5.2 DIAGONALIZATION

THE $n \times n$ MATRICES A AND B ARE SIMILAR IF THERE EXISTS AN INVERTIBLE $n \times n$ MATRIX P SUCH THAT $B = P^{-1}AP$.

AN $n \times n$ MATRIX A IS DIAGONALIZABLE, IF A IS SIMILAR TO A DIAGONAL MATRIX, I.E IF THERE EXISTS AN INVERTIBLE $n \times n$ MATRIX P SUCH THAT $P^{-1}AP$ IS DIAGONAL.

WE WILL SAY THAT THE VECTORS $\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_n$ IN $\mathbb{R}^n$ ARE LINEARLY INDEPENDENT IF THE $n \times n$ MATRIX

$P = [\underline{u}_1 \ \underline{u}_2 \ \ldots \ \underline{u}_n]$ IS INVERTIBLE.

THEOREM 1: LET A BE AN $n \times n$ MATRIX. THEN A IS DIAGONALIZABLE IF AND ONLY IF A HAS n LINEARLY INDEPENDENT EIGENVECTORS.

PROOF:

PROCEDURE FOR DIAGONALIZATION OF A ($n \times n$ MATRIX)

STEP 1: FIND THE EIGENVALUES OF A BY SOLVING

$$\det(\lambda I - A) = 0.$$

STEP 2: FOR EACH EIGENVALUE λ, FIND THE CORRESPONDING BASIC EIGENVECTORS. IF THE TOTAL NUMBER OF BASIC EIGENVECTOR IS n: A IS DIAGONALIZABLE. (IF THE TOTAL IS LESS THAN n, IT IS NOT.)

STEP 3: IF A IS DIAGONALIZABLE: $P = [\underline{u}_1 \ \underline{u}_2, \ldots \ \underline{u}_n]$, AND $P^{-1}AP$ IS DIAGONAL MATRIX WITH DIAGONAL $\lambda_1, \lambda_2, \ldots, \lambda_n$.

<u>NOTE:</u> IN STEP 3. OF THE PROCEDURE $A\underline{u}_1=\lambda_1\underline{u}_1$, $A\underline{u}_2=\lambda_2\underline{u}_2, \dots$ $A\underline{u}_n=\lambda_n\underline{u}_n$, WITH $\underline{u}_1, \underline{u}_2, \dots \underline{u}_n$ BASIC EIGENVECTORS CORRESPONDING TO THE EIGENVALUES $\lambda_1, \lambda_2, \dots, \lambda_n$.

IT COULD BE THAT $\lambda_i=\lambda_j$ FOR SOME i,j WITH $i\neq j$.

<u>EXAMPLE 1:</u> SHOW THAT $A=\begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}$ IS DIAGONALIZAB

AND FIND P SUCH THAT $P^{-1}AP$ IS DIAGONAL.

FIND $P^{-1}AP$.

<u>EXAMPLE 2:</u> SHOW THAT $A=\begin{bmatrix}5&-1\\1&3\end{bmatrix}$ IS NOT DIAGONALIZ

<u>THEOREM 2:</u> IF A IS AN $n\times n$ MATRIX WITH n DISTINCT EIGENVALUES THEN A IS DIAGONALIZABLE.

<u>PROOF:</u>

<u>EXAMPLE 3.</u> SHOW THAT $A=\begin{bmatrix}1&-1\\0&0\end{bmatrix}$ IS DIAGONALIZABLE.

<u>THEOREM 3:</u> IF k IS A POSITIVE INTEGER AND $A\underline{x}=\lambda\underline{x}$ FOR $\underline{x}\neq\underline{0}$, THEN $A^k\underline{x}=\lambda^k\underline{x}$.

<u>PROOF:</u> $A^2\underline{x}=A(A\underline{x})=A(\lambda\underline{x})=\lambda(A\underline{x})=\lambda(\lambda\underline{x})=\lambda^2\underline{x}, \dots\dots$

<u>NOTE:</u> IF $P^{-1}AP=\begin{bmatrix}\lambda_1&0&\cdots&0\\0&\lambda_2&\cdots&0\\ \vdots&&\ddots&\vdots\\0&0&\cdots&\lambda_n\end{bmatrix}$, THEN $(P^{-1}AP)^k=\begin{bmatrix}\lambda_1^k&0&\cdots&0\\0&\lambda_2^k&\cdots&0\\ \vdots&&\ddots&\vdots\\0&0&\cdots&\lambda_n^k\end{bmatrix}=P^{-1}A^kP$

AND SO $A^k=P\begin{bmatrix}\lambda_1^k&0&\cdots&0\\0&\lambda_2^k&\cdots&0\\ \vdots&&\ddots&\vdots\\0&0&\cdots&\lambda_n^k\end{bmatrix}P^{-1}$. (COMPUTING POWERS OF A MATRIX).

Ex.4:...