CHAPTER 5: EIGENVALUES AND EIGENVECTORS

5.1. EIGENVALUES AND EIGENVECTORS

A NONZERO VECTOR $\underline{x}$ IN $\mathbb{R}^n$ IS CALLED AN EIGENVECTOR OF THE $n \times n$ MATRIX A IF THERE EXISTS A SCALAR λ IN $\mathbb{R}$ SUCH THAT $A\underline{x} = \lambda \underline{x}$. ($\lambda$ IS AN EIGENVALUE OF A.)

NOTE: $\underline{x}$ IS AN EIGENVECTOR FOR THE MATRIX TRANSFORM. T_A CORRESPONDING TO THE EIGENVALUE λ OF T_A.

EXAMPLE 1: $\underline{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $A = \begin{bmatrix} 5 & -1 \\ 1 & 3 \end{bmatrix}$

THEOREM 1: FOR AN $n \times n$ MATRIX A, λ IS AN EIGENVALUE OF A IF AND ONLY IF $\det(\lambda I - A) = 0$. (CHARACTERISTIC EQ.)

PROOF:

EXAMPLE 2: FIND THE EIGENVALUES OF $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

NOTE: NOT EVERY $n \times n$ MATRIX HAS (A REAL) EIGENVALUE.

EXAMPLE 3: $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$p_A(\lambda) = \det(\lambda I - A)$ IS CALLED THE CHARACTERISTIC POLYNOMIAL OF A.

THEOREM 2: IF A IS AN $n \times n$ TRIANGULAR MATRIX (UPPER, LOWER OR DIAGONAL) THEN THE EIGENVALUES OF A ARE THE MAIN DIAGONAL ENTRIES.

PROOF:

<u>THEOREM 3</u>: IF A IS AN $n \times n$ MATRIX, THE FOLLOWING STATEMENTS ARE EQUIVALENT:

a) λ IS AN EIGENVALUE OF A.
b) λ IS A SOLUTION OF THE CHARACTER. EQ. $\det(\lambda I - A) = 0$.
c) THE HOMOGENEOUS SYSTEM $(\lambda I - A)\underline{x} = \underline{0}$ HAS NONTRIVIAL SOLUTIONS.

NOTE: IF $A\underline{x} = \lambda \underline{x}$ AND t IS A SCALAR IN $\mathbb{R}$, THEN $A(t\underline{x}) = \lambda(t\underline{x})$, i.e. IF $\underline{x}$ IS AN EIGENVECTOR CORRESPONDING TO THE EIGENVALUE λ OF A, THEN $t\underline{x}$ IS ALSO AN EIGENVECTOR OF A CORRESPONDING TO λ.

HOW TO FIND EIGENVECTORS (CORRESPONDING TO λ)?

<u>EXAMPLE 4</u>: $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, $\lambda_1 = -1$, $\lambda_2 = 2$. (Ex. 2)
(<u>BASIC EIGENVECTORS</u>)

<u>THEOREM 4</u>: IF A IS AN $n \times n$ MATRIX, THEN A IS INVERTIBLE IF AND ONLY IF $\lambda = 0$ IS NOT AN EIGENVALUE OF A.

<u>PROOF</u>: