THE UNIVERSITY OF MANITOBA

DATE: April 16, 2010 **FINAL EXAMINATION**

DEPARTMENT & COURSE NO: MATH 1300 TIME: 2 hours

EXAMINATION: Vector Geom. & Lin. Alg. EXAMINER: Various

Values

[8] 1. a) Use Gaussian Elimination to solve the linear system

$$\begin{aligned} x + y - z &= 3 \\ -x + 3y + z &= 1 \\ 2x + y + z &= 2 \end{aligned}$$

Row reduction for Gaussian elimination:

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ -1 & 3 & 1 & 1 \\ 2 & 1 & 1 & 2 \end{array}\right] \quad \begin{array}{l} R_2 \leftarrow R_2 + R_1 \\ R_3 \leftarrow R_3 - 2R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 0 & 4 & 0 & 4 \\ 0 & -1 & 3 & -4 \end{array}\right] \quad R_3 \leftarrow R_3 + \tfrac{1}{4}R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 0 & 4 & 0 & 4 \\ 0 & 0 & 3 & -3 \end{array}\right]$$

so by back substitution $x_3 = -1$, $x_2 = 1$, $x_1 = 3 - 1 + (-1) = 1$.

b) Check your solutions from part a).

Values

[8] 2. Fill in the blanks. No justification is needed.

a) If A is a 3×3 matrix with $\det(A) = -2$, then $A\,\text{adj}A$ is $-2I$.

b) If A is a 3×3 matrix with $\det(A) = -2$, then $\det(A\,\text{adj}A)$ is -8.

c) Let A, B and C be three $4 \; x \; 4$ matrices with $\det(A)=3$, $\det(B^T)=-2$ and $\det(C)=4$. Then $\det\left(2ABC^{-1}\right)$ is equal to -24.

d) For the matrix $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & -1 & -2 \\ 2 & 0 & 3 \end{bmatrix}$, the cofactor C_{23} is 4.

e) Let $A = (a_{ij})$ be the 17 x 17 matrix with $a_{ij} = \begin{cases} -1, & i \geq j \\ 0, & i < j \end{cases}$. Then $\det(A)$ is -1.

f) Let A, B and C be three $n \times n$ invertible matrices. Given that $AB^{-1}C^{-1} = I$, where I is the identity $n \times n$ matrix, find B in terms of A and C. $B = C^{-1}A$.

g) If A is a 4×4 matrix reduced to B by performing the elementary row operations (i) switch R_2 and R_3, end then (ii) add $-2\,R_4$ to R_1, then the two elementary row matrices E_1 and E_2 such that $B=E_2E_1A$ are

$E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $E_2 = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Values

[12] 3. Let $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 0 \end{bmatrix}$ and let $B = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$.

a) Find the matrix $C = A^2 - B$.

$$A^2 - B = \begin{bmatrix} 1 & -1 & -1 \\ 2 & 0 & 0 \\ 2 & -2 & 0 \end{bmatrix} - \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -2 \\ 1 & 1 & -1 \\ 2 & -3 & -2 \end{bmatrix}$$

b) Find the inverse of A, using row operations.

The row reduction is:

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_1 \leftrightarrow R_3 \\ R_1 \leftarrow \frac{1}{2}R_1 \\ R_2 \leftrightarrow R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right] \quad \begin{array}{l} R_2 \leftarrow R_2 - R_1 \\ R_2 \leftarrow -R_2 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & -1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right]$$

so $A^{-1} = \begin{bmatrix} 0 & 0 & \frac{1}{2} \\ -1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix}$.

c) Find all values of k for which the matrix $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 2 & k & 0 \end{bmatrix}$ is invertible.

$$\det \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 2 & k & 0 \end{bmatrix} = -\det \begin{bmatrix} 1 & -1 \\ 2 & k \end{bmatrix} = -(k+2)$$

so the matrix is invertible if and only if $k \neq -2$.

Values

[8] 4. Let $\mathbf{u} = (1,1,-1,-1)$ and $\mathbf{v} = (1,1,1,1)$ be two vectors in 4-space $\mathbf{R}^4$.

a) Find the length of $\mathbf{u}$. $\|\mathbf{u}\| = \sqrt{1+1+1+1} = 2$

b) Find the dot product of $\mathbf{u}$ and $\mathbf{v}$. $\mathbf{u} \cdot \mathbf{v} = 1+1+(-1)+(-1) = 0$

c) Find a unit vector in $\mathbf{R}^4$ that is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

Let $\mathbf{x} = (x_1, x_2, x_3, x_4)$. Then

$$\begin{aligned} x_1 + x_2 - x_3 - x_4 &= 0 \\ x_1 + x_2 + x_3 + x_4 &= 0 \end{aligned}$$

and so $(x_1 + x_2) = (x_3 + x_4) = 0$. Hence any vector of the form $(t, -t, u, -u)$ is orthogonal to $\mathbf{u}$ and $\mathbf{v}$. The additional condition $2t^2 + 2u^2 = 1$ is necessary to have a unit vector. The most striaghtforward solution is $(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$.

[9] 5. The equation $2x - y + z = 1$ is an equation of the plane π.

a) Find an equation of the plane parallel to the plane π and that passes though the point $(2,2,4)$.

$$2(x-2) - (y-2) + (z-4) = 0, \text{ or } 2x - y + z = 6$$

b) Find parametric equations of the line parallel to the plane π, passing through the point $(2,2,4)$ and perpendicular to the vector $\mathbf{u} = (1,0,1)$.

$(x, y, z) = (2, 2, 4) + t(a, b, c) = (2+ta, 2+tb, 4+tc)$. Being in the plane in (a) implies $2a - b + c = 0$, while being orthogonal to $(1, 0, 1)$ implies $a + c = 0$. This in turn implies $(a, b, c) = (a, a, -a) = a(1, 1, -1)$.

Values

[9] 6. Let $\mathbf{u} = (1,2,3)$ and $\mathbf{v} = (-1,1,0)$.

a) Find the area of the triangle determined by the vectors $\mathbf{u}$ and $\mathbf{v}$.

Since $\mathbf{u} \times \mathbf{v} = (-3,-3,3)$, the area is $\frac{1}{2}\|(-3,-3,3)\| = \frac{1}{2}\sqrt{27} = \frac{3}{2}\sqrt{3}$.

b) Find the vector $\mathbf{w} = proj_{\mathbf{v}}\, \mathbf{u}$.

$\mathbf{w} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = \frac{1}{2}(-1,1,0)$.

c) Find $\mathbf{w}\cdot(\mathbf{u}\times\mathbf{v})$ where $\mathbf{w}$ is the vector from b).

$\mathbf{w}\cdot(\mathbf{u}\times\mathbf{v}) = \frac{1}{2}\det\begin{bmatrix} -1 & 1 & 0 \\ 1 & 2 & 3 \\ -1 & 1 & 0 \end{bmatrix} = -\frac{3}{2}\det\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} = 0$ (or observe $R_1 = R_3$).

[9] 7. Determine if W is a subspace of the vector space V. Show your work.

a) $V = M_{22}$ and $W = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} : ad = 1\right\}$ (W is the set of all matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $ad = 1$).

Since $\mathbf{0}$ is not in W, it is not a subspace.

b) $V = \mathbb{R}^2$ and $W = \{(x,y) : 2x + 5y = -1\}$ (W is the set of all pairs of real numbers (x,y) such that $2x + 5y = -1$).

Since $\mathbf{0}$ is not in W, it is not a subspace.

Values

c) $V = \mathbb{R}$ and $W = \{(x,y,z) : x - y = 0\}$ (W is the set of all triples of real number (x,y,z) such that $x - y = 0$).

Let $\mathbf{u} = (x_1, y_1, z_1)$ and $\mathbf{v} = (x_2, y_2, z_2)$ be in W, that is, $x_1 - y_1 = x_2 - y_2 = 0$. Then $\mathbf{u} + \mathbf{v} = (x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$ and $(x_1 + x_2) - (y_1 - y_2) = x_1 - y_1 + x_2 - y_2 = 0 + 0 = 0$, and so $\mathbf{u} + \mathbf{v}$ is also in W. Similarly $r\mathbf{u} = r(x_1, y_1, z_1) = (rx_1, ry_1, rz_1)$ and $rx_1 - ry_1 = r(x_1 - y_1) = r \cdot 0 = 0$, and $r\mathbf{u}$ is in W. Hence W is a subspace.

[8] 8. Let $p(x) = 2 - x^2$, $q(x) = 1 + x + x^2$ and $r(x) = 1 - x$.

a) Determine if p is a linear combination of q and r. Show your work.

$2 - x^2 = r_1(1 + x + x^2) + r_2(1 - x)$ implies

$$\begin{aligned} 2 &= r_1 + r_2 && (1) \\ 0 &= r_1 - r_2 && (2) \\ -1 &= r_1 && (3) \end{aligned}$$

From (2) and (3) we have $r_1 = r_2 = -1$. Hence (1) is inconsistent, and so p is *not* a linear combination of q and w.

b) Is p in the span of $\{q, r\}$? Explain.

No, by definition, from (a).

Values

[8] 9. Find all values of k such that the set of vectors $\{(1,0,1,0),(k,0,3,0),(0,0,1,-1),(0,1,0,1)\}$ is a basis of $\mathbb{R}^4$. Show your work.

$$\det\begin{bmatrix} 1 & 0 & 1 & 0 \\ k & 0 & 3 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \det\begin{bmatrix} 0 & 3 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} - k\det\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \det\begin{bmatrix} 3 & 0 \\ 1 & -1 \end{bmatrix} - k\det\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= -3 + k$$

so the set of vectors is linearly independent if and only if $k \neq 3$.

[8] 10. Let $R = \begin{bmatrix} 1 & 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ be the reduced row echelon form of a matrix A.

a) Find the solution set of the homogeneous system with coefficient matrix A, i.e. The null space of A.

The free variables are assigned $x_2 = s$ and $x_4 = t$. The the constrained variables are $x_1 = 2t$, $x_3 = -t$, and $x_5 = 0$. Hence $\mathbf{x}$ is in the solution space if and only if $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5) = (2t, s, -t, t, 0) = s(0,1,0,0,0) + t(2,0,-1,1,0)$.

b) Find a basis of the null space of A. $\{(0,1,0,0,0), (2,0,-1,1,0)\}$ is a basis.

c) The dimension of the null space of A is 2.

Values

[8] 11. If A is a **non-invertible** 6×6 matrix, state whether each of the following statements is TRUE or FALSE (No explanation is necessary).

a) The linear system $A\mathbf{x} = \mathbf{b}$ is inconsistent, whenever $\mathbf{b} \neq \mathbf{0}$. F

b) A can be written as a product of elementary matrices. F

c) If R is the reduced row echelon form of A, then $\det(R) = 0$. T

d) If A is a diagonal matrix, all of the diagonal entries are zero. F

e) If $\mathbf{b} \neq \mathbf{0}$, the solution set of the linear system $A\mathbf{x} = \mathbf{b}$ is a vector space. F

f) The column space of A is a subspace of $\mathbb{R}^6$. T

g) The largest possible dimension of the column space of A is 5. T

h) The largest possible dimension of the null space is 5. F

[5] 12. If $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a linearly dependent set, then $\mathbf{w}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

Prove that this statement is true, or give an example to show that it is false.

The statement is false. A simple counterexample from 2-space: $\mathbf{u} = (0, 0)$, $\mathbf{v} = (1, 0)$, and $\mathbf{w} = (0, 1)$.