Chapter 5

Poisson Processes

5.1 Exponential Distribution

The gamma function is defined by

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\, dt, \alpha > 0.$$

Theorem 5.1. *The gamma function satisfies the following properties:*
(a) For each $\alpha > 1$, $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$.
(b) For each integer $n \geq 1$, $\Gamma(n) = (n-1)!$.
(c) $\Gamma(1/2) = \sqrt{\pi}$.

Proof. For each $\alpha > 1$, by an integration by parts

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\, dt = \int_0^\infty t^{\alpha-1} d(-e^{-t})\, dt = t^{\alpha-1}(-e^{-t}) \Big|_0^\infty - \int_0^\infty (-e^{-t}) d(t^{\alpha-1})$$
$$= t^{\alpha-1}(-e^{-t}) \Big|_0^\infty + \int_0^\infty (\alpha-1) t^{\alpha-2} e^{-t}\, dt = (\alpha-1)\Gamma(\alpha-1).$$

Next, we prove by induction that for each integer $n \geq 1$, $\Gamma(n) = (n-1)$. The case $n = 1$ holds because

$$\Gamma(1) = \int_0^\infty e^{-t}\, dt = 1.$$

The case n implies the case $n+1$, because if $\Gamma(n) = (n-1)!$, then $\Gamma(n+1) = n\Gamma(n) = n(n-1)! = n!$.

Finally, by the change of variables $t = x^2/2$, $(dt = x\, dx)$,

$$\Gamma(1/2) = \int_0^\infty t^{-1/2} e^{-t}\, dt = \int_0^\infty \sqrt{\frac{2}{x^2}} e^{-\frac{x^2}{2}} x\, dx = \sqrt{2} \int_0^\infty e^{-\frac{x^2}{2}}\, dx$$
$$= \frac{\sqrt{2}}{2} \int_{-\infty}^\infty e^{-\frac{x^2}{2}}\, dx = \frac{\sqrt{2}}{2}\sqrt{2\pi} = \sqrt{\pi}$$

Q.E.D.

In particular, we have that

$$\int_0^\infty x^n e^{-x}\, dx = \Gamma(n+1) = n! \tag{5.1}$$

By the change of variables $y = \frac{x}{\theta}$,

$$\int_0^\infty x^{\alpha-1}e^{-x/\theta}\,dx = \int_0^\infty x^{\alpha-1}\theta^\alpha e^{-y}\,dy = \theta^\alpha\Gamma(\alpha) \tag{5.2}$$

Exercise 5.1. *Find:*
(i) $\int_0^\infty x^3e^{-x}\,dx$.
(ii) $\int_0^\infty x^{12}e^{-x}\,dx$.
(iii) $\int_0^\infty x^{23}e^{-2x}\,dx$.
(iv) $\int_0^\infty x^{24}e^{-x/3}\,dx$.

Exercise 5.2. *Use integration by parts to show that*

$$\int xe^{-x}\,dx = -e^{-x}(1+x)+c.$$

Exercise 5.3. *Use integration by parts to show that*

$$\int \frac{x^2}{2}e^{-x}\,dx = -e^{-x}(1+x+\frac{x^2}{2})+c.$$

Exercise 5.4. *Prove that for each integer $n \geq 1$,*

$$\int \frac{x^n}{n!}e^{-x}\,dx = -\sum_{j=0}^{n} e^{-x}\frac{x^j}{j!}+c.$$

Hint: use integration by parts and induction.

Definition 5.1.1. *A r.v. X is said to have an exponential distribution with parameter $\lambda > 0$, if the density of X is given by*

$$f(x) = \begin{cases} \frac{e^{-\frac{x}{\lambda}}}{\lambda} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

We denote this by $X \sim \text{Exponential}(\lambda)$.

The above function f defines a bona fide density because it is nonnegative and

$$\int_{-\infty}^\infty f(t)\,dt = \int_0^\infty \frac{e^{-\frac{t}{\lambda}}}{\lambda}\,dt = -e^{-\frac{t}{\lambda}}\Big|_0^\infty = 1.$$

Theorem 5.2. *Let X be a r.v. with an exponential distribution and parameter $\lambda > 0$, then*

$$E[X] = \lambda, \text{Var}(X) = \lambda^2, E[X^k] = \lambda^k k!, M(t) = \frac{1}{1-\lambda t}, \text{ if } t < \lambda^{-1}.$$

Proof. Using (5.2),

$$E[X^k] = \int_0^\infty x^k \frac{e^{-\frac{x}{\lambda}}}{\lambda}\,dx = \frac{1}{\lambda}\Gamma(k+1)\lambda^{k+1} = k!\lambda^k$$

In particular,

$$E[X] = \lambda, E[X^2] = 2\lambda^2$$
$$\mathrm{Var}(X) = E[X^2] - (E[X])^2 = \lambda^2.$$

We have that for $t < \lambda^{-1}$,

$$M(t) = E[e^{tX}] = \int_0^\infty e^{tx} \frac{e^{-\frac{x}{\lambda}}}{\lambda}\, dx = \frac{1}{\lambda} \int_0^\infty e^{-x(\frac{1-\lambda t}{\lambda})}\, dx = \frac{1}{\lambda} \frac{\lambda}{1-\lambda t} = \frac{1}{1-\lambda t}.$$

Q.E.D.

The cumulative distribution function of an exponential distribution with mean $\lambda > 0$ is

$$F(x) = P(X \le x) = \int_{-\infty}^x f(t)\, dt = 1 - e^{-\frac{x}{\lambda}},\ x \ge 0.$$

The exponential distribution satisfies that for each $s, t \ge 0$,

$$\mathrm{P}(X > s + t | X > t) = P(X > s).$$

This is property is called the memoryless property of the exponential distribution.

Given a r.v. X, the cumulative distribution function of X is $F(x) = P(X \le x)$. The survival function of X is $S(x) = P(X > x) = 1 - F(x)$. If X has a continuous distribution with density X, the failure (or hazard) rate function is defined as

$$r(t) = \frac{f(t)}{S(t)} = -\frac{d}{dt} \ln(S(t)).$$

We have that

$$\mathrm{P}(X \in (t, t+dt) | X > t) = \frac{P \in (t, t+dt)}{P(X > t)} \simeq \frac{f(t)dt}{S(t)} = r(t)dt.$$

If X designs a lifetime, the failure rate function is the rate of death for the survivor at time t. Since $S(t)$ is a nonincreasing function, $r(t) \ge 0$.

Assuming that X is a nonnegative r.v.

$$1 - F(t) = e^{-\int_0^t r(s)\, ds}.$$

If X has an exponential distribution with mean λ, then the failure rate function is $r(t) = \frac{1}{\lambda}$ for each $t \ge 0$. Reciprocally, if the failure rate function is constant, the r.v. has an exponential distribution.

Definition 5.1.2. *X has a gamma distribution with parameters $\alpha > 0$ and $\theta > 0$, if the density of X is*

$$f(x) = \begin{cases} \frac{x^{\alpha-1} e^{-\frac{x}{\theta}}}{\theta^\alpha \Gamma(\alpha)} & \text{if } x \ge 0 \\ 0 & \text{if } x < 0 \end{cases}$$

We denote this by $X \sim \mathrm{Gamma}(\alpha, \theta)$.

The above function f defines a bona fide density because, by (5.2),

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^{\infty} \frac{x^{\alpha-1}e^{-\frac{x}{\theta}}}{\Gamma(\alpha)\theta^\alpha}\,dx = 1.$$

A gamma distribution with parameter $\alpha = 1$ is an exponential distribution.

Theorem 5.3. *If X has a gamma distribution with parameters α and θ, then*

$$E[X] = \alpha\theta, \mathrm{Var}(X) = \alpha\theta^2, E[X^k] = \frac{\Gamma(\alpha+k)\theta^k}{\Gamma(\alpha)}, \mathrm{Var}(X) = \alpha\theta^2, M(t) = \frac{1}{(1-\theta t)^\alpha}, \text{ if } t < \frac{1}{\theta}.$$

Proof. Using (5.2),

$$E[X^k] = \int_0^\infty x^k \tfrac{x^{\alpha-1}e^{-\frac{x}{\theta}}}{\Gamma(\alpha)\theta^\alpha}\,dx = \tfrac{1}{\Gamma(\alpha)\theta^\alpha}\int_0^\infty x^{k+\alpha-1}e^{-\frac{x}{\theta}}\,dx = \tfrac{1}{\Gamma(\alpha)\theta^\alpha}\Gamma(k+\alpha)\theta^{k+\alpha} = \tfrac{\Gamma(\alpha+k)\theta^\alpha}{\Gamma(\alpha)}.$$

In particular,

$$\begin{aligned} &E[X] = \tfrac{\Gamma(\alpha+1)\theta^1}{\Gamma(\alpha)} = \alpha\theta, E[X^2] = \tfrac{\Gamma(\alpha+2)\theta^2}{\Gamma(\alpha)} = (\alpha+1)\alpha\theta^2, \\ &\mathrm{Var}(X) = E[X^2] - (E[X])^2 = \alpha\theta^2. \end{aligned}$$

We have that for $t < \frac{1}{\theta}$,

$$\begin{aligned} &M(t) = E[e^{tX}] = \int_0^\infty e^{tx}\tfrac{x^{\alpha-1}e^{-\frac{x}{\theta}}}{\Gamma(\alpha)\theta^\alpha}\,dx = \tfrac{1}{\Gamma(\alpha)\theta^\alpha}\int_0^\infty x^{\alpha-1}e^{-x\left(\frac{1-\theta t}{\theta}\right)}\,dx \\ &= \tfrac{1}{\Gamma(\alpha)\theta^\alpha}\left(\tfrac{\theta}{1-\theta t}\right)^\alpha \Gamma(\alpha) = \tfrac{1}{(1-\theta t)^\alpha}. \end{aligned}$$

Q.E.D.

Example 5.1. *The lifetime in hours of an electronic part is a random variable having a probability density function given by*

$$f(x) = cx^9e^{-x}, \quad x > 0.$$

Compute c and the expected lifetime of such an electronic part.

Solution: First, we find c,

$$1 = \int_0^\infty cx^9e^{-x}\,dx = c\Gamma(10) = c\cdot 9!.$$

So, $c = \frac{1}{9!}$. The expected value of X is

$$E[X] = \int_0^\infty x\frac{1}{9!}x^9e^{-x}\,dx = \int_0^\infty \frac{1}{9!}x^{10}e^{-x}\,dx = \frac{\Gamma(11)}{9!} = \frac{10!}{9!} = 10.$$

Example 5.2. *If X is a random variable with density function*

$$f(x) = \begin{cases} 1.4e^{-2x} + .9e^{-3x} & \text{for } x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Find $E[X]$.

Solution: Using (5.2),

$$E[X] = \int_0^\infty x\,(1.4e^{-2x} + .9e^{-3x})\,dx = (1.4)\int_0^\infty xe^{-2x}\,dx + (0.9)\int_0^\infty xe^{-3x}\,dx$$
$$= (1.4)\Gamma(2)(1/2)^2 + (0.9)\Gamma(2)(1/3)^2 = (0.45).$$

Exercise 5.5. *The number of days that elapse between the beginning of a calendar year and the moment a high-risk driver is involved in an accident is exponentially distributed. An insurance company expects that 30% of high-risk drivers will be involved in an accident during the first 50 days of a calendar year. What portion of high-risk drivers are expected to be involved in an accident during the first 80 days of a calendar year?*

Exercise 5.6. *The time that it takes for a computer system to fail is exponential with mean 17000 hours. If a lab has 20 such computers systems. What is the probability that a least two fail before 17000 hours of use?*

Exercise 5.7. *The service times of customers coming through a checkout counter in retail store are independent random variables with an exponential distribution with mean* 1.5 *minutes. Approximate the probability that 100 customers can be services in less than* 3 *hours of total service time.*

Solution: *Let* $X_1, \dots, X_{100}$ *be the services times of the 100 customers. For each* $1 \le i \le 100$*, we have that* $E[X_i] = 1.5$ *and* $\mathrm{Var}(X_i) = 1.5^2$*. So,* $E[\sum_{j=1}^n X_j] = 100(1.5) = 150$ *and* $\mathrm{Var}(\sum_{j=1}^n X_j) = 100(1.5)^2 = 225$*. Using the central limit theorem,*

$$P\left(\sum_{j=1}^n X_j \le (3)(60)\right) \simeq P\left(N(0,1) \le \frac{180-150}{\sqrt{225}}\right) = P(N(0,1) \le 2) = 0.9772.$$

Exercise 5.8. *One has 100 bulbs whose light times are independent exponentials with mean 5 hours. If the bulbs are used one at time, with a failed bulb being immediately replaced by a new one, what is the probability that there is still a working bulb after 525 hours.*

Exercise 5.9. *A critical submarine component has a lifetime which is exponential distributed with mean 10 days. Upon failure, replacement with anew component with identically characteristic occurs What is the smallest number of spare components that the submarine should stock if it is leaving for one year tour and wishes the probability of an inoperative caused by failure exceeding the spare inventory to be less than 0.02?*

Solution: *Let* $X_1, \dots,$ *be the lifetime of the components. We have to find* n *so that* $P(\sum_{i=1}^n X_i < 365) < 0.02$*. For each* $1 \le i \le n$*, we have that* $E[X_i] = 10$ *and* $\mathrm{Var}(X_i) = 10^2 = 100$*. So,* $E[\sum_{i=1}^n X_i] = 10n$ *and* $\mathrm{Var}(\sum_{i=1}^n X_i) = 100n$*. By the central limit theorem*

$$0.02 > P(\sum_{i=1}^n X_i < 365) \simeq \mathrm{P}(N(0,1) \le \frac{365-10n}{10\sqrt{n}})$$

Since $0.02 = \mathrm{P}(N(0,1) \le 2.05)$*, we have that* $\frac{365-10n}{10\sqrt{n}} < 2.05$*,* $10n - 20.55\sqrt{n} - 365 < 0$*. We get* $n = 52$*.*

Exercise 5.10. *The time T required to repair a machine is an exponential distributed random variable with mean $\frac{1}{2}$.*
(a) What is the probability that a repair time exceeds $\frac{1}{2}$ hours.
(b) What is the probability that a repair time takes at least $12\frac{1}{2}$ hours given that is duration exceeds 12 hours.

Solution: *T has density $f(t) = 2e^{-2t}$, $t \geq 0$. So,*

$$P(T \geq \tfrac{1}{2}) = \int_{\frac{1}{2}}^{\infty} 2e^{-2t}\, dt = e^{-1}$$
$$P(T \geq 12\tfrac{1}{2} | T \geq 12) = \frac{\int_{\frac{25}{2}}^{\infty} 2e^{-2t}\, dt}{\int_{12}^{\infty} 2e^{-2t}\, dt} = \frac{e^{-25}}{e^{-24}} = e^{-1}$$

Exercise 5.11. *Find the density of a gamma random variable with mean 8 and variance* 16.

Theorem 5.4. *Let $X \sim \text{Gamma}(\alpha, \theta)$ and let $Y \sim \text{Gamma}(\beta, \theta)$. Suppose that X and Y are independent. Then, $X + Y \sim \text{Gamma}(\alpha + \beta, \theta)$*

Proof. The moment generating function of X and Y are $M_X(t) = \frac{1}{(1-\theta t)^{\alpha}}$ and $M_Y(t) = \frac{1}{(1-\theta t)^{\beta}}$, respectively. Since X and Y are independent r.v.'s, the moment generating function of $X+Y$ is

$$M_{X+Y}(t) = M_X(t) M_Y(t) = \frac{1}{(1-\theta t)^{\alpha}} \frac{1}{(1-\theta t)^{\beta}} = \frac{1}{(1-\theta t)^{\alpha+\beta}},$$

which is the moment generating function of a gamma distribution with parameters $\alpha + \beta$ and θ. So, $X + Y$ has a gamma distribution with parameters $\alpha + \beta$ and θ. Q.E.D.

Theorem 5.5. *Let $X_1, \ldots, X_n$ be independent identically distributed r.v.'s with exponential distribution with parameter λ. Then, $\sum_{i=1}^{n} X_i$ has gamma distribution with parameters n and λ.*

Proof. Every X_i has $\text{Gamma}(1, \lambda)$ distribution. By the previous theorem, $\sum_{i=1}^{n} X_i$ has a $\text{Gamma}(n, \lambda)$ distribution. Q.E.D.

Exercise 5.12. *Suppose that you arrive at a single-teller bank to find 5 other customers in the bank, one being served and the other waiting in line. You join the end of the line. If the services times are all exponential with rate 2 minutes, what are the expected value and the variance of the amount of time you will spend in the bank?*

Suppose that a system has n parts. The system functions works only if all n parts work. Let X_i is the lifetime of the i–th part of the system. Suppose that $X_1, \ldots, X_n$ be independent r.v.'s and that X_i has an exponential distribution with mean θ_i. Let Y be the lifetime of the system. Then, $Y = \min(X_1, \ldots, X_n)$. Then,

$$P(Y > t) = P(\min(X_1, \ldots, X_n) > t) = P(\cap_{i=1}^{n} \{X_i > t\}) = \prod_{i=1}^{n} P(X_i > t)$$
$$= \prod_{i=1}^{n} e^{-\frac{t}{\theta_i}} = e^{-t \sum_{i=1}^{n} \frac{1}{\theta_i}}.$$

So, Y has an exponential distribution with mean $\frac{1}{\sum_{i=1}^{n} \frac{1}{\theta_i}}$. Let $Z = \max(X_1, \ldots, X_n)$. Then,

$$P(Z \leq t) = P(\max(X_1, \ldots, X_n) \leq t) = P(\cap_{i=1}^{n} \{X_i \leq t\}) = \prod_{i=1}^{n} P(X_i \leq t)$$
$$= \prod_{i=1}^{n} \left(1 - e^{-\frac{t}{\theta_i}}\right).$$

Since $\max(x, y) + \min(x, y) = x + y$,

$$E[\max(X_1, X_2)] = \theta_1 + \theta_2 - \frac{1}{\frac{1}{\theta_1} + \frac{1}{\theta_2}}.$$

Given $x_1, \dots, x_n$, we have that

$$\max(x_1, \dots, x_n)$$
$$= \sum_{i=1}^n x_i - \sum_{i_1<i_2} \min(x_{i_1}, x_{i_2}) + \sum_{i_1<i_2<i_3} \min(x_{i_1}, x_{i_2}, x_{i_3}) - \cdots + (-1)^{n+1} \min(x_1, \dots, x_n).$$

Theorem 5.6. *(i) Let X and let Y be two independent r.v. such that X has an exponential distribution with mean θ_1 and Y has an exponential distribution with mean θ_2. Then,*

$$P(X < Y) = \frac{\frac{1}{\theta_1}}{\frac{1}{\theta_1} + \frac{1}{\theta_2}}.$$

(ii) Let $X_1, \dots, X_n$ be independent r.v.'s such that X_i has an exponential distribution with mean θ_i. Then,

$$P(X_i = \min_{1\le j\le n} X_j) = \frac{\frac{1}{\theta_i}}{\sum_{j=1}^n \frac{1}{\theta_j}}.$$

Proof. The joint density of x and Y is

$$f_{X,Y}(x, y) = \frac{e^{-\frac{x}{\theta_1} - \frac{y}{\theta_2}}}{\theta_1\theta_2}, x, y > 0.$$

So,

$$P(X < Y) = \int_0^\infty \int_x^\infty \frac{e^{-\frac{x}{\theta_1} - \frac{y}{\theta_2}}}{\theta_1\theta_2}\, dy\, dx = \int_0^\infty \frac{e^{-\frac{x}{\theta_1} - \frac{x}{\theta_2}}}{\theta_1}\, dy\, dx = \frac{\frac{1}{\theta_1}}{\frac{1}{\theta_1} + \frac{1}{\theta_2}}$$

As to the second part,

$$P(X_i = \min_{1\le j\le n} X_j) = P(X_i < \min_{1\le j\le n, j\ne i} X_j) = \frac{\frac{1}{\theta_i}}{\sum_{j=1}^n \frac{1}{\theta_j}},$$

because $\min_{1\le j\le n, j\ne i} X_j$ has an exponential distribution with mean $\left(\sum_{1\le j\le n, j\ne i}^n \frac{1}{\theta_j}\right)^{-1}$ and X_i and $\min_{1\le j\le n, j\ne i} X_j$ are independent. Q.E.D.

Exercise 5.13. *John and Peter enter a barbershop simultaneously. John is going to get a haircur and Peter is going to get a shave. It the amount of time it takes to receive a haircut (shave) is exponentially distributed with mean 20 (15) minutes, and if John and Peter are immediately server.*
(a) What is the average time for he first of the two to be finished?
(b) What is the average time for both of them to be done?
(c) What is the probability that John finishes before Peter?

Exercise 5.14. *Let X_1 and let X_2 be independent r.v.'s with exponential distribution and mean $\lambda > 0$. Let $X_{(1)} = \min(X_1, X_2)$ and let $X_{(2)} = \max(X_1, X_2)$. Find the density function, mean and variance of $X_{(1)}$ and $X_{(2)}$.*

Solution: *$X_{(1)}$ has an exponential distribution with mean $\frac{1}{\frac{1}{\lambda}+\frac{1}{\lambda}} = \frac{\lambda}{2}$. So, the mean and the variance of $X_{(1)}$ are*

$$E[X_{(1)}] = \frac{\lambda}{2} \text{ and } \mathrm{Var}(X_{(1)}) = \frac{\lambda^2}{4}.$$

The cdf of $X_{(2)}$ is

$$P(X_{(2)} \le x) = P(X_1 \le x, X_2 \le x) = (1 - e^{-\frac{x}{\lambda}})(1 - e^{-\frac{x}{\lambda}}) = 1 - 2e^{-\frac{x}{\lambda}} + e^{-\frac{2x}{\lambda}}.$$

So, the density function of $X_{(2)}$ is

$$f_{X_{(2)}}(x) = \frac{2e^{-\frac{x}{\lambda}}}{\lambda} - \frac{2e^{-\frac{2x}{\lambda}}}{\lambda}.$$

The mean and the variance of $X_{(2)}$ are

$$\begin{aligned} & E[X_{(2)}] = \int_0^\infty x \left(\frac{2e^{-\frac{x}{\lambda}}}{\lambda} - \frac{2e^{-\frac{2x}{\lambda}}}{\lambda} \right) dx = \frac{2}{\lambda} \int_0^\infty x e^{-\frac{x}{\lambda}}\, dx - \frac{2}{\lambda} \int_0^\infty x e^{-\frac{2x}{\lambda}}\, dx \\ = \; & \frac{2}{\lambda}\Gamma(2)\lambda^2 - \frac{2}{\lambda}\Gamma(2)(\lambda/2)^2 = \frac{3\lambda}{2}, \\ & E[X_{(2)}^2] = \int_0^\infty x^2 \left(\frac{2e^{-\frac{x}{\lambda}}}{\lambda} - \frac{2e^{-\frac{2x}{\lambda}}}{\lambda} \right) dx = \frac{2}{\lambda} \int_0^\infty x^2 e^{-\frac{x}{\lambda}}\, dx - \frac{2}{\lambda} \int_0^\infty x^2 e^{-\frac{2x}{\lambda}}\, dx \\ = \; & \frac{2}{\lambda}\Gamma(3)\lambda^3 - \frac{2}{\lambda}\Gamma(3)(\lambda/2)^3 = \frac{7\lambda^2}{2}, \\ & \mathrm{Var}(X_{(2)}) = \frac{7\lambda^2}{2} - \left(\frac{3\lambda}{2}\right)^2 = \frac{5\lambda^2}{4}. \end{aligned}$$

Note that $E[X_1 + X_2] = E[X_{(1)} + X_{(2)}]$.

Exercise 5.15. *A machine consists of 3 components. The lifetime of these components constitute independent r.v.'s with an exponential distributions and parameters 2, 3 and 6. The machine will work only if only its parts work. Find the expected lifetime of the machine.*

Exercise 5.16. *An insurance company sells two types of auto insurance policies: Basic and Deluxe. The time until the next Basic Policy claim is an exponential random variable with mean two days. The time until the next Deluxe Policy claim is an independent exponential random variable with mean three days. What is the probability that the next claim will be a Deluxe Policy claim?*

Exercise 5.17. *Two painters are hired to a paint a house. The first painter working alone will paint the house in λ_1 days. The second painter working alone will paint the house in λ_1 days. How long it will take both painters working together to paint the house.*

Solution: *In one day, the first painter does $\frac{1}{\lambda_1}$ fraction of the house. In one day, the second painter does $\frac{1}{\lambda_2}$ fraction of the house. So, in one day both painters working together do $\frac{1}{\lambda_1} + \frac{1}{\lambda_2}$ fraction of the house. It will take them $\frac{1}{\frac{1}{\lambda_1}+\frac{1}{\lambda_2}}$ days to finish the house.*

Exercise 5.18. *The lifetimes of Bill's dog and cat are independent exponential random variables with respective means λ_d and λ_c. One of them has just died. Find the expected additional lifetime of the other pet.*

5.2 Poisson process

Definition 5.2.1. *A stochastic process $\{N(t) : t \geq 0\}$ is said to be a counting process if $N(t)$ represents the total number of "events" that have occurred up to time t.*

A counting processes $N(t)$ must satisfy:
(i) $N(t) \geq 0$.
(ii) $N(t)$ is integer valued.
(iii) If $s < t$, then $N(s) \leq N(t)$.
(iv) For $s < t$, $N(t)-N(s)$ equal the number of events that have occurred in the interval $(s, t]$.

A counting process if said to possess independent increments if for each $0 \leq t_1 < t_2 < \cdots < t_m$, $N(t_1), N(t_2) - N(t_1), N(t_3) - N(t_2), \ldots, N(t_m) - N(t_{m-1})$ are independent r.v.'s.

A counting process is said to have stationy increments if for each $0 \leq t_1 \leq t_2$, $N(t_2)-N(t_1)$ and $N(t_2 - t_1)$ have the same distribution.

Definition 5.2.2. *A r.v. X has a Poisson distribution with parameter $\lambda > 0$, if for each integer $k \geq 0$,*

$$P[X = k] = e^{-\lambda}\frac{\lambda^k}{k!}.$$

In particular,

$$P[X = 0] = e^{-\lambda},\ \mathrm{P}[X = 1] = e^{-\lambda}\lambda,\ P[X = 2] = e^{-\lambda}\frac{\lambda^2}{2}.$$

Theorem 5.7. *Let X be a r.v. with a Poisson distribution with parameter $\lambda > 0$, then $E[X] = \lambda$, $\mathrm{Var}(X) = \lambda$ and $M(t) = e^{\lambda(e^t-1)}$.*

Proof. Using that $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$,

$$M(t) = E[e^{tX}] = \sum_{k=0}^{\infty} e^{tk}\frac{e^{-\lambda}\lambda^k}{k!} = e^{-\lambda}\sum_{k=0}^{\infty}\frac{(e^t\lambda)^k}{k!} = e^{-\lambda} + e^{e^t\lambda} = e^{\lambda(e^t-1)}.$$

Hence,

$$\begin{aligned}
&M(t) = e^{-\lambda}e^{\lambda e^t}, \\
&M'(t) = e^{-\lambda}e^{\lambda e^t}\lambda e^t,\ E[X] = M'(0) = \lambda \\
&M''(t) = e^{-\lambda}e^{\lambda e^t}(\lambda e^t)^2 + e^{-\lambda}e^{\lambda e^t}\lambda e^t,\ E[X^2] = M''(0) = \lambda^2 + \lambda \\
&\mathrm{Var}(X) = \lambda
\end{aligned}$$

Q.E.D.

Theorem 5.8. *Let $X \sim \mathrm{Poisson}(\lambda_1)$ and let $Y \sim \mathrm{Poisson}(\lambda_2)$. Suppose that X and Y are independent. Then, $X + Y \sim \mathrm{Poisson}(\lambda_1 + \lambda_2)$.*

Proof. The moment generating function of X is $M_X(t) = e^{\lambda_1(e^t-1)}$. The moment generating function of X is $M_X(t) = e^{\lambda_2(e^t-1)}$. Since X and Y are independent r.v.'s, the moment generating function of $X + Y$ is

$$M_{X+Y}(t) = M_X(t)M_Y(t) = e^{\lambda_1(e^t-1)}e^{\lambda_2(e^t-1)} = e^{(\lambda_1+\lambda_2)(e^t-1)},$$

which is the moment generating function of a Poisson($\lambda_1+\lambda_2$). So, $X+Y$ has a Poisson($\lambda_1+\lambda_2$) distribution. Q.E.D.

Problem 5.1. *(25, May 2001) For a discrete probability distribution, you are given the recursion relation*

$$p(k) = \frac{2}{k} p(k-1), k = 1, 2, \dots$$

Determine $p(4)$. *(A) 0.07 (B) 0.08 (C) 0.09 (D) 0.10 (E) 0.11*

Solution: 0.09.

Exercise 5.19. *An insurance company issues 1250 vision care insurance policies. The number of claims filed by a policyholder under a vision care insurance policy during one year is a Poisson random variable with mean 2. Assume the numbers of claims filed by distinct policyholders are independent of one another. What is the approximate probability that there is a total of between 2450 and 2600 claims during a one–year period?*

Exercise 5.20. *The number of claims received each day by a claims center has a Poisson distribution. On Mondays, the center expects to receive 2 claims but on other days of the week, the claims center expects to receive 1 claim per day. The numbers of claims received on separate days are mutually independent of one another. Find the probability that the claim center receives at least 2 claims in a day week (Monday to Friday).*

A function f is said to be $o(h)$ if $\lim_{h\to 0} \frac{f(h)}{h} = 0$.

Definition 5.2.3. *(Definition 1) The stochastic process* $\{N(t) : t \geq 0\}$ *is said to be a Poisson process with rate function* $\lambda > 0$, *if:*
(i) $N(0) = 0$.
(ii) The process has independent increments.
(iii) $P(N(h) \geq 2) = o(h)$.
(iv) $P(N(h) = 1) = \lambda h + o(h)$.

Definition 5.2.4. *(Definition 2) The stochastic process* $\{N(t) : t \geq 0\}$ *is said to be a Poisson process with rate function* $\lambda > 0$, *if:*
(i) $N(0) = 0$.
(ii) The process has independent increments.
(iii) For each $0 \leq s, t$, $N(s+t) - N(s)$ *has a Poisson distribution with mean* λt.

The two definitions of Poisson processes are equivalent.

In the previous definition, $N(t)$ is the number of occurrences until time. The rate of occurrences per unit of time is a constant. The average number of occurrences in the time interval $(s, s+t]$ is λt.

It follows that for each $0 \leq t_1 < t_2 < \cdots < t_m$, and each $0 \leq k_1 \leq k_2 \leq \cdots \leq k_m$,

$$\begin{aligned}
&P(N(t_1) = k_1, N(t_2) = k_2, \dots, N(t_m) = k_m) \\
&= P(N(t_1) = k_1, N(t_2) - N(t_1) = k_2 - k_1, \dots, N(t_m) - N(t_{m-1}) = k_m - k_{m-1}) \\
&= P(N(t_1) = k_1)P(N(t_2) - N(t_1) = k_2 - k_1) \cdots P(N(t_m) - N(t_{m-1}) = k_m - k_{m-1}) \\
&= e^{-\lambda t_1} \frac{(\lambda t_1)^{k_1}}{k_1!} e^{-\lambda(t_2-t_1)} \frac{(\lambda(t_2-t_1))^{k_2-k_1}}{(k_2-k_1)!} \cdots e^{-\lambda(t_m-t_{m-1})} \frac{(\lambda(t_m-t_{m-1}))^{k_m}}{k_m!}
\end{aligned}$$

It is easy to see that for each $0 \leq s \leq t$,

$$E[N(s)] = \lambda s, \mathrm{Var}(N(s)) = \lambda s,\ \mathrm{Cov}(N(s), N(t)) = \lambda s.$$

Exercise 5.21. *Let $\{N(t) : t \geq 0\}$ be a Poisson process with rate $\lambda = 2$. Compute:*
(i) $P(N(5) = 4)$.
(ii) $P(N(5) = 4, N(6) = 9)$.
(iii) $E[2N(3) - 4N(5)]$.
(iv) $\mathrm{Var}(2N(3) - 4N(5))$.

Solution:

$$P(N(5) = 4) = P(\mathrm{Poiss}(10) = 4) = \frac{e^{-10}(10)^4}{4!},$$

$$\begin{aligned} & P(N(5) = 4, N(6) = 9) = P(N(5) = 4, N(6) - N(4) = 5) \\ = \ & P(N(5) = 4)P(N(6) - N(4) = 5) = \tfrac{e^{-10}(10)^4}{4!}\tfrac{e^{-8}(8)^5}{5!}, \end{aligned}$$

$$E[2N(3) - 4N(5)] = 2E[N(3)] - 4E[N(5)] = (2)(3)(2) - (4)(2)(5) = -28$$

$$\begin{aligned} & \mathrm{Var}(2N(3) - 4N(5)) = \mathrm{Var}(-2N(3) - 4(N(5) - N(3))) \\ = \ & (-2)^2\mathrm{Var}(N(3)) + (-4)^2\mathrm{Var}(N(5) - N(3)) = (-2)^2(2)(3) + (-4)^2(2)(5-3) = 88 \end{aligned}$$

Exercise 5.22. *Let $\{N(t) : t \geq 0\}$ be a Poisson process with rate $\lambda = 2$. Compute:*
(i) $P(N(5) = 4, N(6) = 9, N(10) = 15)$.
(ii) $P(N(5) - N(2) = 3)$.
(iii) $P(N(5) - N(2) = 3, N(7) - N(6) = 4)$.
(iv) $P(N(2) + N(5) = 4)$.
(v) $E[N(5) - 2N(6) + 3N(10)]$.
(vi) $\mathrm{Var}(N(5) - 2N(6) + 3N(10))$.
(vi) $\mathrm{Cov}(N(5) - 2N(6), 3N(10))$.

Theorem 5.9. *Let $\{N(t) : t \geq 0\}$ be a Poisson process with rate λ. Let $0 < s, t$ and let $k \geq j \geq 0$. Then,*

$$P(N(s + t) = k | N(s) = j) = \mathrm{P}(N(t) = k - j),$$

i.e. the distribution of $N(s + t)$ given $N(s) = j$ is that $j + \mathrm{Poisson}(\lambda t)$. So,

$$E[N(s + t)|N(s) = j] = j + \lambda t \text{ and } \mathrm{Var}(N(s + t)|N(s) = j) = \lambda t.$$

Proof. Since $N(s)$ and $N(s + t) - N(s)$ are independent,

$$P(N(s+t) = k|N(s) = j) = P(N(s+t) - N(t) = k - j|N(s) = j) = P(N(s+t) - N(t) = k - j)$$

Q.E.D.

Exercise 5.23. *Let $\{N(t) : t \geq 0\}$ be a Poisson process with rate $\lambda = 3$. Compute:*
(i) $P(N(5) = 7|N(3) = 2)$.
(ii) $E[2N(5) - 3N(7)|N(3) = 2)$.
(iii) $\mathrm{Var}(N(5)|N(2) = 3)$.
(iv) $\mathrm{Var}(N(5) - N(2)|N(2) = 3)$.
(v) $\mathrm{Var}(2N(5) - 3N(7)|N(3) = 2)$.

Theorem 5.10. *(Markov property of the Poisson process) Let $\{N(t) : t \geq 0\}$ be a Poisson process with rate λ. Let $0 \leq t_1 < t_2 < \cdots < t_m < s$ and let $k_1 \leq k_2 \leq \cdots \leq k_m \leq j$. Then,*

$$P(N(s) = j|N(t_1) = k_1, \ldots, N(t_m) = k_m) = P(N(s) = j|N(t_m) = k_m)$$

Proof. Since $N(t_1), N(t_2) - N(t_1), \cdots, N(t_m) - N(t_{m-1}), N(s) - N(t_m)$,

$$P(N(s) = j|N(t_1) = k_1, \ldots, N(t_m) = k_m) = \frac{P(N(t_1)=k_1,\ldots,N(t_m)=k_m,N(s)=j)}{P(N(t_1)=k_1,\ldots,N(t_m)=k_m)}$$
$$= \frac{P(N(t_1)=k_1,N(t_2)-N(t_1)=k_2-k_1,\ldots,N(t_m)-N(t_{m-1})=k_m-k_{m-1},N(s)-N(t_m)=j-k_m)}{P(N(t_1)=k_1,N(t_2)-N(t_1)=k_2-k_1,\ldots,N(t_m)-N(t_{m-1})=k_m-k_{m-1})}$$
$$= \frac{P(N(t_1)=k_1)P(N(t_2)-N(t_1)=k_2-k_1)\cdots \mathrm{P}(N(t_m)-N(t_{m-1})=k_m-k_{m-1})\mathrm{P}(N(s)-N(t_m)=j-k_m)}{P(N(t_1)=k_1)P(N(t_2)-N(t_1)=k_2-k_1)\cdots \mathrm{P}(N(t_m)-N(t_{m-1})=k_m-k_{m-1})}$$
$$= \mathrm{P}(N(s) - N(t_m) = j - k_m) = P(N(s) = j|N(t_m) = k_m).$$

Q.E.D.

Theorem 5.11. *Let $\{N(t) : t \geq 0\}$ be a Poisson process with rate λ. Let $s, t \geq 0$. Then,*

$$P(N(t) = k|N(s + t) = n) = \binom{n}{k}\left(\frac{t}{t + s}\right)^k\left(\frac{s}{t + s}\right)^{n-k},$$

i.e. the distribution of $N(t)$ given $N(s + t) = n$ is binomial with parameters n and $p = \frac{t}{t+s}$. So,

$$E[N(t)|N(s + t) = n] = n\frac{t}{s + t} \text{ and } \mathrm{Var}(N(t)|N(s + t) = n) = n\frac{t}{s + t}\frac{s}{s + t}.$$

The previous theorem can be extended as follows, given $0 \leq t_1 < t_2 < \cdots < t_m$, the conditional distribution of $(N(t_1), N(t_2) - N(t_1), \ldots, N(t_m) - N(t_{m-1}))$ given $N(t_m) = n$ is multinomial distribution with parameter $(\frac{t_1}{t_m}, \frac{t_2-t_1}{t_m}, \ldots, \frac{t_m-t_{m-1}}{t_m})$. Given $N(t_m) = n$, we know that events happens in the interval $[0, t_m]$, each of these events happens independently and the probability that one of these events happens in particular interval is the fraction of the total length of this interval.

Exercise 5.24. *Customers arrive at a bank according with a Poisson process with a rate* 20 *customers per minute. Suppose that two customer arrived during the first hour. What is the probability that at least one arrived during the first 20 minutes?*

Exercise 5.25. *Cars cross a certain point in a highway in accordance with a Poisson process with rate $\lambda = 3$ cars per minute. If Deb blindly runs across the highway, then what is the probability that she sill be injured if the amount of time it takes her to cross the road is s seconds? Do it for $s = 2, 5, 10, 20$.*

Exercise 5.26. *Let $\{N(t) : t \geq 0\}$ be a Poisson process with rate $\lambda = 3$. Compute:*
(i) $E[N(1)|N(3) = 2]$.
(ii) $E[2N(1) - 3N(7)|N(3) = 2]$.
(iii) $\text{Var}(N(1)|N(3) = 2)$.

Exercise 5.27. *Two individuals, A and B, both require kidney transplants. If A does not receive a new kidney, then A will die after an exponential time with mean λ_a and B will die after an exponential time with mean λ_b. New kidneys arrive in accordance with a Poisson process having rate λ. It has been decided that the first kidney will to A or to B if B is alive and A is not at the time and the next one to B (if is still living).*
(a) What is the probability that A obtains a new kidney?
((b) What is the probability that B obtains a new kidney?

5.3 Interarrival times

For $n \geq 1$, let S_n be the arrival time of the n–th event, i.e.

$$S_n = \inf\{t \geq 0 : N(t) = n\}.$$

Let $T_n = S_n - S_{n-1}$ be the elapsed time between the $(n-1)$–th and the n–th event.

Theorem 5.12. *T_n, $n = 1, 2, \ldots$ are independent identically distributed exponential random variables having mean $\frac{1}{\lambda}$.*

Proof. We have that

$$P(T_1 > t) = P(N(t) = 0) = e^{-\lambda t}.$$

So, T_1 has an exponential distribution with mean $\frac{1}{\lambda}$. By the Markov property,

$$\begin{aligned}&P(T_2 > t|T_1 = s) = P(\text{no occurrences in } (s, s+t]|T_1 = s)\\&= P(\text{no occurrences in } (s, s+t]) = e^{-\lambda t}.\end{aligned}$$

So, T_1 and T_2 are independent, and T_2 have an exponential distribution with mean $\frac{1}{\lambda}$. By induction, the claim follows. Q.E.D.

The previous theorem says that if the rate of events is λ events per unit of time, then the expected waiting time between events is $\frac{1}{\lambda}$. A useful relation between $N(t)$ and S_n is

$$\begin{aligned}&\{S_n \leq t\} = \{\text{the } n\text{−th occurrence happens before time } t)\\&= \{\text{there are } n \text{ or more occurrences inthe interval } [0, t]\} = \{N(t) \geq n\}.\end{aligned}$$

Theorem 5.13. *S_n has a gamma distribution with parameters $\alpha = n$ and $\theta = \frac{1}{\lambda}$.*

Proof. $T_1, \ldots, T_n$ are i.i.d.r.v.'s with an exponential distribution with mean $\frac{1}{\lambda}$. So, S_n has a gamma distribution with parameters $\alpha = n$ and $\theta = \frac{1}{\lambda}$. Q.E.D.

By the previous theorem, $E[S_n] = \frac{n}{\lambda}$ and $\text{Var}(S_n) = \frac{n}{\lambda}$ and S_n has density function

$$f_{S_n}(t) = \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!}, t \geq 0.$$

Exercise 5.28. *Prove that for each integer $n \geq 1$,*

$$\int \frac{x^n}{n!} e^{-x}\, dx = (-e^{-x}) \sum_{j=0}^{n} \frac{x^j}{j!} + c.$$

The c.d.f. of S_n is

$$F_{S_n}(t) = \mathrm{P}(S_n \leq t) = \int_0^t \frac{\lambda^n s^{n-1} e^{-\lambda s}}{(n-1)!}\, ds = \int_0^{\lambda t} \frac{s^{n-1} e^{-s}}{(n-1)!}\, ds$$
$$= (-e^{-s}) \sum_{j=0}^{n-1} \frac{s^j}{j!} \Big|_0^{\lambda t} = 1 - e^{-\lambda t} \sum_{j=0}^{n-1} \frac{(\lambda t)^j}{j!} = e^{-\lambda t} \sum_{j=n}^{\infty} \frac{(\lambda t)^j}{j!}.$$

We also have that

$$P(N(t) \geq n) = \sum_{j=n}^{\infty} e^{-\lambda t} \frac{(\lambda t)^j}{j!}.$$

Theorem 5.14. *Given that $N(t) = n$, the n arrival times $S_1, \ldots, S_n$ have the same distribution as the order statistics corresponding to n independent random variables uniformly distributed on the interval $(0, t)$*

Exercise 5.29. *Let $N(t)$ be a Poisson process with rate $\lambda = 3$.*
(i) What is the probability of 4 occurrences of the random event in the interval $(2.5, 4]$?
(ii) What is the probability that $T_2 > 5$ given that $N(4) = 1$?
(iii) What is the distribution of T_n?
(iv) What is the expected time of the occurrence of the 5-th event?

Answer: *(i)* $P(N(4) - N(2.5) = 4) = \frac{e^{-4.5}(4.5)^4}{4!}$.
(ii) $P(T_2 > 5 | N(4) = 1) = P(T_2 > 5 | T_1 \leq 4) = P(T_2 > 5) = e^{-15}$.
(iii) T_n *has an exponential distribution with mean* $\frac{1}{3}$.
(iv) $E[S_5] = 5 \cdot \frac{1}{3}$.

Exercise 5.30. *Let $N(t)$ be a Poisson process with rate λ. let S_n denote the time of the n–th event. Find:*
(i) $E[S_4]$*?*
(ii) $E[S_4 | N(1) = 2]$.
(iii) $E[N(4) - N(2) | N(1) = 3]$.

Answer: *(i)* $E[S_4] = \frac{4}{\lambda}$.
(ii) $E[S_4 | N(1) = 2] = 1 + E[S_2] = 1 + \frac{2}{\lambda}$.
(iii) $E[N(4) - N(2) | N(1) = 3] = E[N(4) - N(2)] = 2\lambda$.

Exercise 5.31. *For a Poisson process the expected waiting time between events is 0.10 years.*
(i) What is the probability that 10 or fewer events occur during a 2-year time span?
(ii) What is the probability that the waiting time between 2 consecutive events is at least 0.2 years?
(iii) If $N(2) = 20$, what is the probability that exactly 10 events occur during $(0, 1]$?.

Answer: *We have that* $\frac{1}{\lambda} = 0.1$. *So,* $\lambda = 10$.
(i) $P(N(2) \leq 10) = \sum_{j=0}^{10} \frac{e^{-20}(20^j}{j!}$.

(ii) $P(T_1 \geq 0.2) = \int_{0.2}^{\infty} \frac{e^{-\frac{x}{0.1}}}{0.10}\,dx = e^{-2}$.
(iii) $P(N(1) = 10|N(2) = 20) = \binom{20}{10}\left(\frac{1}{2}\right)^{10}\left(\frac{1}{2}\right)^{10}$.

Exercise 5.32. *A discount store promises to give a small gift to the million-th customer to arrive after opening. The arrivals of customers form a Poisson process with rate $\lambda = 10$ customers/minute. (a) Find the probability density function of the arrival time T of the lucky customer. (b) Find the mean and the variance of T.*

Exercise 5.33. *A certain theory supposes that mistakes in cell division occur according to a Poisson process with rate 2.5 per year., and that an individual dies when 196 such mistakes have occurred. Assuming this theory find*
(a) the mean lifetime of an individual,
(b) the variance of the lifetime of an individuals.
Also approximate
(c) the probability that an individual dies before age 67.2
(d) the probability that an individual reaches age 90.
(e) the probability that an individual reaches age 100.

Exercise 5.34. *Customers arrive at a bank at a Poisson rate of λ. Suppose two customers arrived during the first hour. What is the probability that*
(a) both arrived during the first 20 minutes?
(b) at least one arrived during the first 20 minutes.?

Problem 5.2. *(# 10, November 2001). For a tyrannosaur with 10,000 calories stored:*
(i) The tyrannosaur uses calories uniformly at a rate of 10,000 per day. If his stored calories reach 0, he dies.
(ii) The tyrannosaur eats scientists (10,000 calories each) at a Poisson rate of 1 per day.
(iii) The tyrannosaur eats only scientists.
(iv) The tyrannosaur can store calories without limit until needed.
Calculate the probability that the tyrannosaur dies within the next 2.5 days.
(A) 0.30 (B) 0.40 (C) 0.50 (D) 0.60 (E) 0.70

Solution: *Let N_i be the number of scientist eating by the tyrannosaur in day i. The probability that the tyrannosaur dies within the next 2.5 days is*

$$P(N_1 = 0) + \mathrm{P}(N_1 = 1, N_2 = 0) = e^{-1} + e^{-1}e^{-1} = 0.503.$$

Problem 5.3. *(# 11, November 2001). For a tyrannosaur with 10,000 calories stored:*
(i) The tyrannosaur uses calories uniformly at a rate of 10,000 per day. If his stored calories reach 0, he dies.
(ii) The tyrannosaur eats scientists (10,000 calories each) at a Poisson rate of 1 per day.
(iii) The tyrannosaur eats only scientists.
(iv) The tyrannosaur can store calories without limit until needed.
Calculate the expected calories eaten in the next 2.5 days.
(A) 17,800 (B) 18,800 (C) 19,800 (D) 20,800 (E) 21,800

Solution: Let X_1 be the number of scientists eating in the first day. Let X_2 be the number of scientists eating in the second day. Let $X_{2.5}$ be the number of scientists eating in the first half of the third day. Let A_1 be the event that the tyrannosaur survives the first day. Let A_2 be the event that the tyrannosaur survives the second day. first day. We have that

$$E[X_1] = 1,$$
$$E[X_2] = P(A_1)E[X_2|A_1] = P(A_1) = P(N(1) \geq 1)1 - e^{-1},$$
$$E[X_{2.5}] = P(A_2)E[X_{2.5}|A_2] = P(A_2)(0.5)$$
$$= (1 - P(N(1) = 0) - P(N(1) = 1, P(N(2) - N(1) = 0))(0.5) = (1 - e^{-1} - e^{-2})(0.5)$$

So, the expected calories eaten in the next 2.5 days is

$$10000(1 + 1 - e^{-1} + (1 - e^{-1} - e^{-2})(0.5)) = 10000(2.5 - (1.5)e^{-1} - (0.5)e^{-2}) = 18805.$$

Exercise 5.35. *Men and women enter a supermarket according to independent Poisson processes having respective rates two and four per minute. Starting at an arbitrary time, compute the probability that at least two men arrive before three women arrive.*

Exercise 5.36. *Two independent Poisson processes generate events at respective rates $\lambda_1 = 4$ and $\lambda_2 = 5$. Calculate the probability that 3 events from process two occur before 2 events from process one.*

Exercise 5.37. *Events occur according to a Poisson process with 2 expected occurrences per hour. Calculate the probability that the time of the occurrence of the second event is after one hour.*

5.4 Superposition and decomposition of a Poisson process

Recall that:

Theorem 5.15. *Let $X \sim \text{Poisson}(\lambda_1)$ and let $Y \sim \text{Poisson}(\lambda_2)$. Suppose that X and Y are independent. Then, $X + Y \sim \text{Poisson}(\lambda_1 + \lambda_2)$.*

Theorem 5.16. *If $\{N_1(t) : t \geq 0\}$ and $\{N_2(t) : t \geq 0\}$ are two independent Poisson processes with respective rates λ_1 and λ_2. Then, $\{N_1(t) + N_2(t) : t \geq 0\}$ is a Poisson process with rate $\lambda_1 + \lambda_2$.*

Proof. Let $N(t) = N_1(t) + N_2(t)$. Given $0 \leq t_1 < t_2 < \cdots < t_m$,

$$N_1(t_1), N_1(t_2) - N_1(t_1), \dots, N_1(t_m) - N_1(t_{m-1}), N_2(t_1), N_2(t_2) - N_1(t_1), \dots, N_2(t_m) - N_1(t_{m-1})$$

are independent r.v.'s. So,

$$N(t_1), N(t_2) - N(t_1), \dots, N(t_m) - N(t_{m-1})$$

are independent r.v.'s. Besides, $N_1(t_j) - N_1(t_{j-1}) \sim \text{Poisson}(\lambda_1(t_j - t_{j-1}))$ and $N_2(t_j) - N_2(t_{j-1}) \sim \text{Poisson}(\lambda_2(t_j - t_{j-1}))$. So, $N(t_j) - N(t_{j-1}) \sim \text{Poisson}((\lambda_1 + \lambda_2)(t_j - t_{j-1}))$. Hence, $\{N(t) : t \geq 0\}$ is a Poisson processes with rate $\lambda_1 + \lambda_2$. Q.E.D.

Theorem 5.17. *Let $\{N_1(t) : t \geq 0\}$ and let $\{N_2(t) : t \geq 0\}$ be two independent Poisson processes with respective rates λ_1 and λ_2. Let $\lambda = \lambda_1 + \lambda_2$. Then, the conditional distribution of $N_1(t)$ given $N(t) = n$ is binomial with parameters n and $p = \frac{\lambda_1}{\lambda}$.*

Proof. We have that

$$P(N_1(t) = k|N(t) = n) = \frac{P(N_1(t)=k,\ N(t)=n)}{P(N(t)=n)} = \frac{P(N_1(t)=k,\ N_2(t)=n-k)}{P(N(t)=n)}$$
$$= \frac{\frac{e^{-\lambda_1}\lambda_1^k}{k!}\cdot\frac{e^{-\lambda_2}\lambda_2^{n-k}}{(n-k)!}}{\frac{e^{-\lambda}\lambda^n}{n!}} = \binom{n}{k}p^k(1-p)^k.$$

Q.E.D.

Decomposition of a Poisson process Consider a Poisson process $\{N(t) : t \geq 0\}$ with rate λ. Suppose that each time an event occurs it is classified as either a type I or a type II event. Suppose further that each event is classified as type I event with probability p or type II event with probability $1 - p$. Let $N_1(t)$ denote respectively the number of type I events occurring in $[0, t]$. Let $N_2(t)$ denote respectively the number of type II events occurring in $[0, t]$. Note that $N(t) = N_1(t) + N_2(t)$.

Theorem 5.18. *$\{N_1(t) : t \geq 0\}$ and $\{N_2(t) : t \geq 0\}$ are independent Poisson processes with respective rates $\lambda_1 = \lambda p$ and $\lambda_2 = \lambda(1 - p)$.*

Proof. Given $0 \leq t_1 < t_2 < \cdots < t_m$,

$$(N_1(t_1), N_2(t_1)), (N_1(t_2)-N_1(t_1), N_2(t_2)-N_2(t_1)), \ldots, (N_1(t_m)-N_1(t_{m-1}), N_1(t_m)-N_1(t_{m-1})),$$

are independent r.v.'s. So, we need to prove that for each $1 \leq j \leq m$, $N_1(t_j)-N_1(t_{j-1})$, $N_2(t_j)-N_2(t_{j-1})$ are independent and $N_1(t_j) - N_1(t_{j-1}) \sim \text{Poisson}(\lambda_1(t_j - t_{j-1}))$, $N_2(t_j) - N_1(t_{j-1}) \sim \text{Poisson}(\lambda_1(t_j - t_{j-1}))$. We have that

$$P(N_1(t_j) - N_1(t_{j-1}) = n, N_2(t_j) - N_2(t_{j-1}) = m)$$
$$= P(N(t_j) - N(t_{j-1}) = n + m,\ \text{there are } n \text{ successes in } n + m \text{ trials})$$
$$= \frac{e^{-\lambda t}(\lambda t)^n}{n!} \cdot \binom{n+m}{n} p^n q^m = \frac{e^{-\lambda pt}(\lambda pt)^n}{n!}\frac{e^{-\lambda qt}(\lambda qt)^m}{m!}.$$

Hence,

$$P(N_1(t_j) - N_1(t_{j-1}) = n) = \sum_{m=0}^{\infty} P(N_1(t_j) - N_1(t_{j-1}) = n, N_2(t_j) - N_2(t_{j-1}) = m)$$
$$= \sum_{m=0}^{\infty} \frac{e^{-\lambda pt}(\lambda pt)^n}{n!}\frac{e^{-\lambda qt}(\lambda qt)^m}{m!} = \frac{e^{-\lambda pt}(\lambda pt)^n}{n!}.$$

So, $N_1(t_j) - N_1(t_{j-1}) \sim \text{Poisson}(\lambda_1(t_j - t_{j-1}))$. Similarly, $N_2(t_j) - N_2(t_{j-1}) \sim \text{Poisson}(\lambda_2(t_j - t_{j-1}))$. Since

$$P(N_1(t_j) - N_1(t_{j-1}) = n, N_2(t_j) - N_2(t_{j-1}) = m)$$
$$= P(N_1(t_j) - N_1(t_{j-1}) = n)P(N_2(t_j) - N_2(t_{j-1}) = m)$$

$N_1(t_j) - N_1(t_{j-1})$ and $N_2(t_j) - N_2(t_{j-1})$ are independent. Q.E.D.

Exercise 5.38. *Arrivals of customers into a store follow a Poisson process with rate $\lambda = 20$ arrivals per hour. Suppose that the probability that a customer buys something is $p = 0.30$.*

(a) Find the expected number of sales made during an eight-hour business day.

(b) Find the probability that 10 or more sales are made in one hour.

(c) Find the expected time of the first sale of the day. If the store opens at 8 a.m.

Answer: *Let $N_1(t)$ be the number of arrivals who buy something. Let $N_2(t)$ be the number of arrivals who do not buy something. N_1 and N_2 are two independent Poisson processes. The rate for N_1 is $\lambda_1 = \lambda p = (20)(0.3) = 6$. The rate for N_2 is $\lambda_1 = \lambda(1-p) = (20)(0.7) = 14$.*

(a) $E[X_1](80) = (8)(6) = 48$.

(b) $P(N_1 \geq 10) = 1 - \sum_{j=0}^{9} P(N_1 = j) = 1 - \sum_{j=0}^{9} \frac{e^{-6}6^j}{j!}$.

(c) $E[T_{1,1}] = \frac{1}{\lambda_1} = \frac{1}{6}$ hours or 10 minutes. The expected time of the first sale is $8:10$ a.m.

Exercise 5.39. *Cars pass a point on the highway at a Poisson rate of one car per minute. If 5 % of the cars on the road are vans, then*

(a) What is the probability that at least one van passes by during an hour?

(b) Given that 10 vans have passed by in an hour, what is the expected number of cars to have passed by in that time.

(c) If 50 cars have passed by in an hour, what is the probability that five of them were vans.

Exercise 5.40. *Motor vehicles arrive at a bridge toll gate according to a Poisson process with rate $\lambda = 2$ vehicles per minute. The drivers pay tolls of $1, $2 or $5 depending on which of the three weight classes their vehicles belong. Assuming that the vehicles arriving at the gate belong to classes 1, 2, and 3 with probabilities $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{6}$.*

(a) Find the mean and the variance of the amount in dollars collected in any given hour.

(b) Estimate the probability that $200 or more is collected in one hour.

Exercise 5.41. *The number of vehicles passing in front of a restaurant follows the Poisson distribution with mean 60 vehicles/hour. In general 10% of all vehicles stop at the restaurant. The number of passengers in a car is 1, 2, 3, 4, and 5 with respective probabilities: 0.30, 0.30, 0.20, 0.10, and 0.10. (a) Find the mean and the variance of the number of passengers entering the car in a period of 8 hours. (b) Find the probability that the number of people entering the restaurant in a period of 8 hours is more than 100 people*

Exercise 5.42. *Motor vehicles arrive at a bridge toll gate according to a Poisson process with rate $\lambda = 2$ vehicles per minute. The drivers pay tolls of $1, $2 or $5 depending on which of three weight classes their vehicles belong. Assuming that the vehicles arriving at the gate belong to classes 1, 2, and 3 with probabilities $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{6}$, respectively, find:*

(a) The mean and the variance of the amount in dollars collected in any given hour.

(b) The probability that exactly $5 is collected in a period of 2 minutes.

(c) The probability that the waiting time until the second vehicle paying $5 vehicles is more than 10 minutes.

Exercise 5.43. *The arrival of customer at an ice cream parlor is a Poisson process with a rate $\lambda = 20$ customers per hour. Every customer orders one and only one ice cream. There are 3 sizes for the ice cream: small, medium and large. The probability that a customer wants a small, medium and large ice cream are $\frac{1}{6}$, $\frac{1}{3}$ and $\frac{1}{2}$, respectively. The prices of the ice creams*

are: \$1 for the small ice cream, \$2 for the medium ice cream, \$3 for the large ice cream.
(i) Find the probability that the first ice cream is sold before 20 minutes.
(ii) Find the mean and the variance of the total amount of money collected in a job turn of 8 hours.

Exercise 5.44. *The arrival of customer at an ice cream parlor is a Poisson process with a rate 20 customer per hour. Every customer orders one and only one ice cream. There are only two choices for the flavor: chocolate and vanilla. Forty % of the customers order vanilla and 60 % of the customers order chocolate.*
(i) Find the probability that the first ice cream sold is a chocolate ice cream.

(ii) Find the probability that two or more chocolate ice creams are sold before the first vanilla ice cream is sold.

Problem 5.4. *(# 2, May, 2000). Lucky Tom finds coins on his way to work at a Poisson rate of 0.5 coins/minute. The denominations are randomly distributed:*
(i) 60% of the coins are worth 1;
(ii) 20% of the coins are worth 5; and
(iii) 20% of the coins are worth 10.
Calculate the conditional expected value of the coins Tom found during his one-hour walk today, given that among the coins he found exactly ten were worth 5 each.
(A) 108 (B) 115 (C) 128 (D) 165 (E) 180

Solution: *Let $N_1(t)$ be the number of coins of value 1 found until time t. Let $N_2(t)$ be the number of coins of value 5 found until time t. Let $N_3(t)$ be the number of coins of value 10 found until time t. Then, N_1, $N_2$and N_3 are independent Poisson processes. Let $\lambda_1, \lambda_2, \lambda_3$ be their respective rates. We have that*

$$\lambda_1 = 0.5(0.60) = 0.3,\ \lambda_2 = 0.5(0.20) = 0.1,\ \lambda_3 = 0.5(0.20) = 0.1.$$

We need to find

$$E[N_1(60) + 5N_2(60) + 10N_3(60)|N_2(60) = 10] = (60)(0.30) + (5)(10) + (10)(60)(0.1) = 128$$

Problem 5.5. *(# 11, November, 2000). Customers arrive at a service center in accordance with a Poisson process with mean 10 per hour. There are two servers, and the service time for each server is exponentially distributed with a mean of 10 minutes. If both servers are busy, customers wait and are served by the first available server. Customers always wait for service, regardless of how many other customers are in line.*
Calculate the percent of the time, on average, when there are no customers being served. (A) 0.8% (B) 3.9% (C) 7.2% (D) 9.1% (E) 12.7%

Solution: 0.090909

Problem 5.6. *(# 10, May, 2000). Taxicabs leave a hotel with a group of passengers at a Poisson rate $\lambda = 10$ per hour. The number of people in each group taking a cab is independent and has the following probabilities:*

Number of People	*Probability*
1	*0.60*
2	*0.30*
3	*0.10*

Using the normal approximation, calculate the probability that at least 1050 people leave the hotel in a cab during a 72-hour period.
(A) 0.60 (B) 0.65 (C) 0.70 (D) 0.75 (E) 0.80

Solution: *Let $N_1(t)$ be the number of taxicabs leaving a hotel with exactly one person until time t. Let $N_2(t)$ be the number of taxicabs leaving a hotel with exactly two persons until time t. Let $N_3(t)$ be the number of taxicabs leaving a hotel with exactly three persons until time t. Then, N_1, N_2 and N_3 are independent Poisson processes. Let $\lambda_1, \lambda_2, \lambda_3$ be their respective rates. We have that*

$$\lambda_1 = (10)(0.60) = 6, \ \lambda_2 = (10)(0.30) = 3, \ \lambda_3 = (10)(0.10) = 1.$$

Let Y be the number of people who leave the hotel in 72 hours. Then, $Y = N_1(72)+2N_2(72)+3N_3(72)$ be their respective rates. We have that

$$E[Y] = E[N_1(72) + 2N_2(72) + 3N_3(72)] = (72)(6) + (2)(72)(3) + (3)(72)(1) = 1080$$
$$\text{Var}(Y) = \text{Var}(N_1(72) + 2N_2(72) + 3N_3(72))$$
$$= \text{Var}(N_1(72)) + 4\text{Var}(N_2(72)) + 9\text{Var}(N_3(72)) = (72)(6) + (4)(72)(3) + (9)(72)(1) = 1944$$

So,

$$\text{P}(Y \geq 1050) \simeq P(N(0,1) \geq \tfrac{1050-1080}{\sqrt{1944}}) = P(N(0,1) \leq 0.68) = 0.75$$

Problem 5.7. *(# 36, May 2001). The number of accidents follows a Poisson distribution with mean 12. Each accident generates 1, 2, or 3 claimants with probabilities $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{6}$, respectively. Calculate the variance in the total number of claimants.*
(A) 20 (B) 25 (C) 30 (D) 35 (E) 40

Solution: *Let X_1 be the number of accidents which generate 1 claimant. Let X_2 be the number of accidents which generate 2 claimants. Let X_3 be the number of accidents which generate 3 claimants. Then, X_1, X_2 and X_3 are independent Poisson r.v.'s. Their respective rates are 6, 4 and 2. Let $\lambda_1, \lambda_2, \lambda_3$ be their respective rates. We have that*

$$\text{Var}(X_1 + 2X_2 + 3X_3) = \text{Var}(X_1) + 4\text{Var}(X_2) + 9\text{Var}(X_3) = 6 + (4)(4) + (9)(2) = 40$$

Problem 5.8. *(# 29, November 2000). Job offers for a college graduate arrive according to a Poisson process with mean 2 per month. A job offer is acceptable if the wages are at least 28,000. Wages offered are mutually independent and follow a lognormal distribution with $\mu = 10.12$ and $\sigma = 0.12$ Calculate the probability that it will take a college graduate more than 3 months to receive an acceptable job offer.*
(A) 0.27 (B) 0.39 (C) 0.45 (D) 0.58 (E) 0.61

Solution: *Let W be the job offer. We have that*

$$P(W \geq 28000) = \text{P}(\ln W \geq \ln 28000) = \text{P}(N(0,1) \geq \tfrac{(\ln 28000)-10.12}{0.12})$$
$$= P(N(0,1) \geq 1) = 0.1587.$$

Let $N_1(t)$ be the number of acceptable jobs received until time t months. N_1 has rate $\lambda_1 = (2)(0.1587) = 0.3174$. let $T_{1,1}$ be the time where the first acceptable job offer is received. We need to find

$$P(T_{1,1} > 3) = P(N(3) = 0) = e^{-(3)(0.3174)} = 0.3859.$$

Problem 5.9. *(# 23, November 2000). Workers' compensation claims are reported according to a Poisson process with mean 100 per month. The number of claims reported and the claim amounts are independently distributed. 2% of the claims exceed 30,000. Calculate the number of complete months of data that must be gathered to have at least a 90% chance of observing at least 3 claims each exceeding 30,000.*
(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution: *Let $N_1(t)$ be the number of claims exceeding 30000 received until time t months. N_1 has rate $\lambda_1 = (100)(0.02) = 2$. We have that*

$$P(N(2) \geq 3) = 1 - e^{-4}(1 + 4 + \tfrac{4^2}{2}) = 0.761,$$
$$P(N(3) \geq 3) = 1 - e^{-6}(1 + 6 + \tfrac{6^2}{2}) = 0.93,$$

So, the answer is 3.

Problem 5.10. *(# 23, Sample Test). You are given:*

- *A loss occurrence in excess of 1 billion may be caused by a hurricane, an earthquake, or a fire.*
- *Hurricanes, earthquakes, and fires occur independently of one another.*
- *The number of hurricanes causing a loss occurrence in excess of 1 billion in a one-year period follows a Poisson distribution. The expected amount of time between such hurricanes is 2.0 years.*
- *The number of earthquakes causing a loss occurrence in excess of 1 billion in a one-year period follows a Poisson distribution. The expected amount of time between such earthquakes is 5.0 years.*
- *The number of fires causing a loss occurrence in excess of 1 billion in a one-year period follows a Poisson distribution. The expected amount of time between such fires is 10.0 years.*

Determine the expected amount of time between loss occurrences in excess of 1 billion.

Solution: *Let $N_1(t)$ be the number of hurricanes causing a loss occurrence in excess of 1 billion until time t years. Let $N_2(t)$ be the number of earthquakes causing a loss occurrence in excess of 1 billion until time t years. Let $N_3(t)$ be the number of fires causing a loss occurrence in excess of 1 billion until time t years. N_1, N_2, N_3 are 3 independent Poisson processes with respective rates $\lambda_1 = \frac{1}{2}$, $\lambda_2 = \frac{1}{5}$, $\lambda_3 = \frac{1}{10}$. Let $N(t) = N_1(t) + N_2(t) + N_3(t)$. N has rate function $\lambda = \lambda_1 + \lambda_2 + \lambda_3 = \frac{1}{2} + \frac{1}{5} + \frac{1}{10} = \frac{4}{5}$. The expected amount of time between loss occurrences in excess of 1 billion is $\frac{5}{4} = 1.25$.*

Problem 5.11. *(# 6, November 2000). An insurance company has two insurance portfolios. Claims in Portfolio P occur in accordance with a Poisson process with mean 3 per year. Claims in portfolio Q occur in accordance with a Poisson process with mean 5 per year. The two processes are independent. Calculate the probability that 3 claims occur in Portfolio P before 3 claims occur in Portfolio Q.*
(A) 0.28 (B) 0.33 (C) 0.38 (D) 0.43 (E) 0.48

Solution: *Let* $N_1(t)$ *be the number of claims in portfolio* P *occur in until time* t *years. Let* $N_2(t)$ *be the number of claims in portfolio* Q *occur in until time* t *years.* N_1, N_2 *are two independent Poisson processes with respective rates* $\lambda_1 = 3$, $\lambda_2 = 5$. *Let* $N(t) = N_1(t) + N_2(t)$. N *has rate function* $\lambda = \lambda_1 + \lambda_2 = 3 + 5 = 8$. *The probability that a claim is from portfolio* P *is* $p = \frac{3}{8}$. *Consider the first claims, each claim is independently of the rest of other claims of type* P *with probability* p. *So, the probability that 3 claims occur in Portfolio P before 3 claims occur in Portfolio Q is*

$$P(3 \text{ or more of the first claims are from Portfolio } P\}$$
$$= \mathrm{P}(\mathrm{Binom}(5, 3/8) \geq 3) = \binom{5}{3}(3/8)^3(5/8)^2 + \binom{5}{4}(3/8)^4(5/8)^1 + \binom{5}{3}(3/8)^5(5/8)^0 = 0.2752$$

Problem 5.12. *(# 19, November 2001). A Poisson claims process has two types of claims, Type I and Type II.*
(i) The expected number of claims is 3000.
(ii) The probability that a claim is Type I is 1/3.
(iii) Type I claim amounts are exactly 10 each.
(iv) The variance of aggregate claims is 2,100,000.
Calculate the variance of aggregate claims with Type I claims excluded.
(A) 1,700,000 (B) 1,800,000 (C) 1,900,000 (D) 2,000,000 (E) 2,100,000

Solution: *Let* C_1 *and let* C_2 *be the amount of claims of type I and II, respectively.* C_1 *and* C_2 *are independent r.v.'s. Let* N_1 *be the number of claims of type I.* N_1 *has a Poisson distribution with parameter* $3000(1/3) = 1000$. *We have that* $C_1 = 10N_1$. *So,* $\mathrm{Var}(C_1) = (10)^2(1000) = 100000$. *So,*

$$\mathrm{Var}(C_2) = 2100000 - 100000 = 2000000.$$

Problem 5.13. *(# 4, May 2001). Lucky Tom finds coins on his way to work at a Poisson rate of 0.5 coins per minute. The denominations are randomly distributed:*
(i) 60% of the coins are worth 1;
(ii) 20% of the coins are worth 5;
(iii) 20% of the coins are worth 10.
Calculate the variance of the value of the coins Tom finds during his one-hour walk to work.
(A) 379 (B) 487 (C) 566 (D) 670 (E) 768

Solution: *Let* $N_1(t)$ *be the number of coins of value 1 which Tom finds until time* t *minutes. Let* $N_2(t)$ *be the number of coins of value 5 which Tom finds until time* t *minutes. Let* $N_3(t)$ *be the number of coins of value 10 which Tom finds until time* t *minutes.* N_1, N_2 *are two independent Poisson processes with respective rates*

$$\lambda_1 = (0.5)(0.6) = 0.30,\ \lambda_2 = (0.5)(0.2) = 0.10,\ \lambda_2 = (0.5)(0.2) = 0.10.$$

The total value of the coins Lucky Tom finds is $Y = N_1(60) + 5N_2(60) + 10N_3(60)$. *The variance of* Y *is*

$$\begin{aligned} &\mathrm{Var}(Y) = \mathrm{Var}(N_1(60)) + 5^2\mathrm{Var}(N_2(60)) + 10^2\mathrm{Var}(N_3(60)) \\ &= (60)(0.3) + 5^2(60)(.1) + 10^2(60)(.1) = 768 \end{aligned}$$

Problem 5.14. *(# 27, November 2001). On his walk to work, Lucky Tom finds coins on the ground at a Poisson rate. The Poisson rate, expressed in coins per minute, is constant during any one day, but varies from day to day according to a gamma distribution with mean 2 and variance 4. Calculate the probability that Lucky Tom finds exactly one coin during the sixth minute of today's walk.*
(A) 0.22 (B) 0.24 (C) 0.26 (D) 0.28 (E) 0.30

Solution: 0.22

Problem 5.15. *(# 9, November 2002). Lucky Tom finds coins on his way to work at a Poisson rate of 0.5 coins/minute. The denominations are randomly distributed:*
(i) 60% of the coins are worth 1 each
(ii) 20% of the coins are worth 5 each
(iii) 20% of the coins are worth 10 each.
Calculate the probability that in the first ten minutes of his walk he finds at least 2 coins worth 10 each, and in the first twenty minutes finds at least 3 coins worth 10 each.
(A) 0.08 (B) 0.12 (C) 0.16 (D) 0.20 (E) 0.24

Solution: *Let* X *be the number of coins worth 10 each Tom finds in the first 10 minutes.* X *has a Poisson distribution with mean* $(0.5)(10)(0.02) = 1$. *Let* Y *be the number of coins worth 10 each Tom finds between time=10 minutes and time=20 minutes.* Y *has a Poisson distribution with mean* $(0.5)(10)(0.02) = 1$. *We need to find*

$$\begin{aligned} &\mathrm{P}(X \geq 2, X + Y \geq 3) = \mathrm{P}(X = 2, Y \geq 1) + \mathrm{P}(X \geq 3) \\ &= \tfrac{e^{-1}}{2}(1 - e^{-1}) + 1 - e^{-1} - e^{-1} - \tfrac{e^{-1}}{2} = 0.1965. \end{aligned}$$

Problem 5.16. *(# 20, November 2002). Subway trains arrive at a station at a Poisson rate of 20 per hour. 25% of the trains are express and 75% are local. The types of each train are independent. An express gets you to work in 16 minutes and a local gets you there in 28 minutes. You always take the first train to arrive. Your co-worker always takes the first express. You both are waiting at the same station. Which of the following is true?*
(A) Your expected arrival time is 6 minutes earlier than your co-worker's.
(B) Your expected arrival time is 4.5 minutes earlier than your co-worker's.
(C) Your expected arrival times are the same.
(D) Your expected arrival time is 4.5 minutes later than your co-worker's.
(E) Your expected arrival time is 6 minutes later than your co-worker's.

Solution: *Your expected waiting for a train is* $\frac{1}{20}$ *hours or 3 minutes. Your expected traveling time is* $\frac{1}{4}16 + \frac{3}{428} = 25$. *Your expected arrival time is in 28 minutes. Your coworker's expected waiting for a train is* $\frac{1}{5}$ *hours or 12 minutes. Your coworker's expected traveling time is* 16. *Your coworker's expected arrival time is in 28 minutes.*

Problem 5.17. *(# 15, November 2002). Bob is an overworked underwriter. Applications arrive at his desk at a Poisson rate of 60 per day. Each application has a 1/3 chance of being a "bad" risk and a 2/3 chance of being a "good" risk. Since Bob is overworked, each time he gets an application he flips a fair coin. If it comes up heads, he accepts the application without looking at it. If the coin comes up tails, he accepts the application if and only if it is a "good" risk. The expected profit on a "good" risk is 300 with variance 10,000. The expected profit on a "bad" risk is -100 with variance 90,000. Calculate the variance of the profit on the applications he accepts today.*
(A) 4,000,000 (B) 4,500,000 (C) 5,000,000 (D) 5,500,000 (E) 6,000,000

Solution: *Let* $N_1(t)$ *be the number of good risk applications Bob accepts in* t *days.* N_1 *is a Poisson process with rate* $\lambda_1 = 40$. *Let* $N_2(t)$ *be the number of bad risk applications Bob accepts in* t *days.* N_2 *is a Poisson process with rate* $\lambda_2 = (60)(1/3)(1/2) = 10$. *Let* $\{X_j\}$ *be the amount of the good risk. Let* $\{Y_j\}$ *be the amount of the good bad. The amount of the applications Bob accepts today are* $U = \sum_{j=1}^{N_1(1)} X_j + \sum_{j=1}^{N_2(1)} Y_j$. *So,*

$$\mathrm{var}(U) = E[N_1(1)]E[X^2] + E[N_2(1)]E[Y^2] = (40)(100000) + (10)(100000) = 5000000.$$

5.5 Nonhomogenous of a Poisson process

Definition 5.5.1. *(Definition 1) The counting process* $\{N(t) : t \geq 0\}$ *is said to be a nonhomogenous Poisson process with intensity function* $\lambda(t) \geq 0$, $t \geq 0$, *if*

(i) $N(0) = 0$.

(ii) $\{N(t) : t \geq 0\}$ *has independent increments.*

(iii) $P(N(t + h) - N(t) \geq 2) = o(h)$.

(iv) $P(N(t + h) - N(t) = 1) = \lambda(t)h + o(h)$.

Definition 5.5.2. *(Definition 1) The counting process* $\{N(t) : t \geq 0\}$ *is said to be a nonhomogenous Poisson process with intensity function* $\lambda(t)$, $t \geq 0$, *if*

(i) $N(0) = 0$.

(ii) For each $t > 0$, $N(t)$ *has a Poisson distribution with mean* $m(t) = \int_0^t \lambda(s)\, ds$.

(iii) For each $0 \leq t_1 < t_2 < \cdots < t_m$, $N(t_1), N(t_2) - N(t_1), \ldots, N(t_m) - N(t_{m-1})$ *are independent r.v.'s.*

$m(t)$ is the mean value function of the non homogeneous Poisson process.

It follows from the previous definition that for each $0 \leq t_1 < t_2 < \cdots < t_m$ and each integers $k_1, \ldots, k_m \geq 0$,

$$\begin{aligned}
&P(N(t_1) = k_1, N(t_2) = k_2, \ldots, N(t_m) = k_m) \\
&= P(N(t_1) = k_1, N(t_2) - N(t_1) = k_1 - k_1, \ldots, N(t_m) - N(t_{m-1}) = k_m - k_{m-1}) \\
&= \tfrac{e^{-m(t_1)}(m(t_1))^{k_1}}{k_1!} \tfrac{e^{-(m(t_2)-m(t_1))}(m(t_2)-(m(t_1))^{k_2-k_1}}{(k_2-k_1)!} \cdots \tfrac{e^{-(m(t_m)-m(t_{m-1}))}(m(t_m)-m(t_{m-1}))^{k_m-k_{m-1}}}{(k_m-k_{m-1})!},
\end{aligned}$$

If $\lambda(t) = \lambda$, for each $t \geq 0$, we have an homogeneous Poisson process.

Let S_n be the time of the n–th occurrence. Then,

$$P(S_n > t) = P(N_t \le n-1) = \sum_{j=0}^{n-1} \frac{e^{-m(t)}(m(t))^j}{j!}.$$

Exercise 5.45. *For a nonhomogenous Poisson process the intensity function is given by*

$$\lambda(t) = \begin{cases} 10 & \text{if } t \text{ is in } (0,1/2], (1,3/2], \dots \\ 2 & \text{if } t \text{ is in } (1/2,1], (3/2,2], \dots \end{cases}$$

(i) How many occurrences are expected in the time period $(0,1]$*? During* $(0,3/2]$*?*

(ii) If $S_{10} = 0.45$ *is given, calculate the probability that* $S_{11} > 0.75$*.*

Answer: *(i) The expected in the time period* $(0,1]$ *is*

$$m(1) = \int_0^1 \lambda(t)\,dt = \int_0^{1/2} 10\,dt + \int_{1/2}^1 2\,dt = 6.$$

The expected in the time period $(0,3/2]$ *is*

$$m(1) = \int_0^{3/2} \lambda(t)\,dt = \int_0^{1/2} 10\,dt + \int_{1/2}^1 2\,dt + \int_1^{3/2} 10\,dt = 11.$$

(ii)

$$P(S_{11} > 0.75 | S_{10} = 0.45) = P(N(0.75) = 10 | S_{10} = 0.45)$$
$$= P(N(0.75) - N(0.45) = 0 | S_{10} = 0.45) = P(N(0.75) - N(0.45) = 0) = e^{-1},$$

because

$$m(0.75) - m(0.45) = \int_{0.45}^{0.75} \lambda(t)\,dt = \int_{0.45}^{0.5} \lambda(t)\,dt + \int_{0.5}^{0.75} \lambda(t)\,dt = \int_{0.45}^{0.5} 10\,dt + \int_{0.5}^{0.75} 2\,dt = 1.$$

Exercise 5.46. *For a non–homogenous Poisson process , the intensity function is given by*

$$\lambda(t) = \begin{cases} t & \text{for } 0 < t \le 10 \\ 10 & \text{for } 10 < t \end{cases}$$

(a) Calculate the expected number of event occurrences during the period from time 5 until time 15.

(b) If the 50*–th event occurs at time 9, calculate the probability that the* 51*-st event occurs by time 9.1.*

5.6 Compound Poisson process

Definition 5.6.1. *A stochastic process* $\{X(t) : t \ge 0\}$ *is said to be a compound Poisson process if it can be represented as*

$$X(t) = \sum_{i=1}^{N(t)} Y_i, t \ge 0,$$

where $\{N(t) : t \ge 0\}$ *is a Poisson process and* $\{Y_i\}$ *is a sequence of i.i.d.r.v.'s independent of* $\{N(t) : t \ge 0\}$*.*

Using the double expectation theorem, we have that

$$\begin{aligned}
&E[X(t)|N(t)=n] = E[\textstyle\sum_{i=1}^{N(t)} Y_i|N(t)=n] = E[\sum_{i=1}^{n} Y_i|N(t)=n] = nE[Y_1]\\
&E[X(t)] = E[E[X(t)|N(t)=n]] = E[N(t)E[Y_1]] = \lambda t E[Y_1],\\
&\mathrm{Var}(X(t)|N(t)=n) = \mathrm{Var}(\textstyle\sum_{i=1}^{N(t)} Y_i|N(t)=n) = \mathrm{Var}(\sum_{i=1}^{n} Y_i|N(t)=n) = n\mathrm{Var}(Y_1)\\
&\mathrm{Var}(X(t)) = E[\mathrm{Var}(X(t)|N(t)=n)] + \mathrm{Var}(E[X(t)|N(t)=n])\\
&= E[N(t)\mathrm{Var}(Y_1)] + \mathrm{Var}(N(t)E[Y_1]) = \lambda t\mathrm{Var}(Y_1) + \lambda t(E[Y_1])^2 = \lambda t E[Y_1^2].
\end{aligned}$$

Hence,

$$E[X(t)] = \lambda t E[Y_1] \text{ and } \mathrm{Var}(X(t)) = \lambda t E[Y_1^2].$$

Problem 5.18. *(# 30, November 2001). The claims department of an insurance company receives envelopes with claims for insurance coverage at a Poisson rate of $\lambda = 50$ envelopes per week. For any period of time, the number of envelopes and the numbers of claims in the envelopes are independent. The numbers of claims in the envelopes have the following distribution:*

Number of Claims	*Probability*
1	*0.20*
2	*0.25*
3	*0.40*
4	*0.15*

Using the normal approximation, calculate the 90th percentile of the number of claims received in 13 weeks.
(A) 1690 (B) 1710 (C) 1730 (D) 1750 (E) 1770

Solution 1: *Let $N(t)$ be the number of envelopes received until time t. $N(t)$ is a Poisson process with rate $\lambda = 50$. Let $\{X_j\}$ be the number of claims received in each envelope. We have that the total number of claims received in 13 weeks is $Y = \sum_{j=1}^{N(13)} X_j$. We have that*

$$\begin{aligned}
&E[X] = (1)(0.2) + (2)(0.25) + (3)(0.4) + (4)(0.15) = 2.5\\
&E[X^2] = (1)^2(0.2) + (2)^2(0.25) + (3)^2(0.4) + (4)^2(0.15) = 7.2\\
&E[Y] = E[N(13)]E[X] = (50)(13)(2.5) = 1625\\
&\mathrm{Var}(Y) = E[N(13)]E[X^2] = (50)(13)(7.2) = 4680
\end{aligned}$$

Let q be the 90th percentile of the number of claims received in 13 weeks. Then,

$$0.90 = P(Y \le q) \simeq \mathrm{P}(N(0.1) \le \frac{q-1625}{\sqrt{4680}}) = \mathrm{P}(N(0,1) \le 1.282).$$

So, $q = 1526 + \sqrt{4680}(1.282)1749.03$.

Solution 2: *Let $N_i(t)$ be the number of envelopes received until time t with i envelopes, where $1 \le i \le 4$. Then, N_1, N_2, N_3, N_4 are independent Poisson processes with respective rates*

$$\lambda_1 = (50)(0.2) = 10,\ \lambda_2 = (50)(0.25) = 12.5,\ \lambda_3 = (50)(0.4) = 20,\ \lambda_4 = (50)(0.15) = 7.5.$$

We have that the total number of claims received in 13 weeks is $Y = N_1(13) + 2N_2(13) + 3N_3(13) + 4N_4(13)$ *and*

$$E[Y] = (1)(10) + (2)(12.5) + (3)(20) + (4)(7.5) = 1625$$
$$E[Y^2] = (1)^2(10) + (2)^2(12.5) + (3)^2(20) + (4)^2(7.5) = 4680$$

Now, we can proceed as before.

Problem 5.19. *(# 11, May 2000). A company provides insurance to a concert hall for losses due to power failure. You are given:*
(i) The number of power failures in a year has a Poisson distribution with mean 1.
(ii) The distribution of ground up losses due to a single power failure is:

x	*Probability of* x
10	*0.3*
20	*0.3*
50	*0.40*

(iii) The number of power failures and the amounts of losses are independent.
(iv) There is an annual deductible of 30.
Calculate the expected amount of claims paid by the insurer in one year.
(A) 5 (B) 8 (C) 10 (D) 12 (E) 14

Solution: *Let* N *be the number of power failures. Let* $\{X_j\}$ *be the ground losses. Let* $S = \sum_{j=1}^{N(1)} X_j$. *We have to find*

$$E[(S-30)_+] = E[S] - E[S \wedge 30].$$

Now,

$$E[S] = E[N]E[X] = (1)((10)(0.3) + (20)(0.3) + (50)(0.4)) = 29,$$
$$\mathrm{P}(S=0) = P(N=0) = e^{-1} = 0.3679$$
$$\mathrm{P}(S=10) = P(N=1, X_1=10) = e^{-1}(0.3) = 0.1104$$
$$\mathrm{P}(S=20) = P(N=1, X_1=20) + P(N=2, X_1+X_2=20) = e^{-1}(0.3) + \frac{e^{-1}}{2}(0.3)^2 = 0.0276$$
$$E[S \wedge 30] = (10)P(S=10) + (20)P(S=20) + (30)P(S \geq 30)$$
$$= (10)(0.1104) + (20)(0.0276) + (30)(1 - 0.3679 - 0.1104 - 0.0276) = 16.479$$

So,

$$E[(S-30)_+] = E[S] - E[S \wedge 30] = 29 - 16.479 = 13.521.$$

Problem 5.20. *(# 14, Sample Test). You are given:*

- *An aggregate loss distribution has a compound Poisson distribution with expected number of claims equal to 1.25.*
- *Individual claim amounts can take only the values 1, 2 or 3, with equal probability.*

Determine the probability that aggregate losses exceed 3.

Solution: *Let N be the number of claims. Let $\{X_j\}$ be the individual claims. Let $S = \sum_{j=1}^{N} X_j$. We have that*

$$\begin{aligned}
&\mathrm{P}(S=0) = P(N=0) = e^{-1.25} = 0.2865\\
&\mathrm{P}(S=1) = P(N=1, X_1=1) = e^{-1.25}(1.25)\tfrac{1}{3} = 0.1194\\
&\mathrm{P}(S=2) = P(N=1, X_1=2) + P(N=2, X_1+X_2=2)\\
&= e^{-1.25}(1.25)\tfrac{1}{3} + \tfrac{e^{-1.25}(1.25)^2}{2}(\tfrac{1}{3})^2 = 0.1194 + 0.0249 = 0.1442\\
&\mathrm{P}(S=3) = P(N=1, X_1=3) + P(N=2, X_1+X_2=3) + P(N=3, X_1+X_2+X_3=3)\\
&= e^{-1.25}(1.25)\tfrac{1}{3} + \tfrac{e^{-1.25}(1.25)^2}{2}2(\tfrac{1}{3})^2 + \tfrac{e^{-1.25}(1.25)^3}{6}(\tfrac{1}{3})^3 = 0.1194 + 0.0488 + 0.0035 = 0.1717
\end{aligned}$$

So,

$$\mathrm{P}(S > 3) = 1 - 0.2865 - 0.1194 - 0.1442 - 0.1717 = 0.2782$$

Problem 5.21. *(# 15, Sample Test). You are given:*

- *An aggregate loss distribution has a compound Poisson distribution with expected number of claims equal to 1.25.*
- *Individual claim amounts can take only the values 1, 2 or 3, with equal probability.*

Calculate the expected aggregate losses if an aggregate deductible of 1.6 is applied.

Solution: *Let N be the number of claims. Let $\{X_j\}$ be the individual claims. Let $S = \sum_{j=1}^{N} X_j$. We have that*

$$\begin{aligned}
&\mathrm{P}(S=0) = P(N=0) = e^{-1.25} = 0.2865\\
&\mathrm{P}(S=1) = P(N=1, X_1=1) = e^{-1.25}(1.25)\tfrac{1}{3} = 0.1194\\
&\mathrm{P}(S=2) = P(N=1, X_1=1) + P(N=2, X_1+X_2=2)\\
&= e^{-1.25}(1.25)\tfrac{1}{3} + \tfrac{e^{-1.25}(1.25)^2}{2}(\tfrac{1}{3})^2 = 0.1194 + 0.0249 = 0.1442\\
&\mathrm{P}(S=3) = P(N=1, X_1=3) + P(N=2, X_1+X_2=3) + P(N=3, X_1+X_2+X_3=3)\\
&= e^{-1.25}(1.25)\tfrac{1}{3} + \tfrac{e^{-1.25}(1.25)^2}{2}2(\tfrac{1}{3})^2 + \tfrac{e^{-1.25}(1.25)^3}{6}(\tfrac{1}{3})^3 = 0.1194 + 0.0488 + 0.0035 = 0.1717
\end{aligned}$$

So,

$$\begin{aligned}
&E[S] = E[N]E[X] = (1.25)((1)(1/3) + (2)(1/3) + (3)(1/3)) = 2.5,\\
&E[S \wedge 1.6] = 1P(S=1) + (1.6)P(S \geq 2) = 0.1194 + (1.6)(1 - 0.2865 - 0.1194) = 1.0700\\
&E[(S-1.6)_+] = E[S] - E[S \wedge 1.6] = 1.43
\end{aligned}$$

Problem 5.22. *(# 31, November 2001). An insurer's losses are given by a Poisson process with mean 12 per year. Severity is constant at 1. Annual premiums of 18 are collected. Initial surplus is 6. Which of the following best describes the insurer's future?*
(A) Ruin is certain within 3 years.
(B) Ruin is certain, but not necessarily within 3 years.
(C) Ruin is possible, but not certain.
(D) Ruin is possible, but can be avoided with certainty by increasing the initial surplus.
(E) The current surplus is adequate so that ruin will not occur.

Solution: $\theta = 0.5$ so (C)

Problem 5.23. *(# 19, November 2000). A community is able to obtain plasma at the continuous rate of 22 units per day. The daily demand for plasma is modeled by a compound Poisson process where the number of people needing plasma has mean 20 and the number of units needed by each person is approximated by an exponential distribution with mean 1. Assume all plasma can be used without spoiling. At the beginning of the period there are 20 units available. Calculate the probability that there will not be enough plasma at some time.*
(A) 0.11 (B) 0.12 (C) 0.13 (D) 0.14 (E) 0.15

Solution: 0.15

Problem 5.24. *(# 20, May 2001). An insurer's claims follow a compound Poisson claims process with two claims expected per period. Claim amounts can be only 1, 2, or 3 and these are equal in probability. Calculate the continuous premium rate that should be charged each period so that the adjustment coefficient will be 0.5.*
(A) 4.8 (B) 5.9 (C) 7.8 (D) 8.9 (E) 11.8

Solution: 7.8 premium rate per period

Problem 5.25. *(# 37, May 2001). For a claims process, you are given: (i) The number of claims $\{N(t) : t \geq 0\}$ is a nonhomogeneous Poisson process with intensity function:*

$$\lambda(t) = \begin{cases} 1 & \text{if } 0 \leq t < 1 \\ 2 & \text{if } 1 \leq t < 2 \\ 3 & \text{if } 2 \leq t \end{cases}$$

(ii) Claims amounts Y_i re independently and identically distributed random variables that are also independent of $N(t)$.
(iii) Each Y_i is uniformly distributed on [200,800].
(iv) The random variable P is the number of claims with claim amount less than 500 by time $t = 3$.
(v) The random variable Q is the number of claims with claim amount greater than 500 *by time $t = 3$.*
(vi) R is the conditional expected value of P, given $Q = 4$.
Calculate R.
(A) 2.0 (B) 2.5 (C) 3.0 (D) 3.5 (E) 4.0

Solution: *P and Q are independent r.v.'s. $N(3)$ is Poisson r.v. with rate $\int_0^3 \lambda(t)\,dt = 6$. We have that $P(Y \leq 500) = \frac{300}{600} = \frac{1}{2}$. So, P is a Poisson r.v. with mean* 3.

Problem 5.26. *(# 27, May 2001). A company insures a fleet of vehicles. Aggregate losses have a compound Poisson distribution. The expected number of losses is 20. Loss amounts, regardless of vehicle type, have exponential distribution with $\theta = 200$. In order to reduce the cost of the insurance, two modifications are to be made:*
(i) a certain type of vehicle will not be insured. It is estimated that this will reduce loss frequency by 20%.
(ii) a deductible of 100 per loss will be imposed.

Calculate the expected aggregate amount paid by the insurer after the modifications.
(A) 1600 (B) 1940 (C) 2520 (D) 3200 (E) 3880

Solution: 1942

Exercise 5.47. *For an individual over 65:*
(i) The number of pharmacy claims in a year is a Poisson random variable with mean 25.
(ii) The amount of each pharmacy claim is uniformly distributed between 5 and 95.
(iii) The amounts of the claims and the number of claims are mutually independent.
(a) Find the mean and the variance of the aggregate claims for this individual in a year.
(b) Determine the probability that aggregate claims for this individual will exceed 2000 using the normal approximation.

Problems

1. Let X be a r.v. with cdf $F(x) = 1 - \sum_{k=0}^{3} \frac{e^{-x}x^k}{k!}$, for $x > 0$. What is the density function of X for $x > 0$?

2. A system made up of 7 components with independent, identically distributed lifetimes will operate until any of 1 of the system's component fails. If the lifetime X of each component has density function

$$f(x) = \begin{cases} \frac{3}{x^4}, & \text{if } 1 < x, \\ 0 & \text{else.} \end{cases}$$

   find the density of the lifetime of the system T and find $E[T]$. Hint: If X_i, $1 \le i \le 7$, are the lifetime of the components, then $T = \min_{1 \le i \le 7} X_i$.

3. Suppose that the distribution of the number of accidents in a certain freeway is a Poisson random variable with rate of two accidents per months. Find the probability that there are 5 or more accidents in this freeway in a period of six months.

4. If X has a Poisson distribution of that $3P[X = 1] = P[X = 2]$. Find $P[X = 4]$.

5. The number of claims received in a year by an insurance company is a Poisson random variable with $\lambda = 500$. The claim amounts are independent and uniformly distributed over $[0, 500]$. If the company has \$140,000 available to pay claims, what is the probability that it will have enough to pay all the claims that come in?

6. An insurance company pays out claims on its life insurance policies in accordance with a Poisson process having rate $\lambda = 5$ per week. If the amount of money paid on each policy is exponentially distributed with mean \$2000, what is the mean and variance of the amount of money paid by the insurance company in a four–week span? Estimate the probability that the insurance company will have to pay less than \$20,000 in a four–week span?

7. A group of policyholders experiences losses, which are exponentially distributed with mean 500. The number of losses until time t follow a Poisson process with 100 expected losses per year. A claim results from a loss if the loss exceeds a deductible of 100. Calculate the expected value of the claims for the group of policyholders over a period of ten years.

8. An average of 500 people pass the California bar exam each year. A California lawyer practices law, on the average, for 30 years. Assuming these members remain steady, how many lawyers would you expect California to have in 2050.

9. Pulses arrive at a Geyser counter in accordance with a Poisson process at a rate of 3 arrivals per minute. Each particle arriving at the counter has a probability $\frac{2}{3}$ of being recorded. Let $X(t)$ denote the number of pulses recorded by time t. Find $P(X(t) = 0)$ and $E[X(t)]$.

10. Events occur according to a non homogenous Poisson process, whose mean value function is given by

$$m(t) = t^2 + 2t, \ t \geq 0$$

What is the probability n events occur between times $t = 4$ and $t = 5$?

11. A store opens at 8 a.m. From 8 until 10 customers arrive a Poisson rate of four an hour. Between 10 and 12 they arrive at a Poisson rate of eight an hour. From 12 to 2 the arrival rate increases steadily from eight per hour at 12 to 10 per hour at 2; and from 2 to 5 the arrival rate drop steadily from ten per hour at 2 to four per hour at 5. Determine the probability distribution of the number o customers that enter the store in a given day.

12. For a compound Poisson process the expected number of claims is 2 and the claim amount distribution is lognormal. That is, the claim amount is a random variable X where $X = e^Y$ and Y is normally distributed with expected value 1 and standard deviation 2. Find the median of X. Find the mean and the variance of the aggregate claims.

13. Customers arrive for service according to a Poisson process with rate $\lambda = 6$ per hour. In the first half-hour of one hour period only 2 customers arrive. Calculate the probability 6 or more will have arrived by the end of this hour.

14. S has a compound Poisson distribution with $\lambda = 0.60$ and individual claims amounts that are 1, 2, 3, with probability 0.20, 0.30 and 0.50 respectively. Determine $P(S \geq 3)$.

15. The number of accidents incurred by an insured driver in a single year has a Poisson distribution with parameter 2. If an accident occurs, the probability the damage amount exceeds the deductible is 0.25. The number of claims that the damage amounts are independent. What is the probability that there will be no damages exceeding the deductible in a single year?

16. Successful lawsuits against a health insurer occur according to a Poisson process with mean 50 per year. One percent of the successful suits cost the insurer a settlements in excess of \$1 million. Calculate the number of complete of years of days that must be gathered to have at least a 50 % chance of observing 2 or more settlements in excess of \$1 million.

17. A Poisson Process has a claims intensity of 0.3. What is the probability of exactly 2 claims from time 0 to time 10?
A. 22% B. 24% C. 26% D. 28% E. 30%

18. A Poisson Process has a claims intensity of 0.3. Given that there have been 2 claims from time 0 to time 10, what is the probability of exactly 2 claims from time 10 to time 20?
A. 22% B. 24% C. 26% D. 28% E. 30%

19. A Poisson Process has a claims intensity of 0.3. Given that there have been 2 claims from time 0 to time 10, what is the probability of exactly 2 claims from time 6 to time 16?
A. 19% B. 21% C. 23% D. 25% E. 27%

20. A Poisson Process has a claims intensity of $\lambda = 0.15$. One observes two claims during the time interval (0, 5). What is the probability that the first claim occurs before time 1? A. 36% B. 37% C. 38% D. 39% E. 40%

21. Claims occur via a Poisson Process with unknown constant intensity λ. Claims are reported after a lag. The lag from occurrence to report is given by an Exponential Distribution with a mean of .6. 480 claims that occurred between time 0 and time 1 have been reported by time 1.25. Estimate λ? A. Less than 680
B. At least 680, but less than 690
C. At least 690, but less than 700
D. At least 700, but less than 710
E. At least 710

22. A Poisson Process has $\lambda = 0.6$. What is the probability of exactly 3 claims from time 5 to time 9?
A. 19% B. 21% C. 23% D. 25% E. 27%

23. A Poisson Process $\lambda = 0.6$. If there are three claims from time 2 to time 8, what is the probability of exactly 3 claims from time 5 to time 9?
A. 19% B. 21% C. 23% D. 25% E. 27%

24. Claims are given by a Poisson Process with $\lambda = 7$. What is probability that the number of claims between time 5 and 15 is at least 60 but no more than 80? Use the Normal Approximation. (A) 73% (B) 75% (C) 77% (D) 79% (E) 81%

Use the following information for the next two questions: The android Data is stranded on the planet Erehwon.

- Data uses energy uniformly at a rate of 10 gigajoules per year.
- If Data's stored energy reach 0, he ceases to function.
- Data gets his energy from dilithium crystals.
- Data gets 6 gigajoules of energy from each dilithium crystal.
- Data finds dilithium crystals at a Poisson rate of 2 per year.
- Data can store dilithium crystals without limit until needed.
- Data currently has 8 gigajoules of energy stored.

25. What is the probability that Data ceases to function within the next 2.5 years?
(A) 30% (B) 33% (C) 36% (D) 39% (E) 42%

26. What is the expected number of gigajoules of energy found by Data in the next 2.5 years?
(A) 20 (B) 22 (C) 24 (D) 26 (E) 28

Use the following information for the next four questions: Claims follow a Poisson Process with $\lambda = 81$. Bob and Ray each make a claim during the year 2002.

27. What is the probability that Ray made his claim before Bob?
(A) 1/16 (B) 1/6 (C) 1/4 (D) 1/3 (E) 1/2

28. What is the probability that Ray made his claim during the first quarter of 2002?
(A) 1/16 (B) 1/6 (C) 1/4 (D) 1/3 (E) 1/2

29. What is the probability that Bob and Ray each made their claims during the last quarter of 2002?
(A) 1/16 (B) 1/6 (C) 1/4 (D) 1/3 (E) 1/2

30. If Ray made his claim on July 23, 2002, what is the probability that Bob made his claim during May or June 2002?
(A) 1/16 (B) 1/6 (C) 1/4 (D) 1/3 (E) 1/2

31. Claims are given by a Poisson Process with $\lambda = 7$. What is probability that the number of claims between time 5 and 11 is greater than the number of claims from time 11 to 15? Use the Normal Approximation.
(A) 87% (B) 89% (C) 91% (D) 93% (E) 95%

32. Claims are given by a Poisson Process with $\lambda = 5$. What is probability that the number of claims between time 2 and 10 is greater than the number of claims from time 7 to 16? Use the Normal Approximation.
(A) 17% (B) 19% (C) 21% (D) 23% (E) 25%

33. Claims are given by a Poisson Process. The fifth loss occurred at time 31. What is the chance that the first loss occurred by time 18?
(A) 95% (B) 96% (C) 97% (D) 98% (E) 99%

34. Claims are given by a Poisson Process. The fifth loss occurred at time 31. What is the chance that the third loss occurred by time 18?
(A) 40% (B) 42% (C) 44% (D) 46% (E) 48%

35. A Poisson Process has a claims intensity of 0.4 per day. How many whole number of days do we need to observe, in order to have at least a 95% probability of seeing at least one claim?
(A) 8 (B) 9 (C) 10 (D) 11 (E) 12

36. A Poisson Process has a claims intensity of 0.4 per day. How many whole number of days do we need to observe, in order to have at least a 95at least two claims?
(A) 11 (B) 12 (C) 13 (D) 14 (E) 15

37. Claims are received by the Symphonic Insurance Company via a Poisson Process with mean .017. During the time interval (0, 100) it receives three claims, one from Beethoven, one from Tchaikovsky, and one from Mahler. The claim from Beethoven was received at time 27 and the claim from Mahler was received at time 91. What is probability that the claim from Tchaikovsky was received between the other two claims?
(A) 64% (B) 66% (C) 68% (D) 70% (E) 72%

Use the following information for the next two questions: For a tyrannosaur with 10,000 calories stored:
(i) The tyrannosaur uses calories uniformly at a rate of 10,000 per day. If his stored calories reach 0, he dies.
(ii) The tyrannosaur eats scientists (10,000 calories each) at a Poisson rate of 1 per day.
(iii) The tyrannosaur eats only scientists.
(iv) The tyrannosaur can store calories without limit until needed.

38. Calculate the probability that the tyrannosaur dies within the next 2.5 days.

39. Calculate the expected calories eaten in the next 2.5 days.
(A) 17,800 (B) 18,800 (C) 19,800 (D) 20,800 (E) 21,800

40. A Poisson Process has a claims intensity of .05. What is the mean time until the first claim?
A. 5 B. 10 C. 15 D. 20 E. 25

41. A Poisson Process has a claims intensity of .05. What is the mean time until the tenth claim?
A. 50 B. 100 C. 200 D. 300 E. 400

42. A Poisson Process has a claims intensity of .05. What is the probability that the time until the first claim is greater than 35?

A. Less than 14% B. At least 14%, but less than 15% C. At least 15%, but less than 16% D. At least 16%, but less than 17% E. At least 17%

43. A Poisson Process has a claims intensity of .05. What is the probability that the time from the fifth claim to the sixth claim is less than 10?
A. Less than 36% B. At least 36%, but less than 38% C. At least 38%, but less than 40% D. At least 40%, but less than 42% E. At least 42%

44. A Poisson Process has a claims intensity of .05. What is the probability that the time from the eighth claim to the tenth claim is greater than 50?
A. Less than 28% B. At least 28%, but less than 30% C. At least 30%, but less than 32% D. At least 32%, but less than 34% E. At least 34%

45. For a claim number process you are given that the elapsed times between successive claims are mutually independent and identically distributed with distribution function: $F(t) = 1 - e^{-t/2}$, $t \geq 0$. Determine the probability of exactly 3 claims in an interval of length 7.
A. Less than 23%
B. At least 23%, but less than 25%
C. At least 25%, but less than 27%
D. At least 27%, but less than 29%
E. At least 29%

46. Claims follow a Poisson Process. The average time between claims is 5. What is the probability that we have observed at least 2 claims by time 9?
A. Less than 53% B. At least 53%, but less than 55% C. At least 55%, but less than 57% D. At least 57%, but less than 59% E. At least 59%

47. Claims follow a Poisson Process. The average time between claims is 5. What is the probability that we have observed exactly 2 claims by time 9?
A. Less than 23% B. At least 23%, but less than 25% C. At least 25%, but less than 27% D. At least 27%, but less than 29% E. At least 29%

48. Claims occur via a homogeneous Poisson Process. The expected waiting time until the first claim is 770 hours. If the claims intensity had been 5 times as large, what would have been the expected waiting time until the first claim?
A. 154 hours B. 765 hours C. 770 hours D. 3850 hours E. None of the above.

49. For a Poisson Process with $\lambda = 5$. What is the probability that we have observed at least 3 claims by time 1.2?
A. Less than 93%
B. At least 93%, but less than 95%
C. At least 95%, but less than 97%
D. At least 97%, but less than 99%
E. At least 99%

50. For a claim number process $\{N(t) : t \geq 0\}$ you are given that the elapsed times between successive claims are mutually independent and identically distributed with distribution function

$$F(t) = 1 - e^{-3t}, t \geq 0$$

Determine the probability of exactly 4 claims in an interval of length 2.
(A) 0.11 (B) 0.13 (C) 0.15 (D) 0.17 (E) 0.19

51. S has a compound-Poisson distribution with $\lambda = 1.0$ and claim amounts that are 1, 2, or 3 with probabilities .50, .25, and .25, respectively. Determine $P(S = 3)$. Determine $P(S \geq 3)$. Determine the mean and the variance of S.

52. You are given:

    - An aggregate loss distribution has a compound Poisson distribution with expected number of claims equal to 1.25.
    - Individual claim amounts can take only the values 1, 2 or 3, with equal probability.

    (i) Determine the probability that aggregate losses exceed 3.

    (ii) Calculate the expected aggregate losses if an aggregate deductible of 1.6 is applied.

53. For a compound Poisson process, the expected number of claims is 2 and the claim amount distribution is lognormal. That is, the claim amount X is a random variable $X = e^Y$ where Y and is normally distributed with expected value 1 and standard deviation 2. Find the expected value and the variance of the aggregate claims.

54. S has a compound Poisson claims distribution with the following properties:
(i) Individual claim amounts equal to 1, 2 or 3
(ii) $E[S] = 56$.
(iii) $\text{Var}(S) = 126$.
(iv) $\lambda = 29$.
Determine the expected number of claims of size 2.d