CHAPTER 1. SYSTEMS OF LINEAR EQUATIONS AND MATRICES

1.1. INTRODUCTION TO SYSTEMS OF LINEAR EQUATIONS

EXAMPLE 1: WE HAVE TO FIND OUT THE NUMBER OF RED AND THE NUMBER OF BLUE BALLOONS IN A BAG.

THERE ARE TWO MORE RED BALLOONS THAN THERE ARE BLUE ONES.

IF THERE WOULD BE TWICE AS MANY BLUE BALLOONS THAN THERE ARE NOW, THERE WOULD BE 8 BALLOONS IN THE BAG ALL TOGETHER.

A **LINEAR EQUATION** IN n VARIABLES (**UNKNOWNS**) $x_1, x_2, \ldots, x_n$ IS AN EQUATION OF THE FORM:

$$a_1 x_1 + a_2 x_2 + \ldots + a_n x_n = b$$

WHERE $a_1, a_2, \ldots, a_n$ AND b ARE REAL NUMBERS.

EXAMPLE 2: a) $2x_1 - 5x_3 + x_4 = 17$

b) $x + 4yz = \cos z$

c) $y = 3x - \frac{2}{7}z$

A **SOLUTION** OF THE EQUATION $a_1x_1+a_2x_2+\ldots+a_nx_n=b$ IS AN n-TUPLE OF NUMBERS $(s_1, s_2, \ldots, s_n)$ THAT SATISFIES THE EQUATION WHEN WE SUBSTITUTE $x_1=s_1, x_2=s_2, \ldots, x_n=s_n$.

THE SET OF ALL OF THE SOLUTIONS OF THE EQUATION IS CALLED THE **SOLUTION SET.**

EXAMPLE 3: $s_1=2$, $s_2=4$ IS A SOLUTION OF $2x+y=8$, BUT SO IS $s_1=0$ AND $s_2=8$. $\{(2,4)$ AND $(0,8)$

SYSTEM OF m EQUATIONS IN n UNKNOWNS:

$$\begin{aligned} a_{11}x_1+a_{12}x_2+\ldots+a_{1n}x_n&=b_1\\ a_{21}x_1+a_{22}x_2+\ldots+a_{2n}x_n&=b_2\\ &\vdots\\ a_{m1}x_1+a_{m2}x_2+\ldots+a_{mn}x_n&=b_m \end{aligned}$$

SOLUTION SET?

EXAMPLE 4: a) $\begin{aligned}2x+y&=8\\ x-y&=-2\end{aligned}$ b) $\begin{aligned}2x+y&=8\\ 2x+y&=4\end{aligned}$ c) $\begin{aligned}2x+y&=8\\ 4x+2y&=16\end{aligned}$

EVERY SYSTEM OF LINEAR EQUATIONS HAS EITHER EXACTLY ONE SOLUTION, NO SOLUTIONS, OR INFINITELY MANY SOLUTIONS.

SYSTEM WITH NO SOLUTIONS: **INCONSISTENT.**

PROBLEM: HOW TO DETERMINE IF A LINEAR SYSTEM HAS SOLUTIONS AND HOW TO FIND THEM?

EXAMPLE 5:

$$\begin{array}{l} \mathrm{I} \\ \mathrm{II} \\ \mathrm{III} \end{array} \quad \begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned} \qquad \begin{array}{c} \begin{array}{ccc|c} x_1 & x_2 & x_3 & \end{array} \\ \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right] \end{array}$$

$\mathrm{III} \to 4\,\mathrm{I} + \mathrm{III}$ $\qquad$ $R_3 \to 4R_1 + R_3$

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -3x_2 + 13x_3 &= -9 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right]$$

$\mathrm{II} \to \frac{1}{2}\,\mathrm{II}$ $\qquad$ $R_2 \to \frac{1}{2} R_2$

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ -3x_2 + 13x_3 &= -9 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array}\right]$$

$\mathrm{III} \to 3\,\mathrm{II} + \mathrm{III}$ $\qquad$ $R_3 \to 3R_2 + R_3$

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ x_3 &= 3 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$\mathrm{II} \to 4\,\mathrm{III} + \mathrm{II}\,;\ \mathrm{I} \to -\mathrm{III} + \mathrm{I}$ $\qquad$ $R_2 \to 4R_3 + R_2\,;\ R_1 \to -R_3 + R_1$

$$\begin{aligned} x_1 - 2x_2 &= -3 \\ x_2 &= 16 \\ x_3 &= 3 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$\mathrm{I} \to 2\,\mathrm{II} + \mathrm{I}$ $\qquad$ $R_1 \to 2R_2 + R_1$

$$\begin{aligned} x_1 &= 29 \\ x_2 &= 16 \\ x_3 &= 3 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

MATRIX IS A RECTANGULAR ARRAY OF NUMBERS.

AUGMENTED MATRIX FOR THE SYSTEM OF m EQUATIONS WITH n UNKNOWNS:

$$\left[\begin{array}{cccc:c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

m - ROWS , $n+1$ COLUMNS ; $m \times (n+1)$ MATRIX

ELEMENTARY ROW OPERATIONS:

1. MULTIPLY A ROW BY A NONZERO CONSTANT.
2. ADD A MULTIPLE OF ONE ROW TO ANOTHER ROW.
3. INTERCHANGE ROWS.

EXAMPLE 6: a) $\begin{aligned} 2x+y&=8 \\ 4x+2y&=16 \end{aligned}$ b) $\begin{aligned} x_2+5x_3&=-4 \\ x_1+4x_2+3x_3&=-2 \\ 2x_1+7x_2+x_3&=-1 \end{aligned}$

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