
Name: (print clearly) ______________________

Student number: ______________________

I understand that cheating is a serious offence ______________________ (Signature – *In Ink*)

Please place a check mark beside your section number and instructor:

- ☐ A01 — J. Chipalkatti (M/W/F 10:30–11:20am, Armes 200)
- ☐ A02 — T. Kucera (Tu/Th 10:00–11:15am, St. Paul's 100)
- ☐ A03 — I. Bilokopytov (M/W/F 1:30–2:20pm, Human Ecology 206)

INSTRUCTIONS

I. No texts or notes or other reference materials are permitted. No calculators, cellphones, electronic translators, or any WiFi-enabled devices are permitted.

II. This exam has a title page, 9 pages of questions and also one blank page for rough work. Please check that you have all the pages. You may remove the blank page if you want, but be careful not to loosen the staple.

III. The value of each question is indicated in the lefthand margin beside the statement of the question. The total value of all questions is 120 points.

IV. **Answer all questions on the exam paper** in the space provided beneath the question. If you need more room, you may continue your work on the reverse side of the page, but CLEARLY INDICATE that your work is continued.

V. For Questions 2 through 10, always show all your work with full explanations.

VI. After the exam is over, a full solution set will be posted on Prof. Kucera's website, which you can find by navigating from the Mathematics Department web site.

Question	Points	Score
1	20	
2	12	
3	12	
4	12	
5	8	
6	12	
7	12	
8	12	
9	8	
10	12	
Total:	120	

1. Answer the following short-answer questions in the spaces provided. Write a Definition, state a Theorem, or do a short computation. Detailed explanations are *not* needed here.

[2] (a) A row echelon form of the system $A\mathbf{x} = \mathbf{0}$, where A is 7×10, has 4 non-zero rows. How many free variables (parameters) does the general solution to this system have?

[2] (b) What is the elementary matrix corresponding to the following elementary row operation?

$$\begin{bmatrix} 2 & 3 & 5 \\ 1 & 2 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 3 & 5 & 8 \\ 1 & 2 & 3 \end{bmatrix}$$

[2] (c) Assume that A, B are matrices such that $B^T A^T = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. Find AB.

[2] (d) Suppose that T is a linear operator $\mathbb{R}^2 \to \mathbb{R}^2$ such that $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y - 2x \\ 3x + 2y \end{bmatrix}$. Find the matrix A so that for all $\mathbf{x} \in \mathbb{R}^2$, $T(\mathbf{x}) = A\mathbf{x}$.

[2] (e) State the Cauchy-Schwarz inequality.

[2] (f) Find the distance between the point $(2, 3)$ and the line $4x - 3y - 1 = 0$ in $\mathbb{R}^2$.

[2] (g) State the formula for the orthogonal projection of a vector $\mathbf{u}$ on a non-zero vector $\mathbf{a}$.

[2] (h) Write the matrix of the linear operator on $\mathbb{R}^2$ corresponding to a rotation through an angle of $\frac{\pi}{3}$.

[2] (i) Write the characteristic polynomial of the matrix $\begin{bmatrix} 3 & -2 \\ 2 & 4 \end{bmatrix}$.

[2] (j) Define "A and B are *similar* matrices".

[12] 2. Let $\mathbf{u} = (2, 0, 2)$, $\mathbf{v} = (0, 1, 1)$, and $\mathbf{w} = (4, -3, 2)$.

Find numbers x, y, z, such that $x\mathbf{u} + y\mathbf{v} + z\mathbf{w} = (1, -1, -2)$.

3. Let $A = \begin{bmatrix} x^2+1 & -11 & x \\ 0 & x^2-3x & 43 \\ 0 & 0 & x^2-1 \end{bmatrix}$.

[4] (a) Find all real numbers x such that A is *not* invertible.

[4] (b) Find the value of the determinant of A^{-1} when $x = 2$.

[4] (c) What are the eigenvalues of A when $x = 0$? In this case, is A diagonalizable?

[12] 4. Let $A = \begin{bmatrix} -1 & 2 & 4 \\ 3 & 0 & 1 \\ 7 & 1 & -2 \end{bmatrix}$. Using the adjoint formula for A^{-1}, find the entry in the 1st row and 2nd column of A^{-1}. [No credit will be given for any other method.]

[8] 5. Consider the system of equations

$$\begin{aligned} x + 2y &= 2 \\ x - y &= 7 \end{aligned}$$

Find the value of x using Cramer's rule. [No credit will be given for any other method.]

6. Consider the vectors

$$\mathbf{u} = (\,-1,\, 2,\, 5\,)\,, \qquad \mathbf{v} = (\,4,\, 1,\, -3\,)$$

in $\mathbb{R}^3$. Let θ denote the angle between them.

[5] (a) Use the dot product of $\mathbf{u}$ and $\mathbf{v}$ to find $\cos\theta$. Is θ acute or obtuse?

[5] (b) Use the cross product of $\mathbf{u}$ and $\mathbf{v}$ to find $\sin\theta$.

[2] (c) Verify that your answers to parts (a) and (b) are correct by checking that

$$\cos^2\theta + \sin^2\theta = 1\,.$$

[12] 7. Find an equation of the plane in $\mathbb{R}^3$ containing the points

$$A = (1, -1, 0)\,, \quad B = (0, 0, -2)\,, \quad C = (1, 1, 1)\,.$$

[12] 8. Consider the vectors

$$\mathbf{u} = (\,k,\ 2,\ 1\,)\ , \qquad \mathbf{v} = (\,3,\ -1,\ 0\,)\ , \qquad \mathbf{w} = (\,5,\ 1,\ -2\,)$$

in $\mathbb{R}^3$.

Find **all** the values of k such that the volume of the parallelopiped generated by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ is 10.

[8] 9. $\mathbf{T}$ is a linear transformation $\mathbb{R}^3 \to \mathbb{R}^3$ such that

$$\mathbf{T}(\mathbf{e}_1) = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}, \quad \mathbf{T}(\mathbf{e}_1) = \begin{bmatrix} -1 \\ 5 \\ -9 \end{bmatrix}, \quad \text{and } \mathbf{T}(\mathbf{e}_1) = \begin{bmatrix} 2 \\ -6 \\ 5 \end{bmatrix}.$$

(a) Find the standard matrix A of $\mathbf{T}$.

(b) Evaluate $\mathbf{T}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$, using part (a) or by any other method.

10. Consider the matrix

$$A = \begin{bmatrix} 22 & -10 \\ 50 & -23 \end{bmatrix},$$

We have chosen the entries carefully so that all the calculations in the following result in reasonably small numbers.

[8] (a) Find all the eigenvalues of A and an eigenvector associated with each.

[4] (b) Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$. You *do not* need to calculate P^{-1} or verify that the equation holds.

This page intended for rough work only