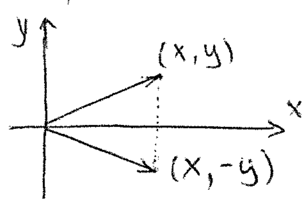
4.9 REFLECTION, ROTATION, DILATION

BASIC MATRIX TRANSFORMATIONS IN $\mathbb{R}^2$ AND $\mathbb{R}^3$.

REFLECTION: MAP EACH POINT INTO ITS SYMMETRIC IMAGE ABOUT A FIXED LINE OR A PLANE.

1. EXAMPLES IN $\mathbb{R}^2$:

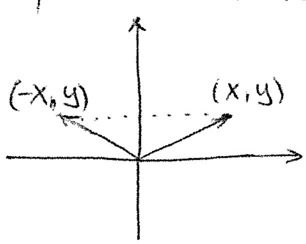
a) REFLECTION ABOUT THE X-AXIS: $T(x,y)=(x,-y)$



$Te_1 = T(1,0)=(1,0)$

$Te_2 = T(0,1)=(0,-1)$; $A=\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $(T=T_A)$

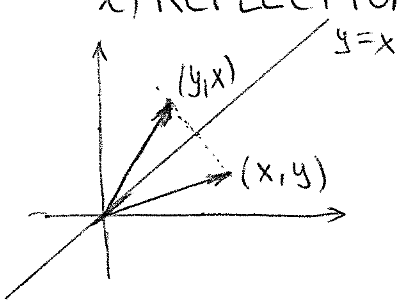
b) REFLECTION ABOUT THE y-AXIS: $T(x,y)=(-x,y)$



$Te_1 = T(1,0)=(-1,0)$

$Te_2 = T(0,1)=(0,1)$; $A=\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ $(T=T_A)$

c) REFLECTION ABOUT THE LINE $y=x$: $T(x,y)=(y,x)$

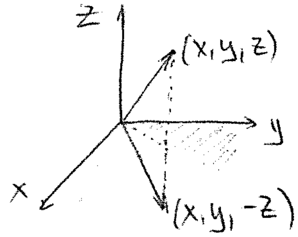


$Te_1 = T(1,0)=(0,1)$

$Te_2 = T(0,1)=(1,0)$; $A=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $(T=T_A)$

2. EXAMPLES IN $\mathbb{R}^3$

a) REFLECTION ABOUT THE XY-PLANE: $T(x,y,z)=(x,y,-z)$



$$Te_1 = T(1,0,0)=(1,0,0)$$
$$Te_2 = T(0,1,0)=(0,1,0)\ ; \quad A=\begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix} \quad (T=T_A)$$
$$Te_3 = T(0,0,1)=(0,0,-1)$$

b) REFLECTION ABOUT THE XZ-PLANE: $T(x,y,z)=(x,-y,z)$

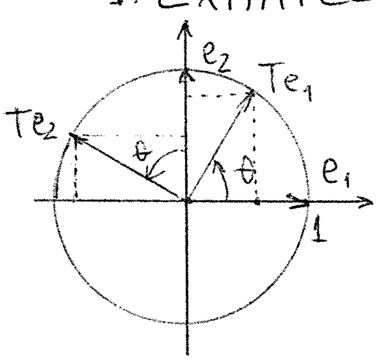
$$\ldots \quad A=\begin{bmatrix}1&0&0\\0&-1&0\\0&0&1\end{bmatrix} \quad \ldots$$

c) REFLECTION ABOUT THE YZ-PLANE: $T(x,y,z)=(-x,y,z)$

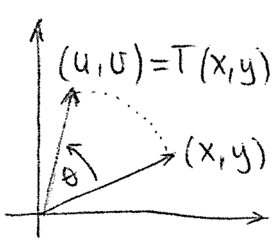
$$\ldots \quad A=\begin{bmatrix}-1&0&0\\0&1&0\\0&0&1\end{bmatrix} \quad \ldots$$

<u>ROTATION</u>: A POINT IS ROTATED ALONG ARCS OF CIRCLES CENTERED AT THE ORIGIN.

1. EXAMPLE IN $\mathbb{R}^2$ (COUNTERCLOCKWISE)



$$Te_1 = T(1,0) = (\cos\theta, \sin\theta)$$
$$Te_2 = T(0,1) = \left(\cos\left(\theta+\tfrac{\pi}{2}\right), \sin\left(\theta+\tfrac{\pi}{2}\right)\right); \quad A=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$$
$$= (-\sin\theta, \cos\theta)$$



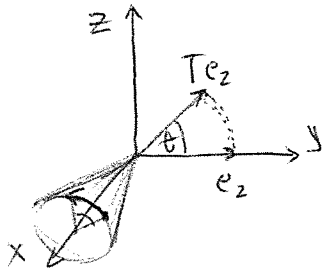
$$T(x,y)=T_A(x,y)=A\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x\cos\theta - y\sin\theta\\ x\sin\theta + y\cos\theta\end{bmatrix}=\begin{bmatrix}u\\v\end{bmatrix}$$

$$u = x\cos\theta - y\sin\theta, \quad v = x\sin\theta + y\cos\theta$$

$$T(x,y) = (x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta)$$

2. EXAMPLES IN $\mathbb{R}^3$ (COUNTERCLOCKWISE)

a) ROTATION ABOUT THE POSITIVE X-AXIS FOR ANGLE θ:



$$Te_1 = T(1,0,0) = (1,0,0)$$
$$Te_2 = T(0,1,0) = (0,\cos\theta,\sin\theta)$$
$$Te_3 = T(0,0,1) = (0,-\sin\theta,\cos\theta)$$

$$; \quad A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

b) ROTATION ABOUT THE POSITIVE y-AXIS FOR ANGLE θ:

$$\ldots \quad A = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \quad \ldots \quad (T = T_A)$$

c) ROTATION ABOUT THE POSITIVE Z-AXIS FOR ANGLE θ:

$$\ldots \quad A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \ldots \quad (T = T_A)$$

"<u>DILATION</u>": FOR k IN $\mathbb{R}$, $T\underline{x} = k\underline{x}$ $\quad (k \geq 0)$

IF $k > 1$, T IS A <u>DILATION</u>; IF $0 \leq k < 1$, T IS A <u>CONTRACTION</u>

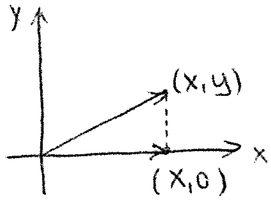
1. IN $\mathbb{R}^2$: $Te_1 = ke_1 = (k,0)$, $Te_2 = ke_2 = (0,k)$ $\quad . \; A = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \quad (T = T_A)$

2. IN $\mathbb{R}^3$ $\ldots \quad A = \begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}.$

PROJECTION (ORTHOGONAL): A POINT IS MAPPED (PROJECTED) ORTHOGONALLY ONTO A FIXED LINE OR A PLANE

1. EXAMPLES IN $\mathbb{R}^2$

a) PROJECTION ONTO THE X-AXIS: $T(x,y)=(x,0)$



$$Te_1 = e_1 = (1,0) \qquad ; \quad A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$
$$Te_2 = (0,0)$$

b) PROJECTION ONTO THE y-AXIS: $T(x,y)=(0,y)$

$$\ldots \quad A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \ldots$$

2. EXAMPLES IN $\mathbb{R}^3$:

a) PROJECTION ONTO THE xy-PLANE: $T(x,y,z)=(x,y,0)$

$$\ldots \quad A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \ldots$$