The University of Manitoba

Vector Geometry and Linear Algebra (MATH 1300)
Final Examination: Winter 2011

Time: Two hours | Total Marks: 100

Family Name (PLEASE PRINT CLEARLY): ______________________

Student Number: ______________________

First Name (PLEASE PRINT CLEARLY): ______________________

Signature: ______________________
(I acknowledge that cheating is an extremely serious offense.)

Place a check mark (✓) in the box corresponding to your class hour.

☐	J. Chipalkatti	MWF 3:30-4:20	in 208 Armes
☐	M. Doob	TTh 10:00-11:20	in 208 Armes
☐	M. Doob	TTh 1:00-2:20	in 200 Armes
☐	C. K. Gupta	MWF 10:30-11:20	in 201 Armes
☐	Y. Zhang	MWF 1:30-2:20	in 204 Armes
☐	Challenge for Credit		

Instructions:
Please ensure that your paper has a total of 9 pages (including this page). Read the questions thoroughly and carefully before attempting them. **Think before you act.** You must **show your work** in order to get marks.

You are **not allowed** to use any of the following: calculators, notes, books, dictionaries or electronic communication devices (e.g., cellular phones, pagers or blackberries). You may use the left-hand pages, as well as the last page for rough work.

	Obtained	Maximum
Page 2		24
Page 3		8
Page 4		13
Page 5		13
Page 6		14
Page 7		14
Page 8		14
Total		**100**

Multiple choice questions

Questions 1-6 are worth 4 marks each. Write the letter corresponding to the correct answer in the box below. You will get −1 (minus one) mark for an incorrect answer, so you may want to leave the box blank if you are unsure.

Q1	Q2	Q3	Q4	Q5	Q6
(d)	(a)	(a)	(c)	(b)	(d)

Q1. Let $\mathbf{u} = (1, -2)$, and $\mathbf{v} = (3, 1)$. Find $||2\,\mathbf{u} + \mathbf{v}||$.

$$(a)\,34, \quad (b)\,4, \quad (c)\,\sqrt{50}, \quad (d)\,\sqrt{34}.$$

Q2. Which of the following is a parametric equation of the line passing through the points $P = (3, -2, 1)$ and $Q = (1, 4, 5)$?

(a) $(x, y, z) = (1 - t, 4 + 3\,t, 5 + 2\,t)$, (b) $(x, y, z) = (2\,t, -6\,t, -4\,t)$,
(c) $(x, y, z) = (3 + 2\,t, -2 - 6\,t, 1 + 4\,t)$, (d) $(x, y, z) = (3\,s + t, -2\,s + 4\,t, s + 5\,t)$.

Q3. It is given that $E = \begin{bmatrix} 1 & -4 & p+q \\ 0 & p-2 & 0 \\ 0 & 0 & q+4 \end{bmatrix}$ is an elementary matrix. What are the values of p and q ?

(a) $(p, q) = (3, -3)$, (b) $(p, q) = (4, -4)$,
(c) $(p, q) = (3, -4)$, (d) There are no such values of p, q.

Q4. Let $A = \begin{bmatrix} 3 & -1 & 4 \\ -1 & 2 & 1 \end{bmatrix}$. Which of the following vectors belongs to the nullspace of A ?

$$(a)\,(3, 1, -2), \quad (b)\,(1, 3), \quad (c)\,(9, 7, -5), \quad (d)\,(0, 8, 2).$$

Q5. Which of the following subsets of $\mathbf{R}^2$ is a subspace of $\mathbf{R}^2$?

(a) $\{(x, y) : 2\,x - 3\,y = 7\}$, (b) $\{(x, y) : x + 3\,y = 0\}$,
(c) $\{(x, y) : x \geqslant 0,\ x + 3\,y = 0\}$, (d) $\{(3 - t, 2 + 5\,t) : t \in \mathbf{R}\}$.

Q6. Consider the vector space $M_{2,2}$ of 2×2 real matrices. Let W be its subspace of matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that

$$a + c = 0, \quad b + d = 0.$$

Which of the following is a basis of W ?

$$(a) \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \quad (b) \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$(c) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \quad (d) \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -2 \\ 0 & 2 \end{bmatrix}$$

For questions 7 through 16, you must show your work in order to get credit.

Q7. The lines L_1 and L_2 in 3-space corresponding to the following parametric equations intersect each other. [8]

$$L_1: \quad x = 1 + s, \quad y = 1 - s, \quad z = -1 + 3\,s,$$
$$L_2: \quad x = 1 + 2\,t, \quad y = 2 - 3\,t, \quad z = 4 + t.$$

Find the general form of the equation of the unique plane containing L_1 and L_2.

Solution: The point of intersection of the two lines is $(3, -1, 5)$. The direction numbers of the two lines are $(1, -1, 3)$ and $(2, -3, 1)$. The cross product of the directions numbers, which is the normal to the desired plane, is $(8, 5, -1)$, Hence the equation of the plane is $8x + 5y - z + d = 0$, and, using the point of intersection, the value of d is determined. The equation of the plane is then $8x + 5y - z - 14 = 0$.

Q8. Let A be a 4×4 matrix such that $\det(A) = -2$. Find $\det(\operatorname{adj}(A))$. [6]

Solution: We use the fact that $A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A)$ and take the determinant of both sides. The equation then becomes

$$-\frac{1}{2} = \det(A^{-1}) = \left(-\frac{1}{2}\right)^4 \det(\operatorname{adj}(A))$$

and so $\det(\operatorname{adj}(A)) = -8$.

Q9. Find the area of the triangle formed by the following points in 3-space: [7]

$$A = (2, -1, 6), \quad B = (3, 1, 5), \quad C = (4, 0, 6).$$

Solution: $\overrightarrow{AB} = (1, 2, -1)$, and $\overrightarrow{AC} = (2, 1, 0)$. The cross product of these vectors is $(1, -2, -3)$ and so the area is half the length of this vector: $\frac{1}{2}\sqrt{1+4+9} = \frac{1}{2}\sqrt{14}$

Q10. Consider the vectors $\mathbf{a} = (1, 5, -2)$ and $\mathbf{u} = (3, -2, 2)$. Find the vector $\text{proj}_{\mathbf{a}}\,\mathbf{u}$. [6]

Solution: $\mathbf{a} \cdot \mathbf{u} = 3 - 10 - 4 = -11$ and $\mathbf{a} \cdot \mathbf{a} = 1 + 25 + 4 = 30$. Hence

$$\text{proj}_{\mathbf{a}}\mathbf{u} = \frac{-11}{30}\mathbf{a} = -\frac{11}{30}(1, 5, -2)$$

Q11. Consider the vectors [7]

$$\mathbf{v}_1 = (-2, 3, 1), \quad \mathbf{v}_2 = (3, 1, -7), \quad \mathbf{v}_3 = (1, 4, -6).$$

Determine whether the set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis of $\mathbf{R}^3$. (You must give adequate justification for your answer.)

Solution:

$$\det \begin{bmatrix} -2 & 3 & 1 \\ 3 & 1 & -7 \\ 1 & 4 & -6 \end{bmatrix} = 12 - 21 + 12 - 1 - 56 + 54 = 0$$

and so $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent and **not** a basis.

Q12. Consider the vectors [8]

$$\mathbf{u} = (1, -3, 2), \quad \mathbf{v} = (2, -4, 0), \quad \mathbf{w} = (-3, 5, k - 4)$$

in $\mathbf{R}^3$. Find the values of k such that the volume of the parallelopiped formed by the vectors $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ is 6.

Solution:

$$\det \begin{bmatrix} 1 & -3 & 2 \\ 2 & -4 & 0 \\ -3 & 5 & k-4 \end{bmatrix} = 2k - 12$$

and so $|2k - 12| = 6$. This reduces to $k - 6 = \pm 3$ and so $k = 9$ or $k = 3$.

Q13. Let θ be the angle between the vectors $\mathbf{u} = (1, -3)$ and $\mathbf{v} = (4, -5)$. Find the value of $\cos\theta$. [6]

Solution: $\mathbf{u} \cdot \mathbf{v} = 4 + 15 = 19$, $\|\mathbf{u}\| = \sqrt{10}$ and $\|\mathbf{v}\| = \sqrt{41}$. Since $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos\theta$, it follows that $\cos\theta = \frac{19}{\sqrt{10}\sqrt{41}}$.

Q14. Consider the polynomials [8]

$$\mathbf{p}(x) = 2\,x^2 - x + 5, \quad \mathbf{q}(x) = x^2 + x + 2, \quad \mathbf{r}(x) = 3\,x^2 - 3\,x + 8.$$

in the vector space $\mathbf{P}_2$. Show that the set $\{\mathbf{p}, \mathbf{q}, \mathbf{r}\}$ is linearly dependent. If possible, write $\mathbf{q}$ as a linear combination of $\mathbf{p}$ and $\mathbf{r}$.

Solution: The equation $\mathbf{q} = r_1\mathbf{p} + r_2\mathbf{r}$ becomes

$$x^2 + x + 2 = r_1(2x^2 - x + 5) + r_2(3x^2 - 3x + 8).$$

Equating the coefficients gives the system of equations

$$\begin{aligned} 2r_1 + 3r_2 &= 1 \\ -r_1 - 3r_2 &= 1 \\ 5r_1 + 8r_2 &= 2 \end{aligned}$$

which has a unique solution $(r_1, r_2) = (2, -1)$. This shows that $\{\mathbf{p}, \mathbf{q}, \mathbf{r}\}$ is linearly dependent.

Q15. Consider the vectors [6]

$$\mathbf{u}_1 = (-3, 1), \quad \mathbf{u}_2 = (2, 1), \quad \mathbf{v} = (4, 5).$$

It is given to you that $S = \{\mathbf{u}_1, \mathbf{u}_2\}$ is a basis of $\mathbf{R}^2$. Find the co-ordinates of $\mathbf{v}$ with respect to S.

Solution: We solve $(4,5) = \mathbf{v} = r_1\mathbf{u}_1 + r_2\mathbf{u}_2 = r_1(-3,1) + r_2(2,1)$. This becomes

$$\begin{aligned} -3r_1 + 2r_2 &= 4 \\ r_1 + r_2 &= 5 \end{aligned}$$

which has a unique solution $(r_1, r_2) = (\frac{6}{5}, \frac{19}{5})$.

Q16. Let A be a matrix whose reduced row-echelon form (RREF) is: [14]

$$\begin{bmatrix} 1 & -1 & 0 & 5 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

For parts (a), (b) and (c), complete the sentence by filling in the appropriate number. (No explanation is necessary.)

(a) The dimension of the columnspace of A is ___3___.

(b) The dimension of the nullspace of A is ___2___.

(c) The dimension of the rowspace of A^T is ___3___.

(d) Find a basis for the rowspace of A.

Solution: The first three rows of the matrix is a basis for the row space.

(e) Find a basis for the nullspace of A.

Solution: The reduced row echelon form implies that the solution to $A\mathbf{x} = \mathbf{0}$ has free variables s and t for columns 2 and 4. The general solution is then $(x_1, x_2, x_3, x_4, x_5) = (s - 5t, s, -3t, t, 0) = s(1,1,0,0,0) + t(-5,0,-3,1,0)$. Hence a basis is $\{(1,1,0,0,0), (-5,0,-3,1,0)\}$.