Math 1230: Midterm Examination

Date: November 1, 2017, 5:30–7:30 PM.
Examiner: S. Sankaran.
Duration: 2 hours.

Instructions:

- This exam consists of two parts and a bonus question. Answer all four questions from Part A (46 points) and two of three questions from Part B (24 points), for a total of 70 points. Clearly indicate which questions you choose to answer.
- Answer all question in the examination booklet. If you use more than one booklet, clearly indicate this on all booklets, and make sure your name and student number appears on the front of all of them.
- No aids (including phones, calculators, notes, textbooks, etc.) allowed.
- Be sure to justify your answers; the clarity of your responses is also important. Questions may not necessarily be of equal length or difficulty.
- Good luck!

PART A:

1. (a) (6 points) Prove, using the definition of the limit, that $\lim_{x \to 1} \frac{1}{x+1} = \frac{1}{2}$
   (b) (6 points) Prove, using the definition of the limit, that $\lim_{x \to \infty} \sin(x)$ does not exist.
2. (a) (3 points) Define what it means for a function f to be differentiable at a point x.
   (b) (7 points) Let $f(x) = \frac{1}{\sqrt{1+2x}}$. Using only the definition of the derivative, find $f'(x)$.
3. Compute the derivatives of the following functions, using any rules you desire. Do not simplify your answers.
   (a) (3 points) $f(x) = \sqrt{x^{11}} - \frac{1}{x^{2017}} + 7x^{\pi^2} + \frac{\pi}{2}$
   (b) (3 points) $g(u) = \cos(\frac{2u}{(1-5u)^2})$
   (c) (3 points) $f(x) = \frac{\sqrt{3+\sin(x)}}{\cos(x)}$
   (d) (3 points) $f(t) = \sin(\cos(\sin(t)))$
4. Consider the function $f(x) = \frac{1}{2x+1}$.
   (a) (6 points) Find $f'(x), f''(x)$ and $f'''(x)$.
   (b) (6 points) Find a general formula for the n'th derivative $f^{(n)}(x)$. Be sure to prove (using induction, or otherwise) that your formula is correct.

PART B:

Answer **two** out of the following **three** questions. Clearly indicate which questions you wish to have marked, otherwise the first two responses will be marked.

5. Consider the function

$$f(x) = \begin{cases} |x|^a, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

where $a \in \mathbb{R}$ is a fixed constant.

(a) (4 points) For which values of a is the function continuous at $x = 0$?

(b) (4 points) For which values of a is the function differentiable at $x = 0$?

(c) (4 points) Find a function $g(x)$ that is differentiable at $x = 0$, but $g'(x)$ is not.

6. (12 points) Suppose f is a continuous function, and a is a point in the interior of the domain of f such that $f(a) \neq 0$. Show that f does not vanish in a small neighbourhood of a.

More precisely, show that there exists $\delta > 0$ such that $f(x) \neq 0$ for all x in the interval $(a - \delta, a + \delta)$.

Hint: Apply the definition of the limit, and take $\epsilon = |f(a)|$ at an appropriate point.

7. (a) (3 points) Given a sequence $a_1, a_2, \dots$ and a real number L, give the precise definition for the statement $\lim_{n\to\infty} a_n = L$.

(b) (9 points) Consider the sequence $a_1, a_2, \dots,$ where a_n is given by the first n entries in the decimal expansion of π. For example,

$$a_1 = 3, \qquad a_2 = 3.1, \qquad a_3 = 3.14, \qquad a_4 = 3.141, \qquad \text{etc.}$$

Prove that $\lim_{n\to\infty} a_n = \pi$.

Bonus problem [+4]: This problem is challenging, and not worth many marks. Only attempt if you are fully satisfied with the rest of the exam.

- Consider the function

$$f(x) = \begin{cases} \frac{1}{|q|} & \text{if } x = \frac{p}{q} \neq 0 \text{ is rational, written in lowest terms, i.e. } p, q \text{ relatively prime} \\ 0, & \text{if } x \text{ is irrational or } x = 0. \end{cases}$$

Show that f is discontinuous at all non-zero rational numbers, and continuous at all irrational numbers. What about at $x = 0$?