THE UNIVERSITY OF MANITOBA

DATE: October 27, 2016 — Midterm Examination

DEPARTMENT & COURSE NO: MATH 1300 — TITLE PAGE

EXAMINATION: Vector Geometry & Linear Algebra — TIME: 1 HOUR

FAMILY NAME: (Print in ink) ______________________________

GIVEN NAME: (Print in ink) ______________________________

STUDENT NUMBER: (Print in ink) ______________________________

SIGNATURE: (Sign in ink) ______________________________
(I understand that cheating is a serious offense)

Please indicate your instructor and section by checking the appropriate box below:

- □ A01 S. Kalajdzievski (9:30-10:20, MWF, 206 Human Ecology)
- □ A02 M. Virgilio (8:30-9:45, TR, 208 Armes)
- □ A03 N. Zorboska (11:30-12:20, MWF, 306 Tier)
- □ A04 G. Moghaddam (11:30-12:45. TR, 206 Human Ecology)
- □ OTHER

INSTRUCTIONS TO STUDENTS:

This is a 1 hour exam. Please show your work clearly.

No texts, notes, or other aids are permitted. There are no calculators, cell phones or electronic translators permitted

Answer all questions on the exam paper in the space provided beneath the question. If you continue your work on the reverse side of the page, please CLEARLY INDICATE that your work is continued. **Justify your answers.**

Total number of problems: 7
Students should answer all problems.

Total 60 points

DO NOT WRITE IN THIS COLUMN

1.	/ 8
2.	/ 10
3.	/ 8
4.	/ 8
5.	/7
6.	/10
7.	/9
TOTAL	/60

Always show (justify) your work unless otherwise stated!

(8) **1.** Solve, by **Gauss elimination**, the following system of linear equations. (Note: no marks will be given if other methods are used.)

$$\begin{aligned} y \quad &= -7 \\ x + y - 2z &= 1 \\ y - z &= 4 \end{aligned}$$

Solution. The augmented matrix for this system is $\begin{bmatrix} 0 & 1 & 0 & \vdots & -7 \\ 1 & 1 & -2 & \vdots & 1 \\ 0 & 1 & -1 & \vdots & 4 \end{bmatrix}$. We find th REF of this matrix using row operations:

$$\begin{bmatrix} 0 & 1 & 0 & \vdots & -7 \\ 1 & 1 & -2 & \vdots & 1 \\ 0 & 1 & -1 & \vdots & 4 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 1 & -2 & \vdots & 1 \\ 0 & 1 & 0 & \vdots & -7 \\ 0 & 1 & -1 & \vdots & 4 \end{bmatrix} \xrightarrow{(-1)R_2 \text{ to } R_3} \begin{bmatrix} 1 & 1 & -2 & \vdots & 1 \\ 0 & 1 & 0 & \vdots & -7 \\ 0 & 0 & -1 & \vdots & 11 \end{bmatrix}$$

$$\xrightarrow{(-1)R_3} \begin{bmatrix} 1 & 1 & -2 & \vdots & 1 \\ 0 & 1 & 0 & \vdots & -7 \\ 0 & 0 & 1 & \vdots & -11 \end{bmatrix}$$

The last matrix is in REF.

The system associated to the last matrix is $\begin{aligned} x + y - 2z &= 1 \\ y \quad &= -7 \\ z &= -11 \end{aligned}$. We subsitute the values of y and z (from the second and third equation) into the first equation to get $x - 7 + 22 = 1$, from where we calculate $x = -14$.

Hence the only solution of the system is $(x, y, z) = (-14, -7, 11)$.

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(10) **2.** In the following system a and b are constants.

$$\begin{aligned} x + 2ay &= 2 \\ 2x + 2y &= b \end{aligned}$$

(a) Find **all values** of a and b, if any, such that the system has a unique solution.
(b) Find **all values** of a and b, if any, such that the system is inconsistent.
(c) Find **all values** of a and b, if any, such that the system has infinitely many solutions.

Note: you are NOT asked to solve the system. So, do NOT solve it.

Solution. The augmented matrix of the system is $\left[\begin{array}{cc|c} 1 & 2a & 2 \\ 2 & 2 & b \end{array}\right]$. We perform one row operation: $\left[\begin{array}{cc|c} 1 & 2a & 2 \\ 2 & 2 & b \end{array}\right] \xrightarrow{(-2)R_1 \text{ to } R_2} \left[\begin{array}{cc|c} 1 & 2a & 2 \\ 0 & 2-4a & b-4 \end{array}\right]$.

(a) If $2-4a \neq 0$ then the system will have a unique solution. This happens for all a, except $a = \frac{1}{2}$.

(b) If $2-4a = 0$ and at the same time $b-4 \neq 0$, then the system is inconsitent. This happens for $a = \frac{1}{2}$ and for any b such that $b \neq 4$.

(c) If $2-4a = 0$ and at the same time $b-4 = 0$, then the system has infinitely many solutions. This happens for $a = \frac{1}{2}$ and for $b = 4$.

(8) **3.** Consider the matrix $A = \begin{bmatrix} k & k & 0 \\ k^2 & 2 & k \\ 0 & k & k \end{bmatrix}$.

(a) Compute $\det A$.
(b) Determine **all values *k*** such that A is **NOT** invertible? Do not forget to show your work and justify your answer.

Solution.

(a) $\det A = 2k^2 - k^4 - k^3$.

(b) A is not invertible if and only if $\det A = 0$. We solve: $2k^2 - k^4 - k^3 = 0 \Leftrightarrow k^2(2-k-k^2) = 0 \Leftrightarrow k^2(1-k)(2+k) = 0$, and the last equation is true if and only if $k = 0$, or $k = 1$, or $k = -2$.

So A is not invertible only if $k = 0$, or $k = 1$, or $k = -2$.

(8) 4. Express $A = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$ as a product of elementary matrices. Show your work and find explicitly the elementary matrices in the product.

Solution. First we row reduce A: $\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \overset{R_1 \leftrightarrow R_2}{\rightarrow} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \overset{(-2)R_1 \text{ to } R_2}{\rightarrow} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. The elementary matrix associated to the first row operation that we have used above is $E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and the elementary matrix associated to the second row operations is $E_2 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$.

Their inverses can be obtrained from I by applying the corresponding reverse row operations: $E_1^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $E_2^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$.

From the theory relating row operations and elementary matrices and from the row reduction performed above we have $E_2 E_1 A = I$. From here we find $A = E_1^{-1} E_2^{-1} I = E_1^{-1} E_2^{-1}$. Subsituting $E_1^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $E_2^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, we have $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$.

(8) 5. Consider the system

$$\begin{array}{rcrcrcr} x & + & y & - & z & = & 0 \\ 2x & - & y & + & 2z & = & 6 \\ -x & - & y & - & z & = & -6 \end{array} .$$

Use Cramer's rule to solve ONLY for y. (So, you should NOT solve for x and z.)

Solution : The coefficient matrix is $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -1 & 2 \\ -1 & -1 & -1 \end{bmatrix}$, and $A_2 = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 6 & 2 \\ -1 & -6 & -1 \end{bmatrix}$. We compute $\det A = 1 - 2 + 2 + 1 + 2 + 2 = 6$ and $\det A_2 = -6 + 12 - 6 + 12 = +12$. Hence $y = \frac{\det A_2}{\det A} = \frac{12}{6} = 2$.

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(9) **6.** Suppose A, B are 3×3 martices with $\det A = \frac{1}{4}$ and $\det B = 5$. Suppose further that C is an invertible 4×4 matrix, and $D = \begin{bmatrix} 1 & 2 & -1 & 0 \\ -1 & -2 & 1 & 1 \\ 2 & 4 & 2 & 4 \\ 0 & 0 & 1 & -7 \end{bmatrix}$.

Evaluate the following expressions.

(a) $\det(2AB)$
(b) $\det(ABA^T)$
(c) $\det(CDC^{-1})$
(d) $\det(\text{Adj}(A))$

Solution.

(a) $\det(2AB) = (2^3)\left(\frac{1}{4}\right)(5) = 10$.

(b) $\det(ABA^T) = \det A \det B \det A^T = \left(\frac{1}{4}\right)(5)\left(\frac{1}{4}\right) = \frac{5}{16}$.

(c) Notice that the second column is twice the first column. Hence $\det D = 0$. Consequently $\det(CDC^{-1}) = 0$ too.

(d) Recall that $A^{-1} = \frac{1}{\det A} Adj(A)$. Apply determinant to both sides, get

$\det(A^{-1}) = \det\left(\frac{1}{\det A} Adj(A)\right)$. We know $\det A^{-1} = \frac{1}{\det A} = 4$, and

$\det\left(\frac{1}{\det A} Adj(A)\right) = \det(4Adj(A)) = 4^3 \det(Adj(A))$. Hence $4 = 4^3 \det(Adj(A))$, form where we compute $\frac{1}{16} = \det(Adj(A))$.

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(9) **7.** (a) Use row-reduction to find the inverse of the matrix $A = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 1 & -2 \end{bmatrix}$.

(b) Find $\text{Adj}(A)$. [Hint: there is a shortcut.]

Solution. (a)

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{(\frac{1}{2})R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{(-1)R_2 \text{ to } R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{(-1)R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right] \xrightarrow[(-1)R_3 \text{ to } R_1]{R_3 \text{ to } R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -1 & 1 \\ 0 & 1 & 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right]$$

Hence $A^{-1} = \begin{bmatrix} \frac{1}{2} & -1 & 1 \\ 0 & 2 & -1 \\ 0 & 1 & -1 \end{bmatrix}$.

(b) Since $A^{-1} = \dfrac{1}{\det A} Adj(A)$, it follows that $(\det A)A^{-1} = Adj(A)$. We compute $\det A = -2$.

Hence $Adj(A) = (\det A)A^{-1} = (-2)A^{-1} = (-2)\begin{bmatrix} \frac{1}{2} & -1 & 1 \\ 0 & 2 & -1 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 2 & -2 \\ 0 & -4 & 2 \\ 0 & -2 & 2 \end{bmatrix}$

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