Mathematics MATH1300
Vector Geometry and Linear Algebra
Midterm Examination
February 28, 2012, 5:30–6:30pm

1. (20%) Consider the following system of linear equations:

$$\begin{aligned} -x + 2y + 3z + w &= -1 \\ x - y - 2z + w &= 3 \end{aligned}$$

(a) Give the augmented matrix of the system.

(b) Put the augmented matrix in reduced row echelon form.

(c) Give **all** solutions to the system of equations.

Solution: The augmented matrix is

$$\left[\begin{array}{cccc|c} -1 & 2 & 3 & 1 & -1 \\ 1 & -1 & -2 & 1 & 3 \end{array}\right].$$

- multiply row 1 by -1 ($R_1 \leftarrow -R_1$) to get
$\left[\begin{array}{cccc|c} 1 & -2 & -3 & -1 & 1 \\ 1 & -1 & -2 & 1 & 3 \end{array}\right]$
- Subtract row 1 from row 2 ($R_2 \leftarrow R_2 - R_1$) to get
$\left[\begin{array}{cccc|c} 1 & -2 & -3 & -1 & 1 \\ 0 & 1 & 1 & 2 & 2 \end{array}\right]$
- Add twice row 2 to row 1 ($R_1 \leftarrow R_1 + 2R_2$) to get
$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 3 & 5 \\ 0 & 1 & 1 & 2 & 2 \end{array}\right]$

Hence z and w are free variables, let $z = t$ and $w = u$. It then follows that $x = 5 + t - 3u$ and $y = 2 - t - 2u$.

2. (15%) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$ and
$C = \begin{bmatrix} -2 & 0 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. If the given expression is defined, calculate the resulting matrix or value. Otherwise, give a reason why it does not exist.

(a) $A^T B$

Solution: $\begin{bmatrix} 1 & -1 \\ 2 & -2 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 6 \\ 3 & 9 \end{bmatrix}$

(b) CA

Solution: Not defined since C has three columns and A has two rows.

(c) $B(A+C)$

Solution: Undefined since A and C do not have the same shape.

3. (5%)

Let A, B, C be square matrices such that $AB = 2AC + I$. Find a formula for A^{-1} in terms of B and C.

Solution: $A(B-2C) = I$, and so $A^{-1} = B - 2C$.

4. (20%) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 1 & 2 \end{bmatrix}$. Find A^{-1}.

Solution:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] R_2 \leftarrow \frac{1}{2} R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] R_1 \leftarrow R_1 - R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & 1 \end{array}\right] R_3 \leftarrow R_3 - R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & 1 \end{array}\right] R_2 \leftarrow R_2 - R_3$$

and so

$$A^{-1} = \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -1 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix}$$

5. (5%) Consider the elementary row operation where row 2 is replaced by row 2 minus twice row 3 (sometimes written $R_2 \leftarrow R_2 - 2R_3$). Let C be the matrix we get after applying this operation to the 3×3 matrix A. Find the elementary matrix E such that $C = EA$.

Solution: $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$.

6. (5%) Some of the entries of the 3×3 matrix A are known:

$$A = \begin{bmatrix} \cdot & 1 & \cdot \\ 1 & \cdot & -2 \\ 1 & \cdot & 2 \end{bmatrix}$$

Suppose that $\det(A) = -2$ and that

$$A^{-1} = \begin{bmatrix} \cdot & \cdot & \cdot \\ x & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}$$

Find x.

Solution: The cofactor $C_{1,2} = (-1)^3 \det \begin{bmatrix} 1 & -2 \\ 1 & 2 \end{bmatrix} = -4$. Hence $x = 2$.

7. (15%)

(a) Find the determinant of the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ -2 & 0 & 1 & 0 \\ 2 & 2 & 1 & 4 \\ 1 & 0 & 2 & 1 \end{bmatrix}$$

Solution: Expanding on the second column gives

$$\det(A) = -2 \det \left(\begin{bmatrix} 1 & 1 & 0 \\ -2 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \right) = (-2) \cdot 3 = -6.$$

(b) Given that $\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = 7$, find

$$\det \begin{bmatrix} a+2d & b+2e & c+2f \\ 3g & 3h & 3i \\ d & e & f \end{bmatrix}$$

Solution: Adding a multiple of one row to another leaves the determinant unchanged. Hence

$$\begin{aligned} \det \begin{bmatrix} a+2d & b+2e & c+2f \\ 3g & 3h & 3i \\ d & e & f \end{bmatrix} &= \det \begin{bmatrix} a & b & c \\ 3g & 3h & 3i \\ d & e & f \end{bmatrix} \\ &= 3 \det \begin{bmatrix} a & b & c \\ g & h & i \\ d & e & f \end{bmatrix} \\ &= -3 \det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \\ &= -21. \end{aligned}$$

8. (15%) Suppose that the augmented matrix of a system of linear equations has been partially reduced using elementary row operations to

$$\left[\begin{array}{ccc|c} 1 & 2 & a+2 & b \\ 0 & 1 & b-1 & a \\ 0 & 0 & a-2 & b \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(a) Find **all values** of a and b for which the system is inconsistent.

> *Solution*: For an inconsistent system, we must have a row with all but the last entry zero and the last entry nonzero, that is, $a = 2$ and $b \neq 0$

(b) Find **all values** of a and b for which the system has exactly one solution.

> *Solution*: If $a \neq 2$, Gaussian elimination guarantees a unique solution, and so b may take on any value.

(c) Find **all values** of a and b for which the system has infinitely many solutions.

> *Solution*: To get a free variable, we must have $a = 2$ and $b = 0$.