COMPLEX NUMBERS

Cartesian Form of Complex Numbers

The fundamental complex number is i, a number whose square is -1; that is, i is defined as a number satisfying $i^2 = -1$. The **complex number system** is all numbers of the form

$$z = x + yi \tag{1}$$

where x and y are real. The number x is called the **real part** of z, and y is called the **imaginary part** of z. For example, real and imaginary parts of $6 - 2i$ are 6 and -2. Both real and imaginary parts of a complex number are themselves real numbers. The real number system is a subset of the complex number system obtained when $y = 0$. We call $x + yi$ the Cartesian form for a complex number.

Complex numbers can be visualized geometrically as points in the complex (Argand) plane. Some fixed point O is chosen to represent the complex number $0+0i$. Through O are drawn two mutually perpendicular axes (Figure 1), one called the real axis, and the other called the imaginary axis. The complex number $x + yi$ is then represented by the point x units in the real direction and y units in the imaginary direction. For example, the complex numbers $1 + 2i$, $-1 - i$, $4 - 3i$, and $-2 + 2i$ are shown in Figure 2. The real number system is represented by points on the real axis.

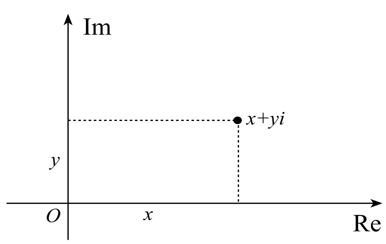

Figure 1

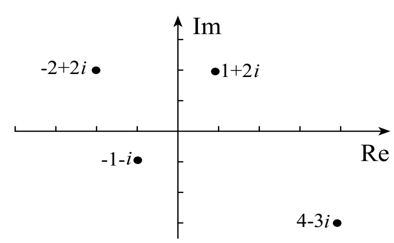

Figure 2

Two complex numbers $x+yi$ and $a+bi$ are said to be equal if their real parts are equal and their imaginary parts are equal; that is,

$$x + yi = a + bi \qquad \Longleftrightarrow \qquad x = a \quad \text{and} \quad y = b. \tag{2}$$

Geometrically, two complex numbers are equal if they correspond to the same point in the complex plane.

We add and subtract complex numbers $z_1 = x + yi$ and $z_2 = a + bi$ as follows:

$$z_1 + z_2 = (x + a) + (y + b)i, \tag{3a}$$
$$z_1 - z_2 = (x - a) + (y - b)i. \tag{3b}$$

In words, complex numbers are added and subtracted by adding and subtracting their real and imaginary parts. For example,

$$(3 - 2i) + (6 + i) = (3 + 6) + (-2 + 1)i = 9 - i,$$
$$(3 - 2i) - (6 + i) = (3 - 6) + (-2 - 1)i = -3 - 3i.$$

Complex numbers are multiplied according to the following definition. If $z_1 = x + yi$ and $z_2 = a + bi$, then

$$z_1 z_2 = (x + yi)(a + bi) = (xa - yb) + (xb + ya)i. \tag{4}$$

For example,

$$(3-2i)(6+i) = [(3)(6)-(-2)(1)] + [(3)(1)+(-2)(6)]i = 20-9i.$$

It is not necessary to memorize equation (4) when we note that this definition is precisely what we would expect if the usual laws for multiplying binomials were applied, together with the fact that $i^2 = -1$:

$$\begin{aligned}(3-2i)(6+i) &= (3)(6)+(3)(i)+(-2i)(6)+(-2i)(i)\\ &= 18+3i-12i-2i^2\\ &= 18-9i-2(-1)\\ &= 20-9i.\end{aligned}$$

With addition, subtraction, and multiplication taken care of, it is natural to turn to division of complex numbers. If we accept that division of any nonzero complex number by itself should be equal to 1, and ordinary rules of algebra should prevail, a definition of division of complex numbers is not necessary; it follows from equation (4). When $z_1 = x+yi$ and $z_2 = a+bi$, we calculate

$$\frac{z_1}{z_2} = \frac{x+yi}{a+bi}$$

by multiplying numerator and denominator by $a-bi$. This results in

$$\begin{aligned}\frac{z_1}{z_2} = \frac{x+yi}{a+bi} &= \frac{(x+yi)(a-bi)}{(a+bi)(a-bi)}\\ &= \frac{(xa+yb)+(-xb+ya)i}{a^2+b^2} \qquad \text{(using (4))}\\ &= \left(\frac{xa+yb}{a^2+b^2}\right) + \left(\frac{ya-xb}{a^2+b^2}\right)i.\end{aligned}$$

For example,

$$\frac{3-2i}{6+i} = \frac{(3-2i)(6-i)}{(6+i)(6-i)} = \frac{16-15i}{37} = \frac{16}{37} - \frac{15}{37}i.$$

In summary, addition, subtraction, multiplication, and division of complex numbers are performed using ordinary rules of algebra with the extra condition that i^2 is always replaced by -1.

Example 1 Write the following complex numbers in Cartesian form:

$$\text{(a)} \quad (3+i)(2-i)^2 - i \qquad \text{(b)} \quad \frac{i^3}{2+i} \qquad \text{(c)} \quad \frac{4-3i^2+2i}{(2-2i^3)^2}$$

Solution

(a) $(3+i)(2-i)^2 - i = (3+i)(3-4i) - i = (13-9i) - i = 13-10i$

(b) $\dfrac{i^3}{2+i} = \dfrac{-i(2-i)}{(2+i)(2-i)} = \dfrac{-1-2i}{5} = -\dfrac{1}{5} - \dfrac{2}{5}i$

(c) $\dfrac{4-3i^2+2i}{(2-2i^3)^2} = \dfrac{4+3+2i}{(2+2i)^2} = \dfrac{7+2i}{8i} = \dfrac{(7+2i)(-i)}{(8i)(-i)} = \dfrac{2-7i}{8} = \dfrac{1}{4} - \dfrac{7}{8}i$ ●

Notice in part (c) of this example that we multiplied numerator and denominator by $-i$ rather than $-8i$; the result is the same in either case. Both lead to a real denominator.

The **complex conjugate** $\overline{z}$ of a complex number $z = x+yi$ is

$$\overline{z} = x - yi. \tag{5}$$

Geometrically, $\overline{z}$ is the reflection of z in the real axis (Figure 3).

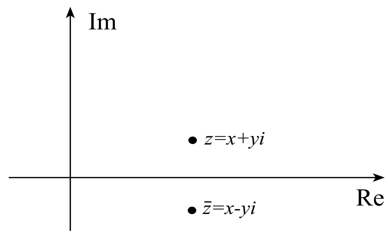

Figure 3

If we multiply a complex number $z = x + yi$ by its complex conjugate, we obtain

$$z\overline{z} = (x + yi)(x - yi) = x^2 + y^2. \tag{6}$$

This real number represents the square of the length of the line segment joining the numbers $z = 0$ and $z = x + yi$ in the complex plane. We use this property in our procedure to divide complex numbers. To divide z_1 by z_2, multiply z_1/z_2 by $\overline{z_2}/\overline{z_2}$,

$$\frac{z_1}{z_2} = \frac{z_1}{z_2}\frac{\overline{z_2}}{\overline{z_2}} = \frac{z_1\overline{z_2}}{z_2\overline{z_2}}.$$

The denominator will be real, and the Cartesian form is immediate.

With complex numbers in place, we can give a complete discussion of quadratic equations. When the discriminant of a quadratic equation is positive, the equation has two real solutions. For example, the discriminant of

$$x^2 + 4x - 2 = 0$$

is $16 + 8 = 24$, and solutions of the equation are

$$x = \frac{-4 \pm \sqrt{16 + 8}}{2} = -2 \pm \sqrt{6}.$$

When the discriminant is zero, we regard the quadratic as having two real solutions which are identical. For example, the discriminant of

$$x^2 + 4x + 4 = 0$$

is zero. The left side may be factored in the form

$$(x + 2)^2 = 0.$$

We say that -2 is a double root of the equation.

For quadratics with negative discriminants, we first consider the equation

$$x^2 + 1 = 0.$$

The complex number i is a solution, but so also is $-i$ since $(-i)^2 + 1 = -1 + 1 = 0$. The quadratic equation

$$x^2 + 16 = 0$$

has two solutions $x = \pm 4i$. If we apply the quadratic equation formula to the equation

$$x^2 + 2x + 5 = 0,$$

the result is

$$x = \frac{-2 \pm \sqrt{4 - 20}}{2} = \frac{-2 \pm \sqrt{-16}}{2}.$$

By $\sqrt{-16}$ we would seem to mean the number that multiplied by itself is -16. But there are two such numbers, namely $\pm 4i$. Let us make the agreement that $\sqrt{-16}$ shall denote that complex number whose square is -16, and which has a positive imaginary part. By this agreement,

$$\sqrt{-16} = 4i \qquad \text{and} \qquad -\sqrt{-16} = -4i.$$

The quadratic formula applied to $x^2 + 2x + 5 = 0$ therefore gives two complex numbers

$$x = \frac{-2 \pm 4i}{2} = -1 \pm 2i.$$

It is straightforward to verify that these two complex conjugates actually satisfy $x^2+2x+5 = 0$.

The agreement made in this last example is worth reiterating as a general principle: When $a > 0$ (is a real number),

$$\sqrt{-a} = \sqrt{a}\, i. \tag{7}$$

We call $\sqrt{a}\, i$ the **principal square root** of $-a$; the other square root is $-\sqrt{a}\, i$.

The above examples lead us to state that every real quadratic equation

$$ax^2 + bx + c = 0 \tag{8a}$$

has two solutions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{8b}$$

When $b^2 - 4ac > 0$, roots are real and distinct; when $b^2 - 4ac = 0$, roots are real and equal; and when $b^2 - 4ac < 0$, roots are complex conjugates. Verification of this is a matter of substituting (2.8b) into (2.8a).

Example 2 Find all solutions of the following equations:

(a) $x^2 + x + 3 = 0$ (b) $x^2 - 6x + 9 = 0$ (c) $2x^2 + 17x - 2 = 0$ (d) $x^4 + 5x^2 + 4 = 0$

Solution (a) By quadratic equation formula (2.8b),

$$x = \frac{-1 \pm \sqrt{1 - 12}}{2} = \frac{-1 \pm \sqrt{-11}}{2} = -\frac{1}{2} \pm \frac{\sqrt{11}}{2} i.$$

(b) This quadratic can be factored, $0 = x^2 - 6x + 9 = (x - 3)^2$, and therefore has a double root $x = 3$.

(c) By (2.8b),

$$x = \frac{-17 \pm \sqrt{289 + 16}}{4} = \frac{-17 \pm \sqrt{305}}{4},$$

two real roots.

(d) If we set $y = x^2$, then

$$0 = x^4 + 5x^2 + 4 = y^2 + 5y + 4 = (y + 4)(y + 1).$$

Consequently, y is equal to -1 or -4. Since $y = x^2$, we set $x^2 = -1$ and $x^2 = -4$, These equations have roots $x = \pm i$ and $x = \pm 2i$.●

EXERCISES

1. Show each of the following complex numbers in the complex plane: $2-i$, $3+4i$, $-1-5i$, $-3+2i$, $5i$, $2(1+i)$

In Exercises 2–26 write the complex expression in Cartesian form.

2. $(2+4i)-(3-2i)$

3. $(1+2i)^2$

4. $(-2+i)(3-4i)$

5. $3i(4i-1)^2$

6. i^3-3i^2+2i+4

7. $(1+i)^6$

8. $\dfrac{1-i}{3+2i}$

9. $\dfrac{(3+i)^2}{2-i}$

10. $i^{24}-3i^{13}+4$

11. $(i-2)[(2+i)(1-i)+3i-2]$

12. $6i\left(\dfrac{1+i}{2-i}\right)+3\left(\dfrac{i-4}{2i+1}\right)$

13. $\overline{2+i}-(3+\overline{4i})$

14. $\overline{1+i}^{\,2}+\overline{(1+i)^2}$

15. $\left(\dfrac{1}{2}-\dfrac{\sqrt{3}}{2}i\right)^3$

16. $(1+\overline{2-i})^2$

17. $\dfrac{(2i+3)(4-i)}{(3+i)(-6+2i)}$

18. $\dfrac{(1+i)^2(2-i)}{(3+2i)^2}$

19. $(1-i)^{12}(2i+3)$

$*$**20.** $\overline{(4-i)^2}$

$*$**21.** $\dfrac{1}{2i}\left(\dfrac{1}{\sqrt{2}}+\dfrac{i}{\sqrt{2}}\right)^{18}$

$*$**22.** $\dfrac{1}{1+\dfrac{1}{1+2i}}$

$*$**23.** $\dfrac{\dfrac{i}{3+i}}{(2-i)^2}$

$*$**24.** $\dfrac{(1+i^3)^2(2-i)}{\overline{4-5i}}$

$*$**25.** $\left(\dfrac{1}{\sqrt{2}}+\dfrac{i}{\sqrt{2}}\right)^4\left(\dfrac{1}{\sqrt{2}}-\dfrac{i}{\sqrt{2}}\right)^4$

$*$**26.** $\left(\dfrac{2i}{1+i}\,\dfrac{3-4i}{3+4i}\right)^2$

In Exercises 27–36 find all solutions of the equation.

27. $x^2+5x+3=0$

28. $x^2+3x+5=0$

29. $x^2+8x+16=0$

30. $x^2+2x-7=0$

31. $x^2+2x+7=0$

32. $4x^2-2x+5=0$

$*$**33.** $\sqrt{3}x^2+5x+\sqrt{15}=0$

$*$**34.** $x^4+4x^2-5=0$

$*$**35.** $x^4+4x^2+3=0$

$*$**36.** $x^4+6x^2+3=0$

$*$**37.** Verify the following properties for the complex conjugation operation:

(a) $\overline{z_1+z_2}=\overline{z_1}+\overline{z_2}$

(b) $\overline{z_1-z_2}=\overline{z_1}-\overline{z_2}$

(c) $\overline{z_1z_2}=\overline{z_1}\,\overline{z_2}$

(d) $\overline{\left(\dfrac{z_1}{z_2}\right)}=\dfrac{\overline{z_1}}{\overline{z_2}}$

(e) $\overline{z^n}=\overline{z}^n$, n a positive integer. Hint: Try mathematical induction.

$*$**38.** Verify that all complex numbers z satisfying the equation $z\overline{z}=r^2$, $r>0$ a real constant, lie on a circle. What is its centre and radius?

$*$**39.** Prove that if $z_1 z_2 = 0$, then at least one of z_1 and z_2 must be zero.

$*$**40.** We have made the agreement that when $a > 0$ is a real number, $\sqrt{-a}$ denotes a complex number with positive imaginary part. Show that with this agreement, $\sqrt{z_1 z_2}$ is not always equal to $\sqrt{z_1}\,\sqrt{z_2}$.

$*$**41.** Explain the fallacy in

$$-1 = \sqrt{-1}\,\sqrt{-1} = \sqrt{(-1)(-1)} = \sqrt{1} = 1.$$

$*$**42.** Find two numbers whose sum is 6 and whose product is 10.

$*$**43.** Verify that the values of x in (2.8b) satisfy (2.8a).

$*$**44.** To find the square roots of a complex number, say i, we could set $(x + yi)^2 = i$, and solve the equation for x and y. Do this by using 2 for equality of complex numbers.

$*$**45.** Use the technique of Exercise 44 to find square roots for (a) $-7 - 24i$ (b) $2 + i$.

Answers

2. $-1 + 6i$ **3.** $-3 + 4i$ **4.** $-2 + 11i$ **5.** $24 - 45i$ **6.** $7 + i$ **7.** $-8i$
8. $1/13 - (5/13)i$ **9.** $2 + 4i$ **10.** $5 - 3i$ **11.** $-4 - 3i$ **12.** $-24/5 + (33/5)i$
13. $-1 + 3i$ **14.** $-4i$ **15.** -1 **16.** $8 + 6i$ **17.** $-7/10 - (1/4)i$
18. $58/169 - (4/169)i$ **19.** $-192 - 128i$ **20.** $15 - 8i$ **21.** $1/2$ **22.** $3/4 + (1/4)i$
23. $-9/250 + (13/250)i$ **24.** $-28/41 - (6/41)i$ **25.** 1 **26.** $-672/625 - (1054/625)i$
27. $(-5 \pm \sqrt{13})/2$ **28.** $(-3/2) \pm (\sqrt{11}/2)i$ **29.** -4 double root **30.** $-1 \pm 2\sqrt{2}$
31. $-1 \pm \sqrt{6}i$ **32.** $(1/4) \pm (\sqrt{19}/4)i$ **33.** $(-5/(2\sqrt{3})) \pm [\sqrt{12\sqrt{5} - 25}/(2\sqrt{3})]i$
34. $\pm 1, \pm\sqrt{5}i$ **35.** $\pm i, \pm\sqrt{3}i$ **36.** $\pm\sqrt{3 + \sqrt{6}}i, \pm\sqrt{3 - \sqrt{6}}i$ **38.** $z = 0, r$
42. $3 \pm i$ **44.** $\pm[(1/\sqrt{2}) + (1/\sqrt{2})i]$
45. (a) $\pm(3 - 4i)$ (b) $\pm[(\sqrt{\sqrt{5} + 2}/\sqrt{2}) + (\sqrt{\sqrt{5} - 2}/\sqrt{2})i]$