



- [8] 1. Determine the value of  $k$  for which the following linear system has infinitely many solutions, and then find those solutions.

$$\begin{aligned}x + 2y + 3z &= 0 \\2x + 3y + z &= 2 \\-x - y + 2z &= k.\end{aligned}$$

The augmented matrix for this system is

$$\left[ \begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 3 & 1 & 2 \\ -1 & -1 & 2 & k \end{array} \right] \begin{matrix} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 + R_1 \end{matrix}$$

$$\left[ \begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -1 & -5 & 2 \\ 0 & 1 & 5 & k \end{array} \right] \begin{matrix} R_3 \rightarrow R_3 + R_2 \end{matrix}$$

$$\left[ \begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -1 & -5 & 2 \\ 0 & 0 & 0 & k+2 \end{array} \right] \begin{matrix} R_2 \rightarrow -1R_2 \end{matrix}$$

$$\left[ \begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 5 & -2 \\ 0 & 0 & 0 & k+2 \end{array} \right]$$

This system has infinitely many solutions when the bottom row is all zeros  
 i.e.  $k+2=0$ ,  $k=-2$

$$\left[ \begin{array}{ccc|c} 1 & 0 & -7 & 4 \\ 0 & 1 & 5 & -2 \\ 0 & 0 & 0 & k+2 \end{array} \right]$$

They may put the matrix in RREF.

When  $k=-2$   $z$  is a free parameter  
 and the solutions to the system are

$$x = 4 + 7t$$

$$y = -2 - 5t$$

$$z = t, t \in \mathbb{R}$$



- [10] 2. Using any method you like, find the inverse of the matrix

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 5 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix}$$

Row Reduction

$$\left[ \begin{array}{ccc|ccc} 3 & 1 & 2 & 1 & 0 & 0 \\ 5 & 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array} \right] R_3 \rightarrow R_3 + 2R_1$$

$$\left[ \begin{array}{ccc|ccc} 3 & 1 & 2 & 1 & 0 & 0 \\ 5 & 0 & 1 & 0 & 1 & 0 \\ -1 & 0 & 0 & -1 & 0 & 1 \end{array} \right] R_3 \rightarrow -1R_3$$

$$\left[ \begin{array}{ccc|ccc} 3 & 1 & 2 & 1 & 0 & 0 \\ 5 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & -1 \end{array} \right] R_1 \rightarrow R_1 + (-3)R_3, R_2 \rightarrow R_2 + (-5)R_3$$

$$\left[ \begin{array}{ccc|ccc} 0 & 1 & 2 & -2 & 0 & 3 \\ 0 & 0 & 1 & -5 & 1 & 5 \\ 1 & 0 & 0 & 1 & 0 & -1 \end{array} \right] R_1 \rightarrow R_1 + 2R_2$$

$$\left[ \begin{array}{ccc|ccc} 0 & 1 & 0 & 8 & -2 & -7 \\ 0 & 0 & 1 & -5 & 1 & 5 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{array} \right] R_1 \leftrightarrow R_2$$

$$\left[ \begin{array}{ccc|ccc} 0 & 0 & 1 & -5 & 1 & 5 \\ 0 & 1 & 0 & 8 & -2 & -7 \\ 1 & 0 & 0 & 1 & 0 & -1 \end{array} \right] R_3 \leftrightarrow R_1$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 8 & -2 & -7 \\ 0 & 0 & 1 & -5 & 1 & 5 \end{array} \right]$$

$$\therefore A^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 8 & -2 & -7 \\ -5 & 1 & 5 \end{bmatrix} \text{ by the row reduction method.}$$

## 25 Adjoint Method

$$C_{11} = (-1)^{1+1} \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} = -1 \quad C_{12} = (-1)^{1+2} \begin{vmatrix} 5 & 1 \\ 2 & 2 \end{vmatrix} = -8 \quad C_{13} = (-1)^{1+3} \begin{vmatrix} 5 & 0 \\ 2 & 1 \end{vmatrix} = 5$$

$$C_{21} = (-1)^{2+1} \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = 0 \quad C_{22} = (-1)^{2+2} \begin{vmatrix} 3 & 2 \\ 2 & 2 \end{vmatrix} = 2 \quad C_{23} = (-1)^{2+3} \begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix} = -1$$

$$C_{31} = (-1)^{3+1} \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 1 \quad C_{32} = (-1)^{3+2} \begin{vmatrix} 3 & 2 \\ 5 & 1 \end{vmatrix} = 7 \quad C_{33} = (-1)^{3+3} \begin{vmatrix} 3 & 1 \\ 5 & 0 \end{vmatrix} = -5$$

$\therefore$  The cofactor matrix of  $A$  is

$$C = \begin{bmatrix} -1 & -8 & 5 \\ 0 & 2 & -1 \\ 1 & 7 & -5 \end{bmatrix}, \quad \text{adj}(A) = \begin{bmatrix} -1 & 0 & 1 \\ -8 & 2 & 7 \\ 5 & -1 & -5 \end{bmatrix}$$

$$\det(A) = 1 \cdot C_{11} + 0 \cdot C_{21} + 1 \cdot C_{31} = -8 + 7 = -1$$

$$\therefore A^{-1} = \frac{\text{adj}(A)}{\det(A)} = \begin{bmatrix} 1 & 0 & -1 \\ 8 & -2 & -7 \\ -5 & 1 & 5 \end{bmatrix}$$



3. Consider the matrix

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 3 & -6 & 3 \end{bmatrix}.$$

- [2] (a) Compute  $\det A$ .

$$\det(A) = 0 \cdot C_{21} + 1 \cdot C_{22} + 0 \cdot C_{23} = (-1)^{2+2} \begin{vmatrix} 1 & 0 \\ 3 & 3 \end{vmatrix} = 3$$

- [10] (b) Is  $A$  row equivalent to  $I_3$ ? If not, explain why; if it is, find elementary matrices  $E_1, E_2, \dots, E_n$  such that  $E_n \cdots E_1 A = I_3$ .

Yes,  $\det(A) \neq 0$  which is equivalent to  $A$  is row equivalent to  $I_3$  by a theorem from class.

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 3 & -6 & 3 \end{bmatrix} R_3 \rightarrow R_3 + (-3)R_1 \quad \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} R_3 \rightarrow \frac{1}{3}R_3$$

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} R_1 \rightarrow R_1 + 2R_2 \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 3 & -6 & 3 \end{bmatrix}$$



- [9] 4. Compute the determinant of the matrix

$$A = \begin{bmatrix} 1 & -3 & -2 & 0 \\ 2 & 1 & 4 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & 2 \end{bmatrix}.$$

Cofactor Expansion

$$\begin{aligned} \det(A) &= 0 \cdot C_{31} + 0 \cdot C_{32} + 1 \cdot C_{33} + 1 \cdot C_{34} \\ &= (-1)^{3+3} \begin{vmatrix} 1 & -3 & 0 \\ 2 & 1 & -1 \\ 0 & 0 & 2 \end{vmatrix} + (-1)^{3+4} \begin{vmatrix} 1 & -3 & -2 \\ 2 & 1 & 4 \\ 0 & 0 & -2 \end{vmatrix} \\ &= (0 \cdot C_{31} + 0 \cdot C_{32} + 2C_{33}) - (0 \cdot C_{31} + 0 \cdot C_{32} + (-2)C_{33}) \\ &= 2(-1)^{3+3} \begin{vmatrix} 1 & -3 \\ 2 & 1 \end{vmatrix} + 2(-1)^{3+3} \begin{vmatrix} 1 & -3 \\ 2 & 1 \end{vmatrix} \\ &= 2 \cdot (1+6) + 2 \cdot (1+6) \\ &= 28 \end{aligned}$$

$\infty$  Row Reduction

$$\left| \begin{array}{cccc} 1 & -3 & -2 & 0 \\ 2 & 1 & 4 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & 2 \end{array} \right| \xrightarrow{R_2 \rightarrow R_2 + (-2)R_1} = \left| \begin{array}{cccc} 1 & -3 & -2 & 0 \\ 0 & 7 & 8 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & 2 \end{array} \right| \xrightarrow{R_4 \rightarrow R_4 + 2R_3} =$$

$$= \begin{vmatrix} 1 & -3 & -2 & 0 \\ 0 & 7 & 8 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 4 \end{vmatrix} = 1 \cdot 7 \cdot 1 \cdot 4 = 28$$



- [9] 5. Consider the matrix

$$A = \begin{bmatrix} 1 & a \\ 1 & 0 \end{bmatrix}.$$

(a) Are there values of  $a$  for which the expression

$$\det(A^2) - \det(3A) - 2a \det(A^{-1})$$

is not defined?

$\det(A) = -a$ ,  $A^{-1}$  is undefined if  $\det(A) = 0$ .

$\therefore$  The expression is undefined if  $a=0$ .

- (b) Suppose that  $\det(A^2) - \det(3A) - 2a \det(A^{-1}) = 2$ . What is the value of  $a$ ?

$$\det(A^2) = (\det(A))^2 = (-a)^2 = a^2$$

$$-\det(3A) = -3^2 \det(A) = -9a$$

$$\det(A^{-1}) = \frac{1}{(-a)} \quad \text{(only well defined if } a \neq 0\text{)}$$

$$\therefore \det(A^2) - \det(3A) - 2a \det(A^{-1}) = 2$$

$$\Rightarrow a^2 + 9a + 2 = 2$$

$$\Leftrightarrow a^2 + 9a = 0$$

$$\Leftrightarrow a(a+9) = 0$$

$$\Leftrightarrow a=0 \text{ or } a=-9$$

$$a \neq 0 \text{ from (a)} \therefore a = -9$$



- [6] 6. A square  $n \times n$  matrix such that  $A^2 = A$  is called *idempotent*. Show that the only nonsingular (i.e., invertible) idempotent matrix is  $A = I_n$ , the  $n \times n$  identity matrix.

Suppose  $A$  is idempotent and non singular.  
Then  $A^2 = A$  and there exists  $A^{-1}$  such that

$$AA^{-1} = A^{-1}A = I_n$$

$$\text{Then } A^2 = A$$

$$\Leftrightarrow A^{-1}A^2 = A^{-1}A$$

$$\Leftrightarrow A^{-1}A \cdot A = I_n \quad (A^{-1}A = I_n)$$

$$\Leftrightarrow I_n \cdot A = I_n$$

$$\Leftrightarrow A = I_n$$

$\therefore$  If  $A$  is idempotent and nonsingular  
and  $n \times n$  then  $A = I_n$ .

- [6] 7. State three conditions equivalent to “The square  $n \times n$  matrix  $A$  is invertible”.

(a)  $\det(A) \neq 0$  / For every  $n \times 1$   $b$   $Ax=b$  is consistent /

(b) For every  $n \times 1$   $b$   $Ax=b$  has a unique solution /  $Ax=0$  has only the trivial soln, /

(c) The RREF of  $A$  is  $I_n$  /  $A$  is a product of elementary matrices /

There is an  $n \times n$   $B$  s.t.  $AB = I_n$  /

There is an  $n \times n$   $B$  s.t.  $BA = I_n$  /