Supporting Information for "A Time-Heterogeneous D-Vine Copula Model for Unbalanced and Unequally Spaced Longitudinal Data"

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This supplement consists of three appendices. Web Appendix A contains the asymptotic results of the IFM and SEQ estimators, the Web Figure and Web Tables cited in Sections 2 and 3 of the main text, and an illustration of model fitting for the D-vine copula with $d = 5$. Web Appendix B gives additional simulation results. Web Appendix C presents some of the results from the analysis of MFUS data.

Appendix A

Asymptotic results

We present the asymptotic behaviour of the IFM and sequential estimators (SEQ) for the time-heterogeneous D-vine copula model focusing on the balanced data. The log-likelihood function in the balanced case is given by

$$\begin{aligned}\mathcal{L}(\boldsymbol{\alpha},\boldsymbol{\beta}|\mathbf{Y},\boldsymbol{x}) &= \sum_{i=1}^{n}\sum_{l=1}^{d}\log(f_l(y_{i,l};\boldsymbol{\alpha}_l)) \\ &\quad + \sum_{i=1}^{n}\sum_{k=1}^{d-1}\sum_{j=1}^{d-k}\log c_{j(j+k);S_{jk}}\Big(F_{j|S_{jk}}(y_{i,j}|\mathbf{y}_{i,S_{jk}};\boldsymbol{\alpha}_{j:j+k-1},\boldsymbol{\beta}_{j,j+k-1}), \\ &\qquad\qquad F_{j+k|S_{jk}}(y_{i,j+k}|\mathbf{y}_{i,S_{jk}};\boldsymbol{\alpha}_{j+1:j+k},\boldsymbol{\beta}_{j+1,j+k});\boldsymbol{\beta}_{j(j+k)}\Big) \\ &= \mathcal{L}^M(\boldsymbol{\alpha}|\mathbf{Y},\boldsymbol{x}) + \mathcal{L}^C(\boldsymbol{\alpha},\boldsymbol{\beta}|\mathbf{Y},\boldsymbol{x}).\end{aligned}$$

It is well known that, under standard regularity conditions (Lehmann, 2004), the maximum likelihood estimator $\hat{\boldsymbol{\beta}}^{\text{ML}}$ obtained by maximizing $\mathcal{L}(\boldsymbol{\alpha},\boldsymbol{\beta}|\mathbf{Y},\boldsymbol{x})$ jointly for $(\boldsymbol{\alpha},\boldsymbol{\beta})$ is consistent and follows an asymptotic normal distribution

$$\sqrt{n}(\hat{\boldsymbol{\beta}}^{\mathrm{ML}} - \boldsymbol{\beta}) \to N\left(0, \mathrm{V}\big(\hat{\boldsymbol{\beta}}^{ML}\big)\right),$$

as $n \to \infty$, where

$$\mathrm{V}\big(\hat{\boldsymbol{\beta}}^{ML}\big) = I_{\boldsymbol{\beta}}^{-1} + I_{\boldsymbol{\beta}}^{-1} I_{\boldsymbol{\alpha},\boldsymbol{\beta}}^{\top} \left[I_{\boldsymbol{\alpha}} - I_{\boldsymbol{\alpha},\boldsymbol{\beta}} I_{\boldsymbol{\beta}}^{-1} I_{\boldsymbol{\alpha},\boldsymbol{\beta}}^{\top}\right]^{-1} I_{\boldsymbol{\alpha},\boldsymbol{\beta}} I_{\boldsymbol{\beta}}^{-1}.$$

Here, $I = -\mathrm{E}\left[\left(\frac{\partial^2 \mathcal{L}(\boldsymbol{\alpha}, \boldsymbol{\beta} \mid \mathbf{Y}, \boldsymbol{x})}{\partial(\boldsymbol{\alpha}, \boldsymbol{\beta})\partial(\boldsymbol{\alpha}, \boldsymbol{\beta})^{\top}}\right)\right]$ is the Fisher information matrix, which can be decomposed into components associated with marginal and dependence parameters as $I = \begin{pmatrix} I_{\boldsymbol{\alpha}} & I_{\boldsymbol{\alpha},\boldsymbol{\beta}} \\ I_{\boldsymbol{\beta},\boldsymbol{\alpha}} & I_{\boldsymbol{\beta}} \end{pmatrix}$.

The IFM and sequential estimators primarily affect the terms involving the block diagonal matrices $I_{\boldsymbol{\alpha}}$ and $I_{\boldsymbol{\beta}}$ in the asymptotic variance expression. Before introducing the main result, we define the following. Let $\boldsymbol{J}_{\boldsymbol{\alpha}}$ and $\boldsymbol{K}_{\boldsymbol{\alpha}}$ be components involving marginal parameters with entries

$$\boldsymbol{J}_{\boldsymbol{\alpha},j,j} = -E\left[\left(\frac{\partial^2 \mathcal{L}_j^M(\boldsymbol{\alpha}|\mathbf{Y}, \boldsymbol{x})}{\partial \boldsymbol{\alpha}_j \partial \boldsymbol{\alpha}_j^{\top}}\right)\right] \quad \text{and} \quad \boldsymbol{K}_{\boldsymbol{\alpha},j,k} = E\left[\left(\frac{\partial \mathcal{L}_j^M(\boldsymbol{\alpha}|\mathbf{Y}, \boldsymbol{x})}{\partial \boldsymbol{\alpha}_j}\right)\left(\frac{\partial \mathcal{L}_k^M(\boldsymbol{\alpha}|\mathbf{Y}, \boldsymbol{x})}{\partial \boldsymbol{\alpha}_k}\right)^{\top}\right],$$

for $j, k = 1, \dots, d$. Similarly, define

$$\boldsymbol{J}_{\boldsymbol{\beta}} = \begin{pmatrix} \boldsymbol{J}_{\boldsymbol{\beta},1,1} & \cdots & \cdots & \mathbf{0} \\ \vdots & \ddots & & \vdots \\ \boldsymbol{J}_{\boldsymbol{\beta},d-2,1} & \cdots & \boldsymbol{J}_{\boldsymbol{\beta},d-2,d-2} & \mathbf{0} \\ \boldsymbol{I}_{\boldsymbol{\beta},d-1,1} & \cdots & \boldsymbol{I}_{\boldsymbol{\beta},d-1,d-2} & \boldsymbol{I}_{\boldsymbol{\beta},d-1,d-1} \end{pmatrix},$$

with $\boldsymbol{J}_{\boldsymbol{\beta},j,k} = -E\left[\left(\frac{\partial^2 \mathcal{L}_j^C(\boldsymbol{\alpha},\boldsymbol{\beta}|\mathbf{Y},\boldsymbol{x})}{\partial\boldsymbol{\beta}_j \partial\boldsymbol{\beta}_k^\top}\right)\right]$, and

$$\boldsymbol{K}_{\boldsymbol{\beta}} = \begin{pmatrix} \boldsymbol{K}_{\boldsymbol{\beta},1,1} & \dots & \dots & \mathbf{0} \\ \vdots & \ddots & & \vdots \\ \mathbf{0}^\top & \dots & \boldsymbol{K}_{\boldsymbol{\beta},d-2,d-2} & \mathbf{0} \\ \mathbf{0}^\top & \dots & \mathbf{0}^\top & \boldsymbol{I}_{\boldsymbol{\beta},d-1,d-1} \end{pmatrix},$$

with $\boldsymbol{K}_{\boldsymbol{\beta},j,k} = E\left[\left(\frac{\partial \mathcal{L}_j^C(\boldsymbol{\alpha},\boldsymbol{\beta}|\mathbf{Y},\boldsymbol{x})}{\partial\boldsymbol{\beta}_j}\right)\left(\frac{\partial \mathcal{L}_k^C(\boldsymbol{\alpha},\boldsymbol{\beta}|\mathbf{Y},\boldsymbol{x}}{\partial\boldsymbol{\beta}_k}\right)^\top\right]$, for $j,k = 1,\dots,d-1$. Here, $\mathcal{L}_j^C(\boldsymbol{\alpha},\boldsymbol{\beta}|\mathbf{Y},\boldsymbol{x})$ implicitly depends on all the parameters associated with the previous trees but not the proceeding ones. Moreover, the full vine structure is used to estimate the copula parameters for the top level, which accounts for the appearance of blocks from the Fisher's information matrix $\boldsymbol{I}_{\boldsymbol{\beta}}$ in the last row of $\boldsymbol{J}_{\boldsymbol{\beta}}$ and $\boldsymbol{K}_{\boldsymbol{\beta}}$.

Theorem 1. *The estimators of dependence parameters have the following asymptotic properties:*

(i) Under conditions (M1)–(M8) of Lehmann (2004, pages 499-501), the IFM estimator $\hat{\boldsymbol{\beta}}^{IFM}$ *is consistent and follows the asymptotic normal distribution*

$$\sqrt{n}(\hat{\boldsymbol{\beta}}^{IFM} - \boldsymbol{\beta}) \to N\left(0,\, V(\hat{\boldsymbol{\beta}}^{IFM})\right),$$

as $n \to \infty$*, where*

$$V(\hat{\boldsymbol{\beta}}^{IFM}) = I_{\boldsymbol{\beta}}^{-1} + I_{\boldsymbol{\beta}}^{-1} I_{\boldsymbol{\alpha},\boldsymbol{\beta}}^\top \boldsymbol{J}_{\boldsymbol{\alpha}}^{-1} \boldsymbol{K}_{\boldsymbol{\alpha}} \boldsymbol{J}_{\boldsymbol{\alpha}}^{-1} I_{\boldsymbol{\alpha},\boldsymbol{\beta}} I_{\boldsymbol{\beta}}^{-1}.$$

(ii) Under conditions of Theorem 1 of Hobæk Haff (2013), the sequential estimator $\hat{\boldsymbol{\beta}}^{SEQ}$ *is consistent and follows the asymptotic normal distribution*

$$\sqrt{n}(\hat{\boldsymbol{\beta}}^{SEQ} - \boldsymbol{\beta}) \to N\left(0,\, V(\hat{\boldsymbol{\beta}}^{SEQ})\right),$$

as $n \to \infty$*, where*

$$V(\hat{\boldsymbol{\beta}}^{SEQ}) = \boldsymbol{J}_{\boldsymbol{\beta}}^{-1}\boldsymbol{K}_{\boldsymbol{\beta}}\big(\boldsymbol{J}_{\boldsymbol{\beta}}^{-1}\big)^{\top} + \boldsymbol{J}_{\boldsymbol{\beta}}^{-1}\boldsymbol{K}_{\boldsymbol{\beta}}\big(\boldsymbol{J}_{\boldsymbol{\beta}}^{-1}\big)^{\top} I_{\boldsymbol{\alpha},\boldsymbol{\beta}}^{\top}\boldsymbol{J}_{\boldsymbol{\alpha}}^{-1}\boldsymbol{K}_{\boldsymbol{\alpha}}\boldsymbol{J}_{\boldsymbol{\alpha}}^{-1} I_{\boldsymbol{\alpha},\boldsymbol{\beta}}\boldsymbol{J}_{\boldsymbol{\beta}}^{-1}\boldsymbol{K}_{\boldsymbol{\beta}}\big(\boldsymbol{J}_{\boldsymbol{\beta}}^{-1}\big)^{\top}.$$

Proof. The result in part *(i)* follows directly from Joe (2005). Here, $\boldsymbol{J}_{\boldsymbol{\alpha}}^{-1}\boldsymbol{K}_{\boldsymbol{\alpha}}\boldsymbol{J}_{\boldsymbol{\alpha}}^{-1}$ is the asymptotic covariance matrix of $\hat{\boldsymbol{\alpha}}^{IFM}$ which quantifies the loss of asymptotic efficiency of the IFM estimator $\hat{\boldsymbol{\beta}}^{IFM}$ due to the estimation of margins in the first step.

To prove the result in part *(ii)*, we can decompose the covariance matrix for the sequential estimator $\hat{\boldsymbol{\beta}}^{SEQ}$ into two parts as $\mathrm{V}(\hat{\boldsymbol{\beta}}^{SEQ}) = \mathrm{V}^{SEQ}_{\text{dependence}} + \mathrm{V}^{SEQ}_{\text{margins}}$. Following Haff (2013) Hobæk Haff (2013), it can be shown that $\mathrm{V}^{SEQ}_{\text{dependence}} = \boldsymbol{J}_{\boldsymbol{\beta}}^{-1}\boldsymbol{K}_{\boldsymbol{\beta}}\big(\boldsymbol{J}_{\boldsymbol{\beta}}^{-1}\big)^{\top}$. This matrix quantifies the asymptotic efficiency loss due to sequential estimation. Hence, the asymptotic variance of $\hat{\boldsymbol{\beta}}^{SEQ}$ is obtained by replacing $I_{\boldsymbol{\beta}}^{-1}$ in Equation (9) with $\boldsymbol{J}_{\boldsymbol{\beta}}^{-1}\boldsymbol{K}_{\boldsymbol{\beta}}\big(\boldsymbol{J}_{\boldsymbol{\beta}}^{-1}\big)^{\top}$.

Remark. *The asymptotic results can be extended to the unbalanced case by assuming that the proportion of available data for different pair copula components remains constant in the limit, i.e.,* $\lim_{n\to\infty} n_k/n = \lambda_k$*. However, the expressions for variance matrices in terms of these quantities is extremely complex, hence a complete derivation is not pursued here.*

Web Table 1: *Parametric families of bivariate copulas and the corresponding Kendall's τ in terms of θ.*

Family	Kendall's τ
Gaussian, Student-t	$2\arcsin(\theta)/\pi, \quad -1 \leq \theta \leq 1$
Clayton	$\theta/(\theta+2), \quad \theta > 0$
Clayton 90°	$-\theta/(\theta+2), \quad \theta > 0$
Clayton 180° (Survival Clayton)	$\theta/(\theta+2), \quad \theta > 0$
Clayton 270°	$-\theta/(\theta+2), \quad \theta > 0$
Gumbel	$1-1/\theta, \quad \theta > 1$
Gumbel 90°	$-1-1/\theta, \quad \theta < -1$
Gumbel 180° (Survival Gumbel)	$1-1/\theta, \quad \theta > 1$
Gumbel 270°	$-1-1/\theta, \quad \theta < -1$
Frank	$1-4/\theta+4/\theta^2\int_\theta^0 \frac{x}{e^x-1}dx, \quad \theta > 0$ $1-4/\theta-4/\theta^2\int_0^\theta \frac{x}{e^x-1}dx, \quad \theta < 0$

Web Table 2: *An example dataset of size $n = 12$ with $d = 5$ where * indicates the observed measurement.*

Group		Time	Measurements 1	2	3	4	5
$\mathcal{Y}^1 = \{\mathbf{y}_i\,;\, i \in I_1\}$	$\mathbf{y}_1$	$\mathbf{t}_1$	*				
	$\mathbf{y}_2$	$\mathbf{t}_2$	*				
$\mathcal{Y}^2 = \{\mathbf{y}_i\,;\, i \in I_2\}$	$\mathbf{y}_3$	$\mathbf{t}_3$	*	*			
	$\mathbf{y}_4$	$\mathbf{t}_4$	*	*			
	$\mathbf{y}_5$	$\mathbf{t}_5$	*	*			
$\mathcal{Y}^3 = \{\mathbf{y}_i\,;\, i \in I_3\}$	$\mathbf{y}_6$	$\mathbf{t}_6$	*	*	*		
	$\mathbf{y}_7$	$\mathbf{t}_7$	*	*	*		
$\mathcal{Y}^4 = \{\mathbf{y}_i\,;\, i \in I_4\}$	$\mathbf{y}_8$	$\mathbf{t}_8$	*	*	*	*	
	$\mathbf{y}_9$	$\mathbf{t}_9$	*	*	*	*	
$\mathcal{Y}^5 = \{\mathbf{y}_i\,;\, i \in I_5\}$	$\mathbf{y}_{10}$	$\mathbf{t}_{10}$	*	*	*	*	*
	$\mathbf{y}_{11}$	$\mathbf{t}_{11}$	*	*	*	*	*
	$\mathbf{y}_{12}$	$\mathbf{t}_{12}$	*	*	*	*	*

Example: D-vine Copula with $d = 5$

To illustrate the fitting of the proposed D-vine copula model, we consider an example with at most $d = 5$ repeated measurements. We order the pseudo-copula data $\mathbf{U} = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}^\top$ of size $n = n_1 + n_2 + n_3 + n_4 + n_5$ as described in Web Table 2 and partition it into groups $\boldsymbol{\mathcal{U}}^g, g = 1, 2, 3, 4, 5$. The D-vine copula model consists of pair copulas $c = (c_{12}, c_{23}, c_{13;2}, c_{34}, c_{24;3}, c_{14;23}, c_{45}, c_{35;4}, c_{25;34}, c_{15;234})$ with the corresponding vector of parameters $\boldsymbol{\beta} = (\boldsymbol{\beta}_{12}, \boldsymbol{\beta}_{23}, \boldsymbol{\beta}_{13;2}, \boldsymbol{\beta}_{34}, \boldsymbol{\beta}_{24;3}, \boldsymbol{\beta}_{14;23}, \boldsymbol{\beta}_{45}, \boldsymbol{\beta}_{35;4}, \boldsymbol{\beta}_{25;34}, \boldsymbol{\beta}_{15;234})$. Using the vine decomposition in Equation (4), we write the copula log-likelihood function as

$$
\begin{aligned}
\mathcal{L}(\boldsymbol{\beta}|\mathbf{U}, \boldsymbol{x}) = &\sum_{i \in I_2 \cup I_3 \cup I_4 \cup I_5} \log c_{12}\Big(u_{i,1}, u_{i,2}; \theta_{12}(x_{i,12}; \boldsymbol{\beta}_{12})\Big) \\
&+ \sum_{i \in I_3 \cup I_4 \cup I_5} \Big[\log c_{23}\big(u_{i,2}, u_{i,3}; \theta_{23}(x_{i,23}; \boldsymbol{\beta}_{23})\big) + \log c_{13;2}\Big(u_{i,1|2}, u_{i,3|2}; \theta_{13;2}(x_{i,13}; \boldsymbol{\beta}_{13})\Big)\Big] \\
&+ \sum_{i \in I_4 \cup I_5} \Big[\log c_{34}\Big(u_{i,3}, u_{i,4}; \theta_{34}(x_{i,34}; \boldsymbol{\beta}_{34})\Big) + \log c_{24;3}\Big(u_{i,2|3}, u_{i,4|3}; \theta_{24;3}(x_{i,24}; \boldsymbol{\beta}_{24})\Big) \\
&\qquad + \log c_{14;23}\Big(u_{i,1|23}, u_{i,4|23}; \theta_{14;23}(x_{i,14}; \boldsymbol{\beta}_{14})\Big)\Big] \\
&+ \sum_{i \in I_5} \Big[\log c_{45}\Big(u_{i,4}, u_{i,5}; \theta_{45}(x_{i,45}; \boldsymbol{\beta}_{45})\Big) + \log c_{35;4}\Big(u_{i,3|4}, u_{i,5|4}; \theta_{35;4}(x_{i,35}; \boldsymbol{\beta}_{35})\Big) \\
&\qquad + \log c_{25;34}\Big(u_{i,2|34}, u_{i,5|34}; \theta_{25;34}(x_{i,25}; \boldsymbol{\beta}_{25})\Big) \\
&\qquad + \log c_{15;234}\Big(u_{i,1|234}, u_{i,5|234}; \theta_{15;234}(x_{i,15}; \boldsymbol{\beta}_{15})\Big)\Big].
\end{aligned}
$$

Appendix B: Additional Simulation Results

Web Table 3: *The proportion of times out of* $M = 1000$ *replications that each candidate model has been selected by the AIC under the six DGPs with sample sizes* $n = 250$ *and* 500.

DGP	n	HOM–P	HOM–T	HOM–V	HET–P	HET–T	HET–V
DGP1	250	**1.000**	0.000	0.000	0.000	0.000	0.000
	500	**0.999**	0.000	0.000	0.001	0.000	0.000
DGP2	250	0.023	**0.970**	0.000	0.000	0.002	0.000
	500	0.020	**0.980**	0.000	0.000	0.000	0.000
DGP3	250	0.005	0.066	**0.845**	0.000	0.001	0.083
	500	0.003	0.065	**0.858**	0.000	0.002	0.072
DGP4	250	0.001	0.000	0.000	**0.999**	0.000	0.000
	500	0.021	0.000	0.000	**0.979**	0.000	0.000
DGP5	250	0.000	0.000	0.000	0.002	**0.998**	0.000
	500	0.000	0.000	0.000	0.005	**0.995**	0.001
DGP6	250	0.000	0.000	0.000	0.000	0.007	**0.993**
	500	0.000	0.000	0.000	0.000	0.012	**0.988**

Web Table 4: *Mean absolute differences* (×100) *of Kendall's* τ *for each of the* 10 *pair copulas, averaged over* $M = 1000$ *samples of size* $n = 250$ *and* 500 *under the different HOM DGPs in the balanced case.*

Scenario	DGP	Model	τ_{12}	τ_{23}	τ_{34}	τ_{45}	$\tau_{13;2}$	$\tau_{24;3}$	$\tau_{35;4}$	$\tau_{14;23}$	$\tau_{25;34}$	$\tau_{15;234}$
HOM	**DGP1**											
	n=250	HOM–P	**1.61**	**0.93**	**0.09**	**0.17**	**0.02**	2.83	**0.88**	**2.55**	**0.11**	3.50
		HOM–T	8.70	8.30	4.30	2.70	0.57	**1.43**	7.43	6.11	7.11	**1.17**
		HOM–V	20.60	37.60	33.60	26.60	4.40	2.40	3.60	15.40	14.40	34.40
		HET–P	1.83	0.99	0.47	0.61	0.86	3.10	1.46	2.95	1.03	3.30
		HET–T	1.83	0.99	0.47	0.61	0.86	3.10	1.46	2.95	1.03	3.30
		HET–V	8.47	22.85	18.99	12.72	6.51	5.15	4.48	8.07	7.16	20.22
	n=500	HOM–P	**0.77**	**0.10**	**0.21**	**0.13**	1.20	1.65	**0.41**	**1.21**	2.58	**1.12**
		HOM–T	8.64	8.36	4.36	2.64	**0.61**	1.39	7.39	4.66	5.66	5.33
		HOM–V	22.09	39.09	35.09	28.09	2.91	**0.91**	5.09	13.91	12.91	32.91
		HET–P	0.97	0.31	0.45	0.43	1.48	1.89	0.87	1.40	**2.57**	1.38
		HET–T	0.97	0.31	0.45	0.43	1.48	1.89	0.87	1.40	**2.57**	1.38
		HET–V	8.68	22.96	19.12	12.87	6.08	4.94	4.92	7.17	6.32	19.10
	DGP2											
	n=250	HOM–P	0.83	0.82	0.87	0.84	2.09	2.02	2.12	2.86	2.89	3.15
		HOM–T	**0.55**	**0.55**	**0.55**	**0.55**	1.20	1.20	1.20	**1.95**	**1.95**	**3.08**
		HOM–V	29.16	29.16	29.16	29.16	**0.89**	**0.89**	**0.89**	20.84	20.84	35.84
		HET–P	1.00	0.99	1.03	0.98	2.32	2.25	2.31	3.22	3.18	3.35
		HET–T	0.68	0.68	0.68	0.68	1.72	1.72	1.72	2.78	2.78	3.31
		HET–V	17.23	17.25	17.24	17.23	3.85	3.85	3.85	15.03	15.03	24.01
	n=500	HOM–P	0.59	0.57	0.57	0.60	1.43	1.44	1.46	2.17	2.12	2.24
		HOM–T	**0.37**	**0.37**	**0.37**	**0.37**	0.81	0.81	0.81	**1.43**	**1.43**	**2.21**
		HOM–V	29.18	29.18	29.18	29.18	**0.83**	**0.83**	**0.83**	20.82	20.82	35.82
		HET–P	0.71	0.69	0.69	0.71	1.62	1.62	1.63	2.44	2.39	2.45
		HET–T	0.49	0.49	0.49	0.49	1.22	1.22	1.23	2.28	2.29	2.50
		HET–V	17.21	17.20	17.22	17.20	3.83	3.83	3.83	14.95	14.95	23.91
	DGP3											
	n=250	HOM–P	2.44	2.44	2.49	2.55	2.53	2.63	2.59	2.62	2.50	2.68
		HOM–T	1.57	1.57	1.57	1.57	1.38	1.38	1.38	1.69	1.69	2.62
		HOM–V	**0.65**	**0.65**	**0.65**	**0.65**	**0.65**	**0.65**	**0.65**	**0.65**	**0.65**	**0.65**
		HET–P	2.97	2.91	3.02	2.97	2.90	2.99	2.95	2.91	2.80	3.06
		HET–T	2.08	2.09	2.09	2.09	2.15	2.15	2.15	2.56	2.56	3.27
		HET–V	0.85	0.85	0.85	0.85	0.84	0.84	0.84	1.05	1.05	1.36
	n=500	HOM–P	1.85	1.79	1.75	1.80	1.90	1.87	1.90	1.97	1.90	1.92
		HOM–T	1.11	1.11	1.11	1.11	1.01	1.01	1.01	1.28	1.28	1.89
		HOM–V	**0.47**	**0.47**	**0.47**	**0.47**	**0.47**	**0.47**	**0.47**	**0.47**	**0.47**	**0.47**
		HET–P	2.15	2.15	2.08	2.10	2.21	2.16	2.20	2.24	2.15	2.20
		HET–T	1.55	1.55	1.55	1.55	1.70	1.70	1.70	2.12	2.12	2.56
		HET–V	0.65	0.65	0.65	0.65	0.69	0.69	0.69	0.90	0.90	1.16

Web Table 5: Mean absolute differences (×100) of Kendall's τ for each of the 10 pair copulas, averaged over $M = 1000$ samples of size $n = 250$ and 500 under different HOM DGPs in the unbalanced case.

Scenario	DGP	Model	τ_{12}	τ_{23}	τ_{34}	τ_{45}	$\tau_{13;2}$	$\tau_{24;3}$	$\tau_{35;4}$	$\tau_{14;23}$	$\tau_{25;34}$	$\tau_{15;234}$
HOM	**DGP1**											
	n=250	HOM–P	**1.61**	**0.93**	**0.85**	**1.35**	1.25	3.24	2.96	**3.35**	**3.35**	4.48
		HOM–T	8.16	8.84	4.84	2.16	**1.24**	**2.74**	8.72	6.78	7.78	**2.69**
		HOM–V	18.69	35.69	31.69	24.69	6.31	4.31	**1.71**	17.31	16.31	36.31
		HET–P	1.83	1.04	1.13	1.76	1.80	3.69	3.45	3.80	3.76	4.44
		HET–T	8.15	8.84	4.84	2.17	1.64	2.95	8.73	6.83	7.81	2.83
		HET–V	7.78	23.97	19.97	13.04	6.78	5.05	3.44	9.48	8.48	22.13
	n=500	HOM–P	**0.77**	**0.31**	**0.64**	**0.97**	1.36	**2.03**	**1.98**	**2.03**	**3.27**	**2.80**
		HOM–T	8.16	8.84	4.84	2.16	**1.06**	2.86	8.86	5.58	6.57	5.21
		HOM–V	19.92	36.92	32.92	25.92	5.08	3.08	2.92	16.08	15.08	35.08
		HET–P	0.97	0.48	0.87	1.24	1.72	2.40	2.42	2.30	3.44	2.80
		HET–T	8.15	8.84	4.85	2.16	1.33	2.95	8.86	5.64	6.61	5.14
		HET–V	8.00	23.98	20.00	13.10	6.12	4.60	3.83	8.37	7.37	20.83
	DGP2											
	n=250	HOM–P	0.83	0.92	1.12	1.37	2.28	2.62	3.14	3.79	4.61	4.82
		HOM–T	**0.62**	**0.62**	**0.62**	**0.62**	**1.49**	**1.49**	**1.49**	**2.77**	**2.77**	**4.72**
		HOM–V	27.20	27.20	27.20	27.20	2.80	2.80	2.80	22.80	22.80	37.80
		HET–P	1.00	1.11	1.33	1.56	2.54	2.95	3.45	4.16	4.90	5.07
		HET–T	0.69	0.69	0.69	0.69	1.69	1.68	1.68	3.06	3.07	4.86
		HET–V	18.32	18.35	18.34	18.34	4.13	4.12	4.12	17.75	17.74	27.60
	n=500	HOM–P	0.59	0.64	0.75	0.92	1.56	1.84	2.21	2.82	3.31	3.54
		HOM–T	**0.44**	**0.44**	**0.44**	**0.44**	**1.05**	**1.05**	**1.05**	**2.05**	**2.05**	**3.48**
		HOM–V	27.23	27.23	27.23	27.23	2.77	2.77	2.77	22.77	22.77	37.77
		HET–P	0.71	0.78	0.91	1.08	1.77	2.06	2.52	3.07	3.63	3.77
		HET–T	0.49	0.49	0.49	0.49	1.18	1.18	1.18	2.29	2.30	3.64
		HET–V	18.39	18.39	18.40	18.39	4.05	4.04	4.04	17.68	17.68	27.52
	DGP3											
	n=250	HOM–P	2.44	2.77	3.32	4.04	2.86	3.44	3.87	3.39	3.97	4.20
		HOM–T	1.80	1.80	1.80	1.80	1.80	1.80	1.80	2.49	2.49	4.02
		HOM–V	**0.85**	**0.85**	**0.85**	**0.85**	**0.85**	**0.85**	**0.85**	**0.85**	**0.85**	**0.85**
		HET–P	2.97	3.31	4.04	4.78	3.21	3.85	4.29	3.70	4.28	4.85
		HET–T	2.03	2.03	2.03	2.03	2.13	2.12	2.13	2.78	2.78	4.21
		HET–V	1.06	1.06	1.06	1.06	0.88	0.88	0.88	1.00	1.00	1.28
	n=500	HOM–P	1.85	1.98	2.31	2.74	2.08	2.39	2.84	2.50	2.92	2.93
		HOM–T	1.32	1.32	1.32	1.32	1.29	1.29	1.29	1.80	1.80	2.87
		HOM–V	**0.61**	**0.61**	**0.61**	**0.61**	**0.61**	**0.61**	**0.61**	**0.61**	**0.61**	**0.61**
		HET–P	2.15	2.39	2.78	3.18	2.42	2.74	3.25	2.78	3.22	3.28
		HET–T	1.44	1.44	1.44	1.44	1.54	1.54	1.54	2.05	2.05	3.06
		HET–V	0.76	0.76	0.76	0.76	0.63	0.63	0.63	0.71	0.71	0.91

Web Table 6: *Mean absolute differences* $(\times 100)$ *of Kendall's* τ *for each of the* 10 *pair copulas, averaged over* $M = 1000$ *samples of size* $n = 250$ *and* 500 *under the different HET DGPs in the balanced case.*

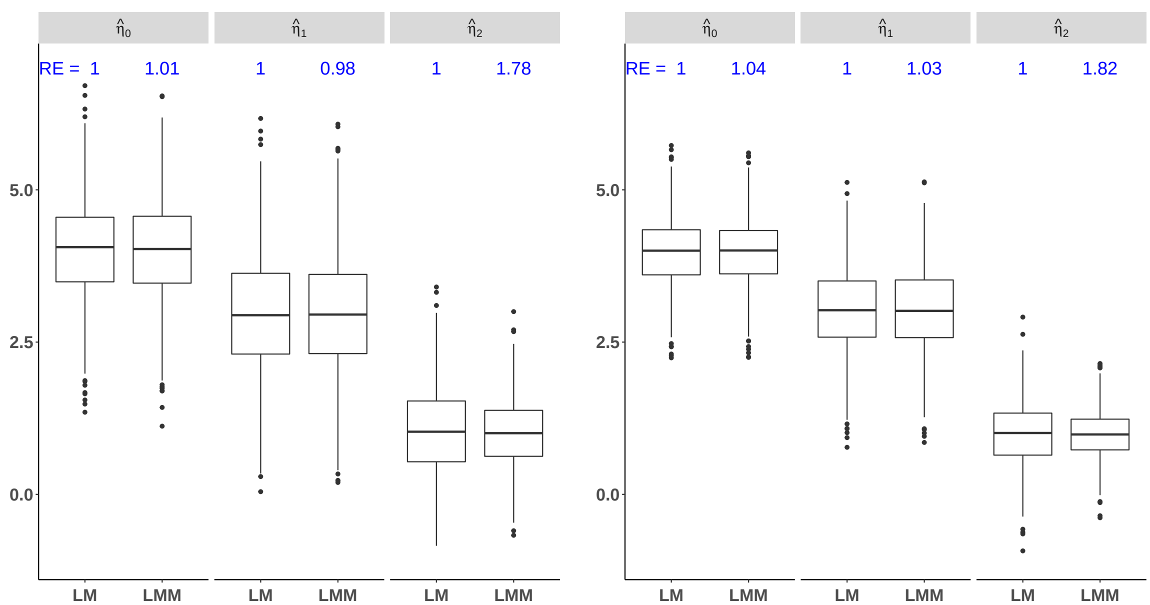
Scenario	DGP	Model	τ_{12}	τ_{23}	τ_{34}	τ_{45}	$\tau_{13;2}$	$\tau_{24;3}$	$\tau_{35;4}$	$\tau_{14;23}$	$\tau_{25;34}$	$\tau_{15;234}$
HET	**DGP4**											
	n=250	HOM–P	2.20	2.76	2.49	2.69	3.27	3.33	3.43	3.53	3.52	3.46
		HOM–T	7.80	5.86	3.00	2.69	2.71	5.28	4.23	5.91	**2.73**	4.51
		HOM–V	35.01	21.41	29.70	26.04	**1.93**	4.01	3.03	13.34	18.17	28.31
		HET–P	**1.51**	**2.00**	**1.89**	**1.78**	3.16	**2.58**	3.09	**3.22**	3.30	**3.20**
		HET–T	7.76	6.02	3.12	2.68	3.12	5.53	4.53	5.73	3.02	4.38
		HET–V	19.35	7.42	14.29	10.94	3.04	3.17	**2.92**	5.84	10.50	13.93
	n=500	HOM–P	2.06	2.54	2.28	2.52	2.63	3.01	2.92	2.87	2.85	2.60
		HOM–T	7.73	5.90	2.90	2.64	2.61	5.32	4.23	5.89	**2.24**	4.09
		HOM–V	34.99	21.41	29.69	26.04	**1.88**	3.98	3.00	13.35	18.18	28.32
		HET–P	**1.12**	**1.40**	**1.37**	**1.31**	2.32	**1.97**	**2.42**	**2.46**	2.56	**2.38**
		HET–T	7.69	6.05	2.99	2.60	2.94	5.48	4.46	5.57	2.56	3.92
		HET–V	19.33	7.38	14.27	10.91	3.04	3.16	2.92	5.78	10.47	13.87
	DGP5											
	n=250	HOM–P	3.15	3.15	3.17	3.17	3.46	3.35	3.48	3.88	4.00	4.11
		HOM–T	3.01	3.01	3.01	3.00	2.93	2.92	2.93	3.22	3.21	4.03
		HOM–V	28.46	28.44	28.44	28.45	2.45	2.44	2.44	17.70	17.71	31.23
		HET–P	1.61	1.62	1.67	1.65	2.60	2.50	2.63	3.35	3.36	**3.23**
		HET–T	**0.71**	**0.71**	**0.71**	**0.70**	**1.63**	**1.63**	**1.63**	**2.32**	**2.32**	3.64
		HET–V	14.00	14.01	14.00	14.00	2.50	2.51	2.51	10.26	10.26	17.09
	n=500	HOM–P	3.04	3.03	3.03	3.03	3.11	3.06	3.13	3.30	3.47	3.41
		HOM–T	2.95	2.96	2.96	2.95	2.83	2.82	2.82	2.95	2.95	3.39
		HOM–V	28.46	28.46	28.45	28.46	2.42	2.41	2.41	17.69	17.69	31.22
		HET–P	1.17	1.16	1.17	1.20	1.95	1.94	1.95	2.60	2.63	**2.39**
		HET–T	**0.67**	**0.67**	**0.67**	**0.66**	**1.49**	**1.48**	**1.48**	**2.15**	**2.15**	2.97
		HET–V	13.98	13.97	13.98	13.97	2.48	2.48	2.48	10.19	10.19	17.00
	DGP6											
	n=250	HOM–P	8.17	8.17	8.16	8.17	5.25	5.11	5.26	4.33	4.33	4.19
		HOM–T	7.94	7.93	7.94	7.91	4.52	4.51	4.53	3.66	3.66	4.12
		HOM–V	18.99	18.94	18.94	18.96	4.29	4.28	4.28	7.87	7.87	13.70
		HET–P	**2.69**	**2.79**	**2.72**	**2.81**	3.47	3.35	3.36	3.42	3.42	3.42
		HET–T	7.29	7.28	7.29	7.26	3.90	3.89	3.90	3.25	3.25	3.78
		HET–V	5.75	5.72	5.72	5.73	**2.15**	**2.15**	**2.15**	**1.84**	**1.84**	**1.99**
	n=500	HOM–P	7.99	8.00	8.00	7.99	4.78	4.74	4.80	3.76	3.88	3.38
		HOM–T	7.87	7.87	7.88	7.86	4.40	4.38	4.39	3.38	3.39	3.38
		HOM–V	18.95	18.94	18.91	18.95	4.25	4.24	4.24	7.88	7.88	13.71
		HET–P	**1.95**	**1.97**	**2.01**	**2.01**	2.54	2.49	2.51	2.55	2.60	2.56
		HET–T	7.27	7.27	7.27	7.26	3.97	3.95	3.96	3.07	3.07	3.06
		HET–V	5.68	5.68	5.67	5.68	**2.07**	**2.07**	**2.07**	**1.73**	**1.73**	**1.86**

Web Table 7: Mean absolute differences (×100) of Kendall's τ for each of the 10 pair copulas, averaged over $M = 1000$ samples of size $n = 250$ and 500 under different HET DGPs in the unbalanced case.

Scenario	DGP	Model	τ_{12}	τ_{23}	τ_{34}	τ_{45}	$\tau_{13;2}$	$\tau_{24;3}$	$\tau_{35;4}$	$\tau_{14;23}$	$\tau_{25;34}$	$\tau_{15;234}$
HET	**DGP4**											
	n=250	HOM–P	2.20	2.85	2.77	3.17	3.50	3.79	4.48	4.38	5.40	5.03
		HOM–T	7.21	6.43	2.72	2.95	3.55	6.30	5.23	5.49	3.67	5.36
		HOM–V	32.57	18.97	27.26	23.60	**2.53**	2.75	2.26	15.79	20.61	30.76
		HET–P	**1.51**	**2.21**	2.41	2.74	3.39	3.26	4.32	**4.09**	5.21	**4.63**
		HET–T	7.13	6.43	**2.05**	**2.01**	3.00	5.66	4.60	4.78	**3.67**	5.22
		HET–V	18.38	5.52	13.14	9.55	3.18	**1.34**	**1.78**	7.91	12.73	16.19
	n=500	HOM–P	2.06	2.59	2.42	2.76	2.79	3.26	3.63	3.35	3.94	3.89
		HOM–T	7.13	6.48	2.59	2.89	3.42	6.34	5.22	5.10	**2.72**	4.88
		HOM–V	32.58	18.99	27.27	23.62	2.46	2.70	2.18	15.77	20.59	30.74
		HET–P	**1.12**	**1.57**	**1.74**	1.98	**2.53**	2.43	3.35	**3.00**	3.80	**3.62**
		HET–T	7.07	6.50	1.84	**1.94**	2.77	5.73	4.61	4.27	2.77	4.65
		HET–V	18.43	5.50	13.16	9.56	3.10	**1.23**	**1.66**	7.87	12.69	16.15
	DGP5											
	n=250	HOM–P	3.15	3.22	3.35	3.55	3.63	3.75	4.33	4.78	5.50	5.49
		HOM–T	3.04	3.04	3.05	3.04	3.09	3.08	3.09	3.95	3.94	5.08
		HOM–V	26.31	26.30	26.29	26.30	2.07	2.07	2.07	19.85	19.85	33.38
		HET–P	1.61	1.81	2.12	2.54	2.82	3.14	3.69	4.21	4.99	**4.72**
		HET–T	**1.07**	**1.07**	**1.07**	**1.07**	**1.99**	**1.99**	**1.99**	**3.23**	**3.23**	4.77
		HET–V	10.69	10.71	10.71	10.70	3.42	3.41	3.41	13.98	13.97	20.72
	n=500	HOM–P	3.04	3.07	3.12	3.22	3.18	3.31	3.62	3.86	4.38	4.49
		HOM–T	2.98	2.98	2.98	2.97	2.91	2.90	2.90	3.27	3.27	4.35
		HOM–V	26.33	26.33	26.32	26.33	1.98	1.98	1.98	19.82	19.82	33.35
		HET–P	1.17	1.30	1.52	1.83	2.09	2.39	2.80	3.17	3.70	**3.63**
		HET–T	**0.77**	**0.77**	**0.77**	**0.77**	**1.44**	**1.44**	**1.44**	**2.45**	**2.45**	3.65
		HET–V	10.74	10.75	10.76	10.75	3.28	3.27	3.27	13.85	13.85	20.59
	DGP6											
	n=250	HOM–P	8.17	8.27	8.42	8.75	5.47	5.69	6.31	5.17	5.17	5.70
		HOM–T	7.98	7.98	7.99	7.96	4.72	4.72	4.73	4.36	4.36	5.19
		HOM–V	16.84	16.79	16.79	16.81	4.29	4.29	4.30	10.02	10.02	15.86
		HET–P	2.69	3.11	3.57	4.44	3.79	4.29	4.85	4.38	4.38	4.91
		HET–T	1.82	1.82	1.82	1.82	2.45	2.44	2.45	3.35	3.35	5.06
		HET–V	**1.24**	**1.24**	**1.24**	**1.24**	**1.19**	**1.19**	**1.19**	**1.16**	**1.16**	**1.01**
	n=500	HOM–P	7.99	8.05	8.12	8.26	4.90	5.01	5.48	4.30	4.80	4.46
		HOM–T	7.89	7.90	7.90	7.88	4.50	4.49	4.50	3.74	3.74	4.36
		HOM–V	16.79	16.79	16.76	16.79	4.23	4.22	4.23	10.04	10.03	15.88
		HET–P	1.95	2.19	2.57	3.04	2.75	3.14	3.65	3.24	3.84	3.78
		HET–T	1.32	1.32	1.32	1.32	1.74	1.74	1.74	2.45	2.45	3.96
		HET–V	**0.88**	**0.88**	**0.88**	**0.88**	**0.87**	**0.87**	**0.87**	**0.86**	**0.86**	**0.75**

Marginal model comparison: LM vs LMM

We use the LMM framework to specify the marginal models for the repeated outcomes primarily to estimate the fixed effect parameters and obtain residuals. The residuals are then used to obtain the copula data for modeling dependence among repeated measurements. It is also possible to consider other approaches to estimate the marginal distributions. For instance, one can use the ordinary least-squares (OLS) estimator under the linear model (LM) to obtain the residuals and the copula data. Here, using the setting under DGP4 with marginal models in Section 4.4, we compare the fixed effect parameters estimates from the OLS estimator under linear model (LM) and from the REML estimator under the linear mixed model (LMM). As shown in Web Figure 1, there can be a considerable efficiency gain in some of the fixed effect parameters if one uses the LMM over the LM. Furthermore, while the estimated copula data obtained under the LM and LMM both yield very high correlation with the true copula data, sum of square differences are, on average, smaller for the LMM compared to the LM (as shown in Web Table 8).

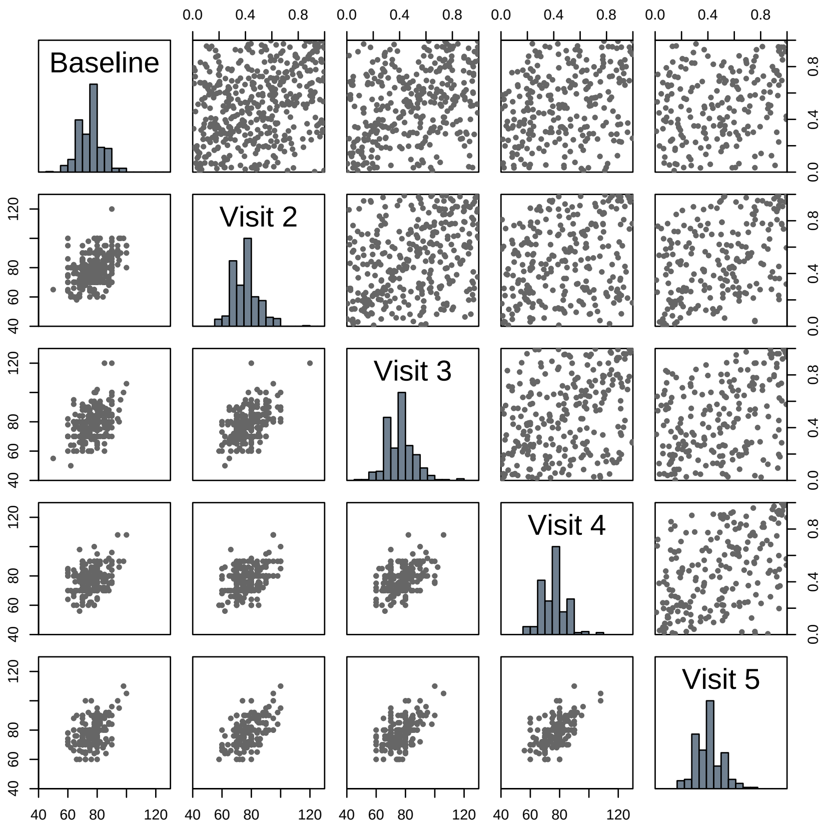


Web Figure 1: Boxplots and the relative efficiency (RE) of the fixed effect parameter estimates under the LM and LMM evaluated over $M = 1000$ samples of size $n = 250$ (left) and 500 (right) generated from DGP4.

	n=250				n=500			
	Correlation		SSD		Correlation		SSD	
	OLS	LMM	OLS	LMM	OLS	LMM	OLS	LMM
$(u_1, \hat{u}_1)$	0.998	0.998	0.185	0.184	0.999	0.999	0.183	0.178
$(u_2, \hat{u}_2)$	0.998	0.998	0.177	0.176	0.999	0.999	0.174	0.170
$(u_3, \hat{u}_3)$	0.998	0.998	0.151	0.144	0.999	0.999	0.145	0.137
$(u_4, \hat{u}_4)$	0.998	0.998	0.113	0.099	0.999	0.999	0.112	0.098
$(u_5, \hat{u}_5)$	0.999	0.999	0.086	0.067	0.999	0.999	0.091	0.072
Overall	0.997	0.998	0.713	0.671	0.998	0.999	0.706	0.654

Web Table 8: Average correlation and average sum of square difference (SSD) between the true and estimated copula data under the LM and LMM over $M = 1000$ samples of size $n = 250$ and 500 under DGP4.

Appendix C: Data Application



Web Figure 2: Pairwise scatter plots of DBP measurements (lower matrix plot) across visits, histograms of DBP measurements at each visit (diagonal panel) and pairwise scatter plots of copula data (upper matrix plot).

Web Table 9: *Group sizes with exactly j and more than j measurements for* $j = 1, \ldots, 5$

j	1	2	3	4	5
subjects with j measurements	19	101	82	54	206
subjects with $\geq j$ measurements	462	443	342	260	206

Web Table 10: *Estimate, standard error (SE), and lower bound (LB) and upper bound (UB) of 95% PI of model parameter estimates for best fitted LMM.*

Parameter	**Estimate**	**SE**	**LB**	**UB**
α_0	51.75	2.29	47.26	56.25
α_{Age}	0.20	0.04	0.13	0.27
α_{BMI}	0.74	0.10	0.55	0.93
α_{IHD}	0.77	0.58	-0.37	1.91
σ_γ	4.85	0.25	4.39	5.36
σ_ϵ	6.39	0.13	6.14	6.64
d_ϵ	0.02	0.009	0.004	0.05

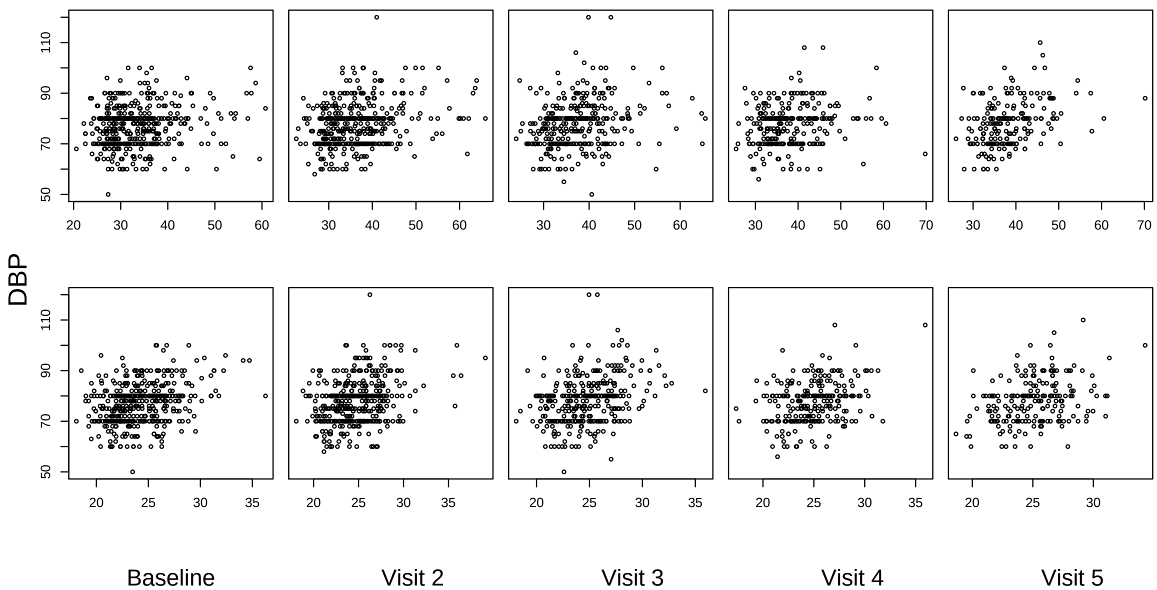
Web Table 11: *Selected copula families under different fitted models for MFUS data.*

Model	**Tree 1**				**Tree 2**			**Tree 3**		**Tree 4**
	c_{12}	c_{23}	c_{34}	c_{45}	$c_{13;2}$	$c_{24;3}$	$c_{35;4}$	$c_{14;23}$	$c_{25;34}$	$c_{15;234}$
HOM–P	G	G	G	G	SG	N	SC	F	N	F
HET–P	G	G	G	SC	SG	SG	SC	F	N	N
HOM–T	G	G	G	G	SG	SG	SG	SC	SC	G
HET–T	G	G	G	G	SG	SG	SG	SC	SC	G
HOM–V	G	G	G	G	G	G	G	G	G	G
HET–V	G	G	G	G	G	G	G	G	G	G

N: Gaussian, G: Gumbel, F: Frank, SC: Survival Clayton, SG: Survival Gumbel

Web Table 12: *The estimated median (50% quantile) and the estimated 90% prediction intervals (5% and 95% quantiles) of the fifth measurement under the three fitted models with corresponding length of prediction interval.*

	Model	5%	50%	95%	Length of interval
Subject 1	LMM	61.46	72.84	84.23	22.77
	HOM–P	63.59	74.76	87.64	24.05
	HET–P	67.47	79.54	86.79	19.32
Subject 2	LMM	65.22	76.61	87.99	22.77
	HOM–P	67.78	80.49	92.74	24.96
	HET–P	74.82	86.18	92.37	17.55
Subject 3	LMM	64.12	75.51	86.89	22.77
	HOM–P	65.95	77.50	89.47	23.53
	HET–P	66.51	81.35	93.47	26.96



Web Figure 3: Scatter plots of MFUS data at different visit times displaying the bivariate relationship of diastolic blood pressure with age (upper panel) and body mass index (lower panel).

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