Fast and backward stable transforms between spherical harmonic expansions and bivariate Fourier series

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July 3, 2017

Spherical Harmonics

Let μ be a positive Borel measure on $D \subset \mathbb{R}^n$. The inner product:

$$\langle f, g \rangle = \int_D \overline{f(x)} g(x) \, \mathrm{d}\mu(x),$$

induces the norm $\|f\|_2 = \sqrt{\langle f, f \rangle}$ and the Hilbert space $L^2(D, \mathrm{d}\mu(x))$.

Let $\mathbb{S}^2 \subset \mathbb{R}^3$ denote the unit 2-sphere and let $\mathrm{d}\Omega = \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi$.
Then any function $f \in L^2(\mathbb{S}^2, \mathrm{d}\Omega)$ may be expanded in spherical harmonics:

$$f(\theta, \varphi) \sim \sum_{\ell=0}^{+\infty} \sum_{m=-\ell}^{+\ell} f_\ell^m Y_\ell^m(\theta, \varphi) = \sum_{m=-\infty}^{+\infty} \sum_{\ell=|m|}^{+\infty} f_\ell^m Y_\ell^m(\theta, \varphi),$$

where the expansion coefficients are:

$$f_\ell^m = \int_{\mathbb{S}^2} \overline{Y_\ell^m(\theta, \varphi)} f(\theta, \varphi) \, \mathrm{d}\Omega.$$

Spherical Harmonics

Spherical harmonics are defined by:

$$Y_\ell^m(\theta,\varphi) = \frac{e^{\mathrm{i}m\varphi}}{\sqrt{2\pi}} \underbrace{\mathrm{i}^{m+|m|}\sqrt{(\ell+\tfrac{1}{2})\frac{(\ell-m)!}{(\ell+m)!}}P_\ell^m(\cos\theta)}_{\tilde{P}_\ell^m(\cos\theta)}, \quad \ell \in \mathbb{N}_0, \quad -\ell \leq m \leq \ell.$$

Associated Legendre functions are defined by ultraspherical polynomials:

$$P_\ell^m(\cos\theta) = (-2)^m(\tfrac{1}{2})_m \sin^m\theta\, C_{\ell-m}^{(m+\frac{1}{2})}(\cos\theta).$$

The notation $\tilde{P}_\ell^m$ is used to denote orthonormality, and:

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}$$

is the Pochhammer symbol for the rising factorial.

Spherical Harmonics

Consider the Laplace–Beltrami operator on $\mathbb{S}^2$:

$$\Delta_{\theta,\varphi} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}.$$

For $\ell \in \mathbb{N}_0$ and $|m| \leq \ell$, the surface spherical harmonics are the eigenfunctions of $\Delta_{\theta,\varphi}$:

$$\Delta_{\theta,\varphi} Y_\ell^m(\theta,\varphi) = -\ell(\ell+1) Y_\ell^m(\theta,\varphi).$$

Spherical harmonics diagonalize the Laplace–Beltrami operator.

Real-(World) Applications

- Global Numerical Weather Prediction (NWP). Global climate models at the *European Centre for Medium-Range Weather Forecasts* use spherical harmonics to represent the world's climate with a horizontal resolution of approximately $10\,\mathrm{km}$, corresponding to roughly 64 million degrees of freedom.
- Analysis of the Planck experiment. The *European Space Agency* sent the *PLANCK* satellite in orbit in 2013 to collect cosmic background radiation in an attempt to observe the first light of the universe. High resolution data are analyzed by spherical harmonics.
- Time-dependent Schrödinger equation in angular coordinates:

$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\Psi(t)\rangle = \left(\hat{\mathcal{T}} + \hat{\mathcal{V}}(t)\right)|\Psi(t)\rangle, \qquad |\Psi(t_0)\rangle = |\Psi_0\rangle,$$

models polarization effects.

Time Evolution

- Many real-world applications may be abstracted to semi-linear PDEs:

$$u_t = \mathcal{L}u + \mathcal{N}(u,t), \qquad u(t_0) = u_0.$$

  where $\mathcal{L}$ is a linear (differential) operator, and $\mathcal{N}$ is not.
- Usually, $\mathcal{L} = \Delta$ or something similar and $\mathcal{N}(u,t) = u^2$, for example.
- Higher-order derivatives in time are reformulated as a system.
- Many algorithms exist for semi-linear time-stepping:
  - Operator splitting preserves unitarity;
  - Exponential integrating integrates the stiff linear term exactly; and,
  - Implicit-explicit schemes are designed for versatility & efficiency.
- Problem: $\mathcal{L}$ is localized in momentum (coefficient) space and $\mathcal{N}$ is localized in physical (value) space.

Problem

Fast transforms are required to convert between representations in momentum and physical spaces:

Synthesis Convert an expansion in spherical harmonics to function values on the sphere.

Analysis Convert function values on the sphere to spherical harmonic expansion coefficients.

For a band-limit of $\ell \leq n$, the naïve cost is $\mathcal{O}(n^4)$ but it can be trivially reorganized to $\mathcal{O}(n^3)$. The goal is a run-time of $\mathcal{O}(n^2 \log^{\mathcal{O}(1)} n)$.

This has a rich history, including works by Driscoll and Healy, Mohlenhamp, Suda and Takami, Kunis and Potts, Rokhlin and Tygert and Tygert.

Questions about current approaches:

- Which grids should be chosen? If tensor-product grids, should they be Gaussian or equispaced-in-angle?
- Is the method numerically stable? Important for time evolution.

Solution: Change the Problem

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The sphere is doubly periodic and supports a bivariate Fourier series.

- The FFT, DCT, and DST solve synthesis and analysis.
- Nonuniform variants extend this to arbitrary grids.
- $Y_\ell^m(\theta,\varphi) = \dfrac{e^{\mathrm{i}m\varphi}}{\sqrt{2\pi}}\tilde{P}_\ell^m(\cos\theta) \Rightarrow$ in longitude, we are done.
- The problem is to convert $\tilde{P}_\ell^m(\cos\theta)$ to Fourier series.
- Since $\tilde{P}_\ell^m(\cos\theta) \propto \sin^m\theta\, C_{\ell-m}^{(m+\frac{1}{2})}(\cos\theta)$,
  - even-ordered $\tilde{P}_\ell^m$ are trigonometric polynomials in $\cos\theta$; and,
  - odd-ordered $\tilde{P}_\ell^m$ are trigonometric polynomials in $\sin\theta$.
- Conversions are *not* one-to-one.

The SH Connection Problem

Definition

- Let $\{\phi_n(x)\}_{n\geq 0}$ be a family of orthogonal functions with respect to $L^2(\tilde{D}, \mathrm{d}\tilde{\mu}(x))$; and,
- let $\{\psi_n(x)\}_{n\geq 0}$ be another family of orthogonal functions with respect to $L^2(D, \mathrm{d}\mu(x))$.

The connection coefficients:

$$c_{\ell,n} = \frac{\langle \psi_\ell, \phi_n \rangle_{\mathrm{d}\mu}}{\langle \psi_\ell, \psi_\ell \rangle_{\mathrm{d}\mu}},$$

allow for the expansion:

$$\phi_n(x) \sim \sum_{\ell=0}^{\infty} c_{\ell,n} \psi_\ell(x).$$

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Theorem

Let $\{\phi_n(x)\}_{n\geq 0}$ *and* $\{\psi_n(x)\}_{n\geq 0}$ *be two families of* orthonormal *functions with respect to* $L^2(D, \mathrm{d}\mu(x))$*. Then the connection coefficients satisfy:*

$$\sum_{\ell=0}^{\infty} \overline{c_{\ell,m}} c_{\ell,n} = \delta_{m,n}.$$

- Any matrix $A \in \mathbb{R}^{m\times n}$, $m \geq n$, with orthonormal columns is well-conditioned and Moore–Penrose pseudo-invertible $A^+ = A^\top$.
- For every m, the $\tilde{P}_\ell^m(x)$ are a family of orthonormal functions for the same Hilbert space $L^2([-1,1], \mathrm{d}x)$.

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Definition

Let G_n denote the Givens rotation:

$$G_n = \begin{bmatrix} 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & c_n & 0 & s_n & \cdots & 0 \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & \cdots & -s_n & 0 & c_n & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{bmatrix},$$

where the sines and the cosines are in the intersections of the n^{th} and $n+2^{\mathrm{nd}}$ rows and columns, embedded in the identity of a conformable size.

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Theorem

The connection coefficients between $\tilde{P}^{m+2}_{n+m+2}(\cos\theta)$ and $\tilde{P}^{m}_{\ell+m}(\cos\theta)$ are:

$$c^m_{\ell,n} = \begin{cases} (2\ell+2m+1)(2m+2)\sqrt{\dfrac{(\ell+2m)!}{(\ell+m+\frac{1}{2})\ell!}\dfrac{(n+m+\frac{5}{2})n!}{(n+2m+4)!}}, & \textit{for} \quad \ell \le n, \quad \ell+n \textit{ even}, \\ -\sqrt{\dfrac{(n+1)(n+2)}{(n+2m+3)(n+2m+4)}}, & \textit{for} \quad \ell = n+2, \\ 0, & \textit{otherwise}. \end{cases}$$

Furthermore, the matrix of connection coefficients $C^{(m)} = G^{(m)}_0 G^{(m)}_1 \cdots G^{(m)}_{n-2} G^{(m)}_{n-1} I_{(n+2)\times n}$, *where the sines and cosines are:*

$$s^m_n = \sqrt{\frac{(n+1)(n+2)}{(n+2m+3)(n+2m+4)}}, \quad \text{and} \quad c^m_n = \sqrt{\frac{(2m+2)(2n+2m+5)}{(n+2m+3)(n+2m+4)}}.$$

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"Proof." W.l.o.g., consider $m = 0$ and $n = 6$.

$$C^{(0)} = \begin{pmatrix} 0.91287 & 0.0 & 0.31623 & 0.0 & 0.17593 & 0.0 \\ 0.0 & 0.83666 & 0.0 & 0.39641 & 0.0 & 0.24398 \\ -0.40825 & 0.0 & 0.70711 & 0.0 & 0.3934 & 0.0 \\ 0.0 & -0.54772 & 0.0 & 0.60553 & 0.0 & 0.37268 \\ 0.0 & 0.0 & -0.63246 & 0.0 & 0.5278 & 0.0 \\ 0.0 & 0.0 & 0.0 & -0.69007 & 0.0 & 0.46718 \\ 0.0 & 0.0 & 0.0 & 0.0 & -0.73193 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -0.76376 \end{pmatrix}$$

The SH Connection Problem

"Proof." Apply the Givens rotation $G_0^{(0)\top}$:

$$G_0^{(0)\top} C^{(0)} = \begin{pmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.83666 & 0.0 & 0.39641 & 0.0 & 0.24398 \\ 0.0 & 0.0 & 0.7746 & 0.0 & 0.43095 & 0.0 \\ 0.0 & -0.54772 & 0.0 & 0.60553 & 0.0 & 0.37268 \\ 0.0 & 0.0 & -0.63246 & 0.0 & 0.5278 & 0.0 \\ 0.0 & 0.0 & 0.0 & -0.69007 & 0.0 & 0.46718 \\ 0.0 & 0.0 & 0.0 & 0.0 & -0.73193 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -0.76376 \end{pmatrix}$$

The SH Connection Problem

"Proof." And again:

$$G_1^{(0)\top} G_0^{(0)\top} C^{(0)} = \begin{pmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.7746 & 0.0 & 0.43095 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.72375 & 0.0 & 0.44544 \\ 0.0 & 0.0 & -0.63246 & 0.0 & 0.5278 & 0.0 \\ 0.0 & 0.0 & 0.0 & -0.69007 & 0.0 & 0.46718 \\ 0.0 & 0.0 & 0.0 & 0.0 & -0.73193 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -0.76376 \end{pmatrix}$$

The SH Connection Problem

"Proof." And again:

$$G_2^{(0)\top} G_1^{(0)\top} G_0^{(0)\top} C^{(0)} = \begin{pmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.72375 & 0.0 & 0.44544 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.68139 & 0.0 \\ 0.0 & 0.0 & 0.0 & -0.69007 & 0.0 & 0.46718 \\ 0.0 & 0.0 & 0.0 & 0.0 & -0.73193 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -0.76376 \end{pmatrix}$$

The SH Connection Problem

"Proof." And again:

$$G_3^{(0)\top} G_2^{(0)\top} G_1^{(0)\top} G_0^{(0)\top} C^{(0)} = \begin{pmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.68139 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.6455 \\ 0.0 & 0.0 & 0.0 & 0.0 & -0.73193 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -0.76376 \end{pmatrix}$$

The SH Connection Problem

"Proof." And again:

$$G_4^{(0)\top} G_3^{(0)\top} G_2^{(0)\top} G_1^{(0)\top} G_0^{(0)\top} C^{(0)} = \begin{pmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.6455 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -0.76376 \end{pmatrix}$$

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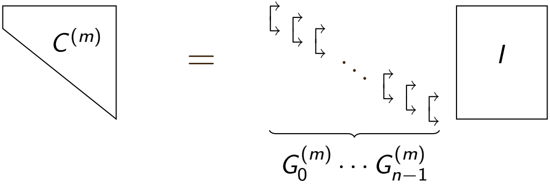
The SH Connection Problem

"Proof." And finally:

$$G_5^{(0)\top} G_4^{(0)\top} G_3^{(0)\top} G_2^{(0)\top} G_1^{(0)\top} G_0^{(0)\top} C^{(0)} = \begin{pmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \end{pmatrix}$$

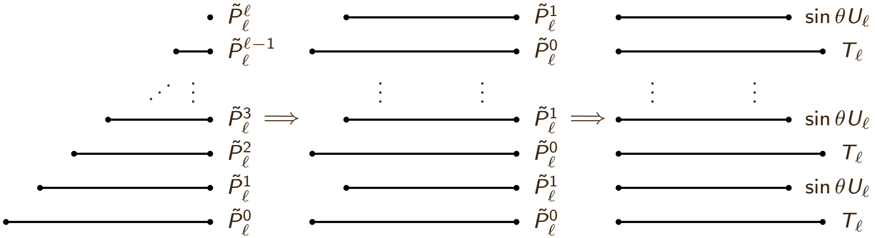
The SH Connection Problem

"Proof." Schematically:



- Conversion between neighbouring layers is $\mathcal{O}(n)$ flops and storage.
- The Givens rotations are computed to high relative accuracy due to analytical expressions of sines and cosines $\Rightarrow$ **backward stable.**

Spherical Harmonics to Fourier



1. Convert high-order layers to layers of order 0 and 1 in $\mathcal{O}(n^3)$ flops and $\mathcal{O}(n^2)$ storage; and,
2. Convert low-order layers to Fourier series in $\mathcal{O}(n^2 \log n)$ flops and $\mathcal{O}(n \log n)$ storage à la Fast Multipole Method.

To make an algorithm fast, we usually need to make approximations.

A Fast Transform

To convert high-order layers to layers of order 0 and 1 in $\mathcal{O}(n^2 \log^2 n)$ flops, we need something more.

For $m \in \mathbb{N}$, the connection coefficients between $\tilde{P}_{\ell+2m}^{2m}$ and $\tilde{P}_n^0$ are given by the inner product:

$$c_{\ell,n}^{2m} = \int_{-1}^{1} \tilde{P}_{\ell+2m}^{2m}(x)\tilde{P}_n^0(x)\,\mathrm{d}x.$$

Using the Fourier transform of $\tilde{P}_n^0$:

$$c_{\ell,n}^{2m} = \frac{(-\mathrm{i})^n\sqrt{n+\frac{1}{2}}}{\pi}\int_{\mathbb{R}} j_n(k)\,\mathrm{d}k \int_{-1}^{1} e^{\mathrm{i}kx}\tilde{P}_{\ell+2m}^{2m}(x)\,\mathrm{d}x.$$

The matrix of connection coefficients is an operator composition with the variables $(n \times k) \times (k \times x) \times (x \times \ell)$. The Fourier integral operator is special.

The Butterfly Algorithm

Purpose: Abstract the algebra of the FFT.

Technique: Divide-and-conquer $\Leftrightarrow$ merge-and-split.

Technology: The interpolative decomposition.

Proof: Fourier integral operators have *rank-proportional-to-area.*

The ranks of operator compositions are bounded by the smallest rank in the composition, extending applicability beyond Fourier integral operators.

The Interpolative Decomposition

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Lemma

Let $A \in \mathbb{R}^{m \times n}$. For any k, there exist $A_{\mathrm{CS}} \in \mathbb{R}^{m \times k}$ whose columns are a unique subset of the columns of A and $A_{\mathrm{I}} \in \mathbb{R}^{k \times n}$ such that:

1. *some subset of the columns of A_{I} makes up the $k \times k$ identity matrix;*
2. $\|\mathrm{vec}(A_{\mathrm{I}})\|_\infty \leq 1$;
3. *the spectral norm of A_{I} satisfies $\|A_{\mathrm{I}}\|_2 \leq \sqrt{k(n-k)+1}$;*
4. *the least singular value of A_{I} is at least 1;*
5. $A_{\mathrm{CS}} A_{\mathrm{I}} = A$ *whenever $k = m$ or $k = n$; and,*
6. *when $k < \min\{m, n\}$, the spectral norm of $A - A_{\mathrm{CS}} A_{\mathrm{I}}$ satisfies:*

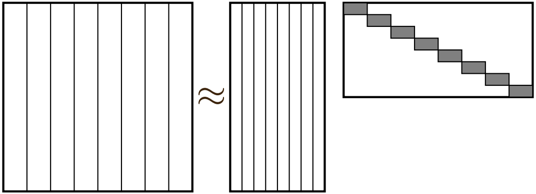
$$\|A - A_{\mathrm{CS}} A_{\mathrm{I}}\|_2 \leq \sqrt{k(n-k)+1}\,\sigma_{k+1}.$$

where σ_{k+1} is the $k+1^{\mathrm{st}}$ singular value of A.

We say that $A \approx A_{\mathrm{CS}} A_{\mathrm{I}}$ and any structure in A is also in A_{CS}.

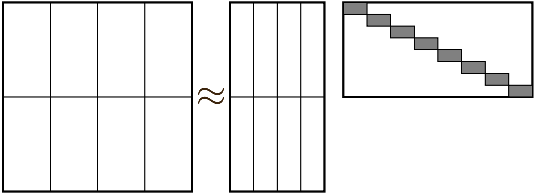
The Butterfly Algorithm

Step 1: Partition $A \in \mathbb{R}^{n \times n}$ into thin strips. Compute IDs of each subblock.



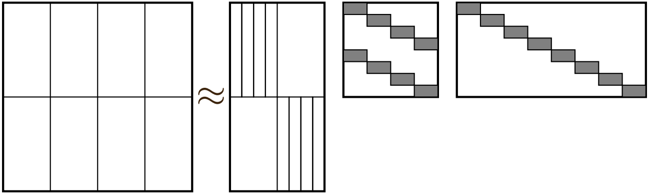
The Butterfly Algorithm

Step 2 (a): Merge the strips and split them approximately in half.



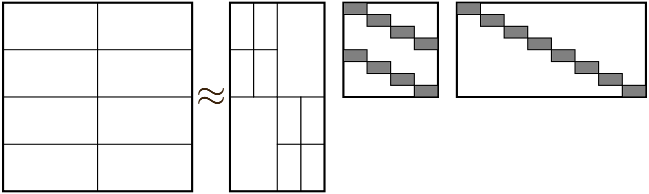
The Butterfly Algorithm

Step 2 (b): Compute IDs of each subblock.



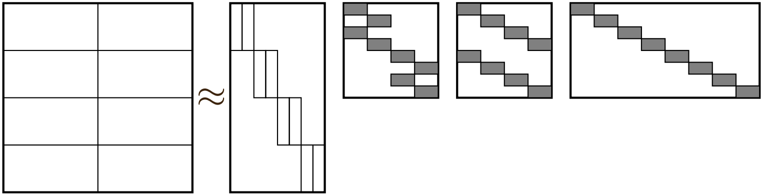
The Butterfly Algorithm

Step 3 (a): Again, merge the strips and split them approximately in half.



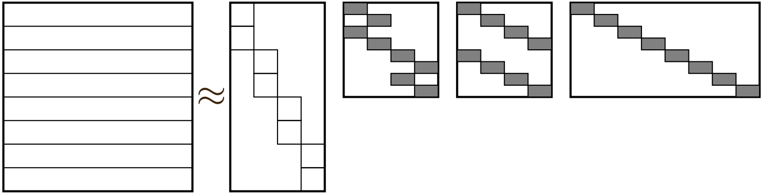
The Butterfly Algorithm

Step 3 (b): Compute IDs of each subblock.



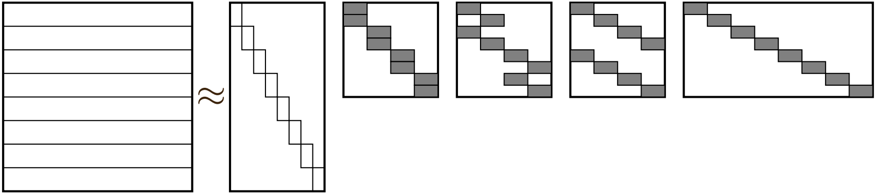
The Butterfly Algorithm

Step 4 (a): Final step, merge the strips and split them approximately in half.



The Butterfly Algorithm

Step 4 (b): Final step, compute IDs of each subblock.



The Butterfly Algorithm

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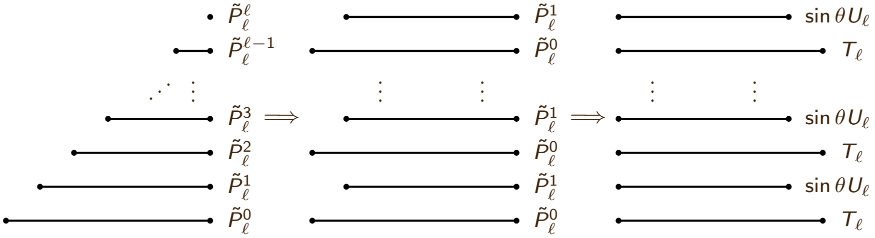
Let $A \in \mathbb{R}^{n \times n}$ have *rank-proportional-to-area*.

- Every time we merge-and-split, the complexity of a matrix-vector product is approximately halved.
- We have created a permuted and sparse block-diagonal factorization.
- Costs $\mathcal{O}(k_{\mathrm{avg}} n^2)$ to compute the factorization and $\mathcal{O}(k_{\mathrm{avg}} n \log n)$ for a matrix-vector product, where k_{avg} is the average rank of all IDs.

To convert all associated Legendre functions to orders 0 and 1, requires $\mathcal{O}(k_{\mathrm{avg}} n^3)$ flops to pre-compute and $\mathcal{O}(k_{\mathrm{avg}} n^2 \log n)$ to apply.

Spherical Harmonics to Fourier



1. Convert high-order layers to layers of order 0 and 1 in $\mathcal{O}(k_{\mathrm{avg}} n^2 \log n)$ flops and $\mathcal{O}(k_{\mathrm{avg}} n^2 \log n)$ storage; and,
2. Convert low-order layers to Fourier series in $\mathcal{O}(n^2 \log n)$ flops and $\mathcal{O}(n \log n)$ storage à la Fast Multipole Method.

Chebyshev–Legendre Transform

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Definition

The meromorphic function $\Lambda : \mathbb{C} \to \mathbb{C}$ is defined by:

$$\Lambda(z) := \frac{\Gamma(z+\frac{1}{2})}{\Gamma(z+1)}.$$

For z sufficiently large,

$$\Lambda(z) \approx \left(1 - \frac{1}{64(z+\frac{1}{4})^2} + \frac{21}{8{,}192(z+\frac{1}{4})^4} + \cdots + \right) \Big/ \sqrt{z+\tfrac{1}{4}}\,,$$

and otherwise:

$$\frac{\Lambda(z+1)}{\Lambda(z)} = \frac{z+\frac{1}{2}}{z+1}.$$

Chebyshev–Legendre Transform

Conversion from $\tilde{P}_n^0$ to cosines is given by:

$$\tilde{P}_n^0(\cos\theta) = \sqrt{n+\tfrac{1}{2}} \sum_{\ell=n,-2}^{0} \Lambda(\tfrac{n-\ell}{2})\Lambda(\tfrac{n+\ell}{2}) \frac{2-\delta_{\ell,0}}{\pi} \cos\ell\theta.$$

Inversely:

$$\cos n\theta = -n \sum_{\ell=n,-2}^{0} \frac{\Lambda(\frac{n-\ell-2}{2})\Lambda(\frac{n+\ell-1}{2})}{(n-\ell)(n+\ell+1)} \sqrt{\ell+\tfrac{1}{2}}\,\tilde{P}_\ell^0(\cos\theta).$$

Similar expressions exist for converting $\tilde{P}_n^1$ to sines.

Chebyshev–Legendre Transform

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Many methods convert between Chebyshev and Legendre expansions:

- Using asymptotics, Legendre polynomial oscillations are localized in frequency;
- Using a fast partial Cholesky decomposition, the diagonally-scaled Toeplitz-dot-Hankel structure may be exploited; or,
- Using an adaptation of the **Fast Multipole Method**, subblocks of connection coefficients well-separated from the main diagonal are well-approximated by low-rank matrices.

Fast Multipole Method

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The method, due to Greengard and Rokhlin, originates from the *multipole expansion* of the Coulombic potential:

$$\frac{1}{|\mathbf{r}-\mathbf{r}_0|} = \frac{1}{\sqrt{r^2 - 2rr_0\cos\theta + r_0^2}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r_0}{r}\right)^n P_n(\cos\theta).$$

Expansion for sufficiently small $r_0/r \ll 1$,

$\Leftrightarrow$ $\mathcal{O}(\log(\varepsilon^{-1}))$ terms in the multipole expansion for approximation to precision ε,

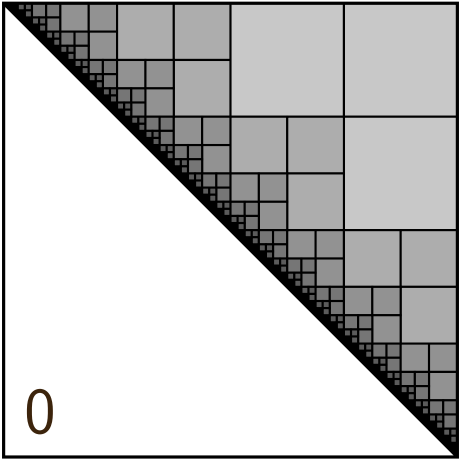
$\Leftrightarrow$ Subblocks well-separated from the main diagonal.

Alpert and Rokhlin use FMM to accelerate the Chebyshev–Legendre transform.

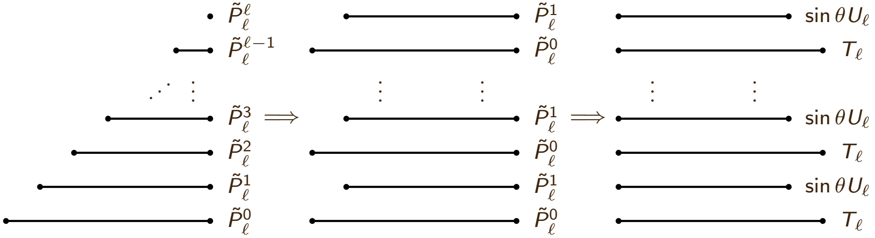
What does well-separation resemble?

Fast Multipole Method

An upper-triangular matrix with subblocks well-separated from the main diagonal:



Spherical Harmonics to Fourier

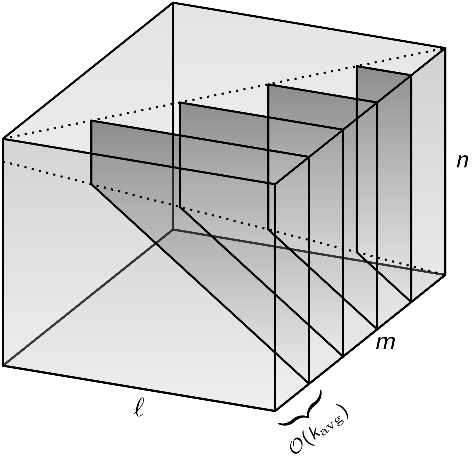


1. Convert high-order layers to layers of order 0 and 1 in $\mathcal{O}(k_{\text{avg}} n^2 \log n)$ flops and $\mathcal{O}(k_{\text{avg}} n^2 \log n)$ storage; and,
2. Convert low-order layers to Fourier series in $\mathcal{O}(n^2 \log n)$ flops and $\mathcal{O}(n \log n)$ storage à la Fast Multipole Method.

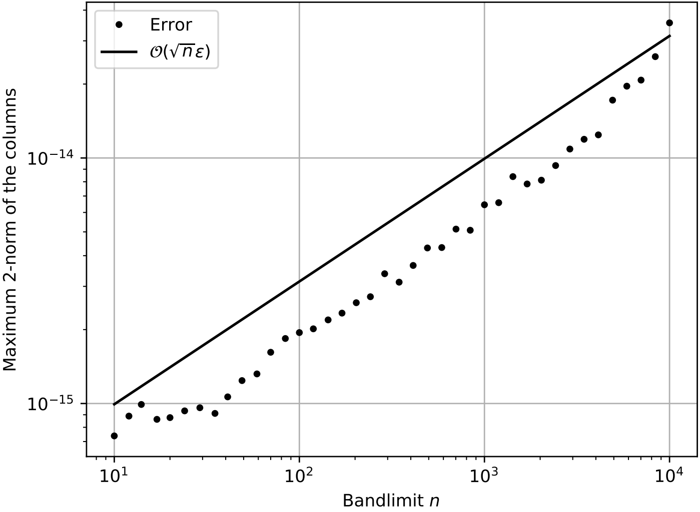
Can we beat $\mathcal{O}(k_{\text{avg}} n^3)$ pre-computation?

Skeletonizing the Pre-Computation

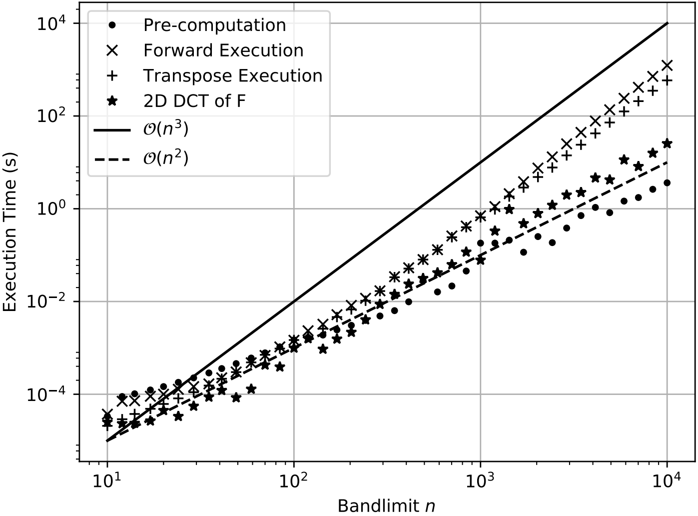
No, but skeletonizing the pre-computation makes it practical for a laptop. Neighbouring layers are converted via Given rotations.



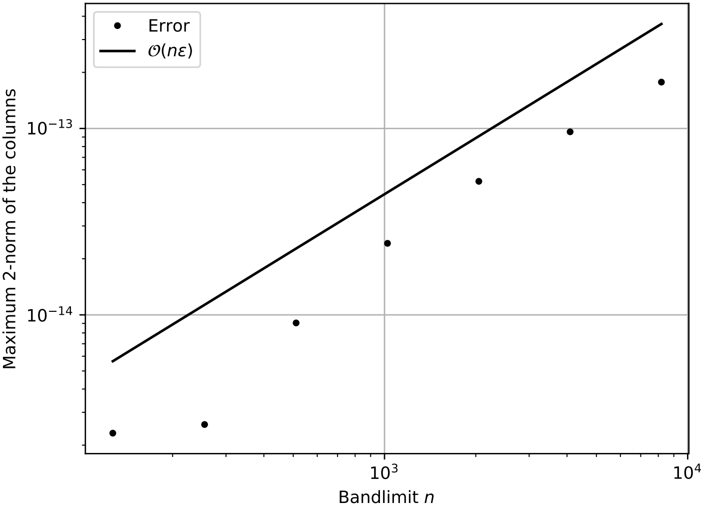
Numerical Results: Slow



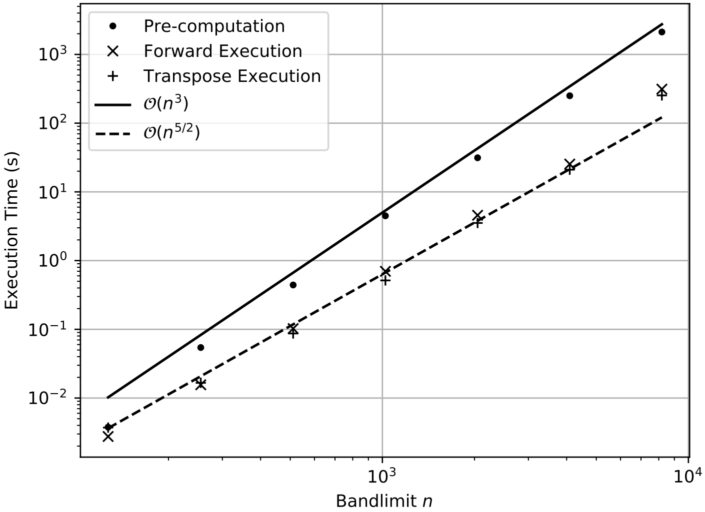
Numerical Results: Slow



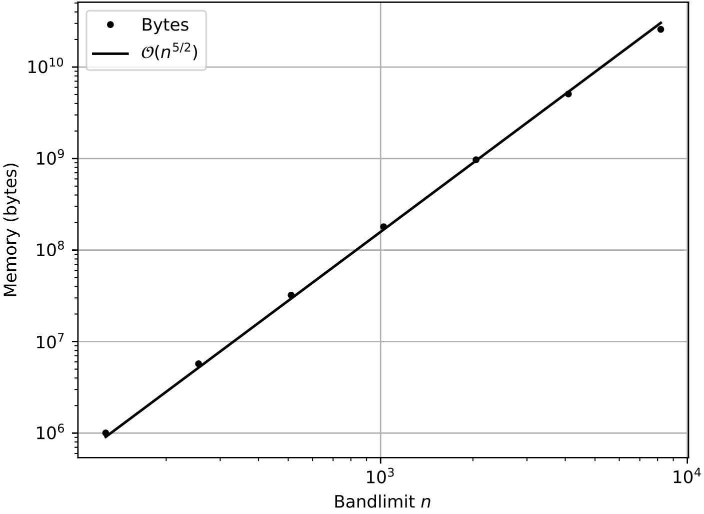
Numerical Results: Fast



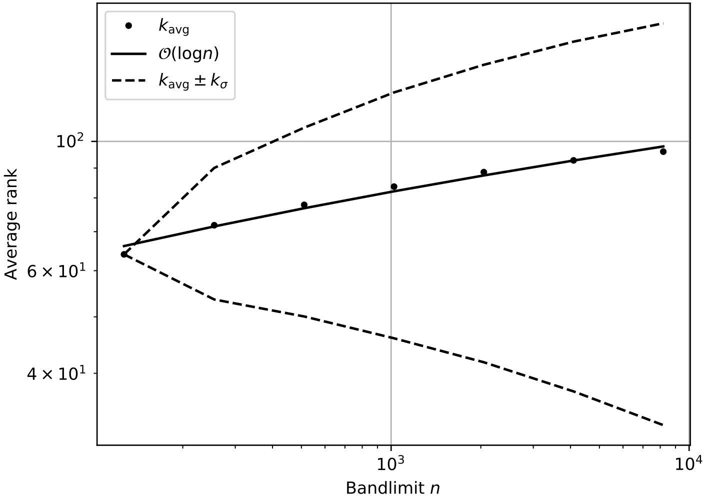
Numerical Results: Fast



Numerical Results: Fast



Numerical Results: Fast



Conclusion & Outlook

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- New transforms are created to convert spherical harmonic expansions into bivariate Fourier series.
- They are asymptotically fast, $\mathcal{O}(n^2 \log^2 n)$, and backward stable by construction. For practical band-limits of $n \leq \mathcal{O}(10,000)$, the asymptotically optimal complexity does not yet appear.
- They are freely available in JULIA in `FastTransforms.jl`.
- A straightforward extension to conversion of Zernike polynomials to Fourier–Chebyshev series on the unit disk.
- A similar approach might extend to more exotic bivariate orthogonal polynomials on triangles.
- Is there a pre-computation-free method?

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