Value

1. Find the following, showing all of your work:

[3]

a) $\lim_{x\to 0^-} \frac{|4x|-\sin^2 x}{3x} \underset{x<0}{=} \lim_{x\to 0^-}\left(\frac{-4x}{3x} - \frac{x}{3}\,\frac{\sin^2 x}{x^2}\right) =$

$= \lim_{x\to 0^-}\left(-\frac{4}{3} - \frac{x}{3}\left(\frac{\sin x}{x}\right)^2\right) = -\frac{4}{3} - \frac{1}{3}\lim_{x\to 0^-} x \left(\lim_{x\to 0^-}\frac{\sin x}{x}\right)^2$

$= -4/3 - \frac{1}{3}\cdot 0 \cdot 1^2 = -4/3$

[3]

b) $\lim_{x\to -\infty} \frac{1}{\sqrt{x^2-5x}+x} = \lim_{x\to -\infty} \frac{\sqrt{x^2-5x}-x}{x^2-5x-\cancel{x^2}} = \lim_{x\to -\infty}\left(-\frac{1}{5}\,\frac{\sqrt{x^2-5x}}{x} + \frac{1}{5}\right)$

$\underset{\sqrt{x^2}=|x|=-x}{\overset{x<0}{=}} \lim_{x\to -\infty}\left(-\frac{1}{5}(-1)\sqrt{\frac{x^2-5x}{x^2}} + \frac{1}{5}\right) =$

$= \lim_{x\to -\infty}\left(\frac{1}{5}\sqrt{1-\underbrace{\frac{5}{x}}_{0}} + \frac{1}{5}\right) = \frac{1}{5} + \frac{1}{5} = \frac{2}{5}$

[3]

c) Horizontal asymptotes of $f(x) = \cos\left(\frac{1}{x}+\pi\right) + \frac{1+2x^3}{x^2-4x^3}$.

$\lim_{x\to \pm\infty}\left(\cos\left(\underbrace{\frac{1}{x}}_{0}+\pi\right) + \frac{1+2x^3}{x^2-4x^3}\right) = \cos\pi + \frac{2}{-4} = -1-\frac{1}{2} = -3/2$

So: $y = -3/2$ is A HORIZONTAL ASYMPTOTE.

2. Find the derivatives of the functions. Do not simplify.

a) $f(x) = \dfrac{2x^3 - 5}{\sqrt{x}+1}$.

[3]

$$f'(x) = \frac{6x(\sqrt{x}+1) - (2x^3-5)\frac{1}{2\sqrt{x}}}{(\sqrt{x}+1)^2}$$

b) $f(x)=\left((x^2+e^{-x})\tan 3x\right)^{125}$.

[4]

$$f'(x) = 125\left((x^2+e^{-x})\tan 3x\right)^{124}\left((2x-e^{-x})\tan 3x + (x^2+e^{-x})\sec^2 3x \cdot 3\right)$$

c) $f(x)=\sin^2(\cos(\pi+2x))$.

[4]

$$f'(x) = 2\sin(\cos(\pi+2x))\cdot\cos(\cos(\pi+2x))\cdot(-\sin(\pi+2x))\cdot 2$$

3. a) Write the definition for $\lim_{x \to a} f(x) = L$.

[2]

b) Prove that $\lim_{x \to 2} \left(\frac{4}{x} - 5 \right) = -3$ by using the definition of limit.

[5] LET $\varepsilon > 0$. WE WANT $\delta > 0$ s.t. $\forall x \in D_f$ WITH $0 < |x-2| < \delta$, WE HAVE

$\left| \frac{4}{x} - 5 + 3 \right| < \varepsilon$

$\left| \frac{4}{x} - 5 + 3 \right| = \left| \frac{4}{x} - 2 \right| = \frac{|4-2x|}{|x|} = \frac{2|x-2|}{|x|} \overset{?}{\leq} 2 \cdot c\, |x-2|$

TAKE $\delta \leq 1$. THEN $|x-2| < \delta \Rightarrow x \in (1,3)$, i.e. $|x| = x > 1$,

i.e. $\frac{2|x-2|}{|x|} < 2|x-2| \overset{?}{<} \varepsilon$. TAKE $\delta = \min\left\{1, \frac{\varepsilon}{2}\right\}$. THEN

$\forall x \in D_f = \mathbb{R} \setminus \{0\}$, IF $0 < |x-2| < \delta \Rightarrow$ (i) $|x| > 1$, AND (ii) $|x-2| < \varepsilon/2$;

AND SO $\left| \frac{4}{x} - 5 - (-3) \right| < 2 \cdot 1\, |x-2| < 2 \cdot \varepsilon/2 = \varepsilon$. $\boxtimes$

Bonus question: c) Write the definition for $\lim_{x \to -\infty} \frac{1}{\sqrt{x^2+1}} = 0$, and show that 0 is the value of the limit by using the definition.

[4] $\forall \varepsilon > 0, \exists M > 0$ s.t. $\forall x \in D_f = \mathbb{R}$, WITH $x < -M$, WE HAVE THAT $\left| \frac{1}{\sqrt{x^2+1}} - 0 \right| < \varepsilon$.

LET $\varepsilon > 0$. (WE HAVE TO FIND $M > 0, \dots$)

$\left| \frac{1}{\sqrt{x^2+1}} - 0 \right| = \left| \frac{1}{\sqrt{x^2+1}} \right| = \frac{1}{\sqrt{x^2+1}} \overset{?}{<} \varepsilon \Leftrightarrow \sqrt{x^2+1} > \frac{1}{\varepsilon} \Leftrightarrow x^2 + 1 > \frac{1}{\varepsilon^2}$

$\Leftrightarrow x^2 > \frac{1}{\varepsilon^2} - 1$.

CASE 1: IF $\frac{1}{\varepsilon^2} - 1 \leq 0$, i.e. $\frac{1}{\varepsilon^2} \leq 1$, i.e. $\varepsilon \geq 1$, THEN ANY $M > 0$ WOULD, SINCE $x^2 \geq 0 > \frac{1}{\varepsilon^2} - 1$. TAKE $M = 1$.

CASE 2: $\frac{1}{\varepsilon^2} - 1 > 0$.

THEN $|x| \overset{?}{>} \sqrt{\frac{1}{\varepsilon^2} - 1}$. SINCE WE WANT $x < 0$: $|x| = -x$, i.e.

$-x > \sqrt{\frac{1}{\varepsilon^2} - 1} \Leftrightarrow x < -\sqrt{\frac{1}{\varepsilon^2} - 1}$. TAKE $M = \sqrt{\frac{1}{\varepsilon^2} - 1} > 0$.

4. a) Write the definition of when is f continuous at the point c. [2]

b) Let $f(x)=\begin{cases} \frac{1}{x}, & x<0 \\ 1-x, & 0\le x<1 \\ 2 & x=1 \\ \frac{x^2-1}{x+3}, & x>1 \end{cases}$. Find the points of discontinuity for f, and determine if they are removable or not. Explain. (You can use the theorems on limits and continuity of rational functions.)

[7]

SINCE RATIONAL FUNCTIONS $\frac{P(x)}{Q(x)}$ ARE CONTINUOUS WHEN EVER $Q(x)\neq 0$, WE HAVE THAT f IS CONTINUOUS WHENEVER $x<0$, OR $0<x<1$, OR $x>1$.

AT $x=0$: $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{1}{x} = \left[\frac{1}{0^-}\right] = -\infty$; LIMIT D.N.E AND

SO f IS NOT CONTINUOUS AT $x=0$; (NOT REMOVABLE)

AT $x=1$: $\left.\begin{array}{l} \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (1-x) = 1-1 = 0 \\ \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{x^2-1}{x+3} = \frac{0}{4} = 0 \end{array}\right\} \lim_{x\to 1} f(x) = 0 \neq 2 = f(1)$

f IS NOT CONTINUOUS AT $x=1$; BUT $x=1$ IS A REMOVABLE DISCONTIN.

(DEFINE $F(x)=\begin{cases} f(x), & x\neq 1 \\ 0, & x=1 \end{cases}$; F IS CONTIN. AT $x=1$)

5. a) Write the definition of the derivative of f at the point c.

[2]

b) Find the derivative of $f(x)=\sqrt{3x}$ by using the definition of the derivative. What is the domain of f, and what is the domain of its derivative?

[6]

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{3(x+h)}-\sqrt{3x}}{h} = \lim_{h\to 0} \frac{\cancel{3x}+3h-\cancel{3x}}{h\left(\sqrt{3(x+h)}+\sqrt{3x}\right)} =$$

$$= \lim_{h\to 0} \frac{3}{\sqrt{3(x+\underset{\downarrow 0}{h})}+\sqrt{3x}} = \frac{3}{\sqrt{3x}+\sqrt{3x}} = \frac{3}{2\sqrt{3x}}.$$

$D_f = [0,\infty)$, $D_{f'} = (0,\infty)$.

6. a) Write the equation of the tangent line to the curve $y = f(x)$ at the point (x_0, y_0).
[2]

b) Find the equation of the tangent line to the curve $y = f(x)$ at $x_0 = 0$, given that $(f(x))^2 + 2e^x f(x) - 3 = 0$, and that f(0) < 0. (Hint: first find $f(0)$, and then $f'(0)$.)
[5]

$(f(0))^2 + 2e^0 f(0) - 3 = 0$, $(f(0))^2 + 2f(0) - 3 = 0$; $(f(0)-1)(f(0)+3) = 0$

$f(0) = 1$, OR $f(0) = -3$. But $f(0) < 0 \Rightarrow \boxed{f(0) = -3}$.

BY CHAIN RULE AND PRODUCT RULE: $2(f(x))f'(x) + 2e^x f(x) + 2e^x f'(x) = 0$

$f'(x) = \dfrac{-2e^x f(x)}{2f(x) + 2e^x}$, $f'(0) = \dfrac{-2e^0 f(0)}{2f(0) + 2e^0}$

$f'(0) = \dfrac{6}{-6+2} = -\dfrac{3}{2}$

$y = -\frac{3}{2}(x - 0) - 3$, $\boxed{y = -\frac{3}{2}x - 3}$.

7. a) State the Intermediate Value Theorem.
[2]

b) If f is continuous on [0, 1] and if the range of f on [0,1] is contained in [0, 1], show that f must have a fixed point in [0, 1], namely $\exists c \in [0,1]$ with $f(c) = c$. (Hint: use a) for $g(x) = f(x) - x$, and check $g(0)$ and $g(1)$. Explain first why is g continuous on [0, 1].)
[5]

g IS CONTINUOUS ON $[0,1]$ AS A DIFFERENCE OF TWO CONTIN. FUNCTIONS, (I.E. $f(x)$ AND x CONTINUOUS ON $[0,1]$).

$g(0) = f(0) - 0 = f(0)$. IF $f(0) = 0$, THEN $c = 0$. IF $f(0) \neq 0$, THEN $f(0) > 0$, SINCE $f(0) \in [0,1]$, AND SO $g(0) > 0$

$g(1) = f(1) - 1$. IF $f(1) = 1$, THEN $c = 1$. IF $f(1) \neq 1$, THEN $f(1) < 1$, SINCE $f(1) \in [0,1]$, AND SO $g(1) = f(1) - 1 < 1 - 1 = 0$.

BY IVT FOR g ON $[0,1]$: $g(0) > 0$, $g(1) < 1 \Rightarrow \exists c \in (0,1)$ S.T. $g(c) = 0$, I.E. $f(c) = c$.

c) Let f be continuous on $[0,1]$, differentiable on $(0,1)$ and such that $f(0) = f(1)$. State the general Mean Value Theorem and then use it to show that $\exists c \in (0,1)$ such that $f'(c) = 0$.
[2]

MVT: . . . .

So: $\exists c \in (0,1)$ S.T. $f'(c) = \dfrac{f(1) - f(0)}{1 - 0} = \dfrac{0}{1} = 0$.