 3$. Noting that

$$\frac{d}{dk}[k^{1/(k-1)}] = k^{1/(k-1)} \cdot \frac{k(\log k - 1) + 1}{k(k-1)^2}$$

we have

$$f'(k) = \frac{2}{k} + k^{1/(k-1)} \exp\left(\frac{5}{8k}\right)\left(\frac{5}{8k^2} + \frac{k(\log k - 1) + 1}{k(k-1)^2}\right).$$

As $\log k - 1 > 0$ for $k \geq 3$, we see $f'(k) > 0$. Now for $k = 3$ we have

$$2\log 3 - 3^{1/(3-1)} \exp\left(\frac{5}{8(3)}\right) > 0.06 > 0$$

and therefore for $k \geq 3$ we have

$$\begin{aligned} & 2\log k - k^{1/(k-1)} \exp\left(\frac{5}{8k}\right) &>& \quad 0 \\ \implies & \log k^2 &>& \quad k^{\frac{1}{k-1}} \exp\left(\frac{5}{8k}\right) \\ \implies & \log\log k^2 &>& \quad \frac{\log k}{(k-1)} + \frac{5}{8k} \\ \implies & \log\log k^2 - \frac{5}{8k} &>& \quad \frac{\log k}{(k-1)}. \end{aligned}$$

□

Appendix B

List of Notation

$\mathcal{A}_r$: The family of graphs with chromatic number at most r

B_d : The bouquet graph, a single vertex with d loops

$\mathcal{B}_r$: The family of r-partite extremal graphs

$\binom{A}{k}$: Collection of subsets of A of size k

C_k : A cycle of length k

$c_k(G)$: The number of cycles of length k in G

$\mathcal{C}_r$: The family of complete r-partite graphs

$\Delta(G)$: The maximum degree of G

$\delta(G)$: The minimum degree of G

$D_{KL}(P||Q)$: The Kullback-Leibler divergence from Q to P

$d(H, G)$: The density of H in G

$\det(M)$: The determinant of matrix M

$A \sqcup B$: The disjoint union of A and B

$E(G)$: The set of edges of G

$e(G)$: The number of edges of G

$\mathbf{E}[X]$: The expected value of X

$\exp(f(x))$: The natural base e raised to the power $f(x)$

$\mathrm{ex}(n, F)$: The extremal number of edges in an F-free graph

$\mathrm{ex}(n, T, F)$: The extremal number of copies of T in an F-free graph

$\mathcal{G}_{n,d}$: The uniform distribution of d-regular graphs on n vertices

$G[U]$: The subgraph of G induced by $U \subseteq V$

$\langle \boldsymbol{u}, \boldsymbol{v} \rangle$: The inner product of $\boldsymbol{u}$ with $\boldsymbol{v}$

$K_{a,b}$: The complete bipartite subgraph with parts of size a and b

K_r : The complete graph on r vertices

$k_r(G)$: The number of cliques of size r in G

$\mathcal{L}_n(G)$: The graph model of random n-lifts of G

$\mathcal{K}_r$: The family of K_{r+1}-free graphs

$n(G)$: The number of vertices in G

$n_T(G)$: The number of copies of T in G

$O(f(n))$: Order of growth at most $f(n)$

$o(f(n))$: Order of growth strictly smaller than $f(n)$

$\Omega(f(n))$: Order of growth at least $f(n)$

$\otimes$: The Kronecker product

P_ℓ : The path on ℓ edges and $\ell + 1$ vertices

$p_\ell(G)$: The number of paths of length ℓ in G

$\mathrm{poly}(n)$: A function with growth at most n^k for some integer k

Pr : Probability

$\mathbb{R}$: The set of real numbers

S_t : The star graph with t leaves

$s_t(G)$: The number of star S_t in G

$f(x) \sim g(x)$: The functions f and g are asymptotically equal

$u \sim v$: The vertices u and v are adjacent

$\mathrm{spec}(M)$: The spectrum of the matrix M

$\tau(G)$: The number of spanning trees in G

$\Theta(f(n))$: Order of growth precisely $f(n)$

$T_r(n)$: The Turán graph on n vertices with r parts

$\mathrm{tr}(M)$: The trace of the matrix M

$\boldsymbol{u}!$: For a vector $\boldsymbol{u}$, the product of its entries

$\binom{n}{\boldsymbol{u}}$: For a vector $\boldsymbol{u}$ whose entries sum to n, $\frac{n!}{\boldsymbol{u}!}$

$V(G)$: The set of vertices of G

$\chi(G)$: The chromatic number of G

$\xi(n)$: From Stirling's approximation, a function satisfying $\xi(n) \to \sqrt{2\pi n}$ as $n \to \infty$

$\mathbb{Z}$: The set of integers

$Z_{u\to v}(G)$: The Zykov symmetrization of v by u

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